A planetary wave model for Saturn’s 10.7-hour periodicities

C. G. A. Smith\textsuperscript{a,b}, L. C. Ray\textsuperscript{b}, N. A. Achilleos\textsuperscript{b}

\textsuperscript{a}Physics Department, The Brooksbank School, Elland, West Yorkshire, UK, HX5 0QG
\textsuperscript{b}Department of Physics and Astronomy, Centre for Planetary Sciences, University College London, Gower St., London WC1E 6BT

Abstract

A proposed resolution of the unexplained 10.7-hour periodicities in Saturn’s magnetosphere is a system of atmospheric vortices in the polar regions of the planet. We investigate a description of such vortices in terms of planetary-scale waves. Approximating the polar regions as flat, we use theory developed originally by Haurwitz (1975) to find circumpolar Rossby wave solutions for Saturn’s upper stratosphere and lower thermosphere. We find vertically propagating twin vortex solutions that drift slowly westwards at < 1\% of the deep planetary angular velocity and are thus ideal candidates for explaining the observed periodicities. To produce integrated field-aligned currents of the order of 1MA we require wind velocities of \( \sim 70\text{ms}^{-1} \). A particular class of vertically propagating solutions are potentially consistent with wave energy being ‘trapped’ between the deep atmosphere and lower thermosphere, at altitudes suited to the production of the necessary field-aligned current systems.

Keywords: Atmospheres, dynamics, Aurorae, Ionospheres, Saturn,

Email address: cgasmith@gmail.com (C. G. A. Smith)

Preprint submitted to Icarus November 28, 2015
Figure 1: Sketches of the proposed vortex system. In each sketch the dotted line indicates the central line of the main auroral oval, and the shaded region the zone of enhanced ionisation associated with auroral electron precipitation. (a) Sketch of twin vortex flows. (b) Sketch of Pedersen currents. (c) Sketch of Hall currents.

1. Introduction

The $\sim 10.7$-hour modulation of various phenomena in Saturn’s magnetosphere (see review by [Carbary and Mitchell, 2013]) has yet to be fully explained. The idea of a vortex-like structure in the neutral atmosphere driving magnetospheric periodicities was first proposed by [Smith, 2006] and investigated further by [Smith, 2011] and [Smith and Achilleos, 2012]. The conclusion of these studies was that a thermospheric vortex could drive approximately the observed magnetic perturbations in the magnetosphere, but that the energy required to sustain magnetic perturbations of the observed magnitude was improbably large.

A complementary approach to the same conceptual model ([Jia et al., 2012], [Jia and Kivelson, 2012]) imposed twin-vortex flows directly on the atmosphere, Saturn, magnetosphere.
ionospheric plasma and calculated the detailed implications for the magnetosphere, using a magnetohydrodynamic model of this region. This approach reproduced many of the observed phenomena, but the thermospheric flow speeds prescribed by the model as a boundary condition were implausibly large. More recently, Southwood and Cowley (2014) presented a qualitative model of twin vortices in both northern and southern polar ionospheres, able to explain the ‘mixed’ northern and southern signals observed on closed field lines and the ‘pure’ northern and southern signals observed on open field lines.

Most recently, Smith (2014) synthesised the Southwood and Cowley (2014) model with lessons learnt from thermosphere modelling (Smith et al., 2005; Müller-Wodarg et al., 2006; Smith and Aylward, 2008; Smith, 2011; Smith and Achilleos, 2012), proposing that the vortices are located not in the thermosphere but in the upper stratosphere, around an altitude of ∼750km above the 1-bar level. Two reasons were given for this suggestion. First, the polar thermosphere substantially subcorotates and so cannot sustain a vortex system with a steady ∼10.7-hour rotation period. Second, a thermospheric vortex system of the required magnitude would entail an unrealistically large thermal energy input, the heating effect of which would produce thermospheric temperatures far greater than those that are observed.

A vortex system located in the upper stratosphere would interact with the ionisation produced at these altitudes by the particle precipitation associated with the main auroral oval, thus generating horizontally divergent currents that flow into and drive the magnetosphere. This scenario is sketched in Fig. 1. Panel (a) shows a simple twin-cell vortex system. Panels (b) and (c)
then indicate the currents driven by the interaction between these vortices and a region of enhanced conductance (indicated by the shaded regions). Panel (b) shows Pedersen currents and panel (c) Hall currents.

A number of studies have also examined empirical evidence for a neutral atmosphere source. Cowley and Provan (2013) examined the rotation periods of a number of neutral atmospheric features and searched for correlations with the observed periodicities in the magnetosphere. They found no convincing correlation that might indicate a direct causal link. Fischer et al. (2014) investigated a possible correlation between the presence of the Great White Spot in the northern hemisphere and a pronounced shift in the period of the 10.7-hour signal, but were unable to find a physical link between the two phenomena. While both of these studies were inconclusive, they dealt with tropospheric and lower stratospheric phenomena. They thus in no way rule out a source in the upper stratosphere or thermosphere. A different type of evidence was presented by Hunt et al. (2014) who analysed observed field-aligned currents in the southern auroral region, concluding that they provide evidence for energy flow outwards from the planet. This indicates an atmospheric location for the original source of energy. All of this evidence taken together – no evidence for a lower atmosphere source but positive evidence for an atmospheric source – points towards an upper atmosphere source as proposed by the recent theoretical studies referenced above (Jia et al., 2012; Southwood and Cowley, 2014; Smith, 2014).

Despite this evidence, as yet there has been no detailed model of how a twin vortex system could be generated or sustained in the upper atmosphere. A possible description of such a global vortex system is in terms of planetary-
scale waves. The purpose of this paper is to explore such a description of the
required vortices in terms of circumpolar Rossby waves. In Section 2 we will
outline how the properties of Rossby waves make them suitable candidates.
In Section 3 we will then develop a theoretical description of circumpolar
Rossby waves using the work of Haurwitz (1975). In Section 4 we will then
analyse explicit solutions of our equations, including predictions of the mag-
nitude of magnetospheric current systems produced. Finally, in Section 5 we
will summarise and conclude.

2. Outline of model

In a rigidly rotating atmosphere, the restoring force mechanism for Rossby
waves arises from the variation of the Coriolis parameter with latitude. In
these circumstances they propagate westwards in the corotating frame at
a small fraction of the planetary rotation velocity (e.g. Houghton, 1986).
Rossby waves are thus good candidates for explaining the ∼10.7-hour peri-
odicities because, provided the background atmosphere on which they prop-
agate is almost in rigid corotation with the deep atmosphere, they will also
almost corotate with the deep atmosphere.

Furthermore, there is evidence that the ∼10.7-hour periodicities corre-
spond to angular velocities slightly slower than the deep rotation velocity of
the planet (Gurnett et al., 2010), consistent with a small westwards prop-
agation velocity. The westwards motion of Rossby waves in these circum-
stances also suggests that Rossby waves in the already strongly subcorotat-
ing thermosphere region are unlikely to be responsible for the periodicities: a
westwards-propagating Rossby wave superimposed on the already westwards-
Figure 2: Sketch of proposed ‘wave cavity’ in the polar regions. The grey shaded area show the regions where ion densities are enhanced by ‘hard’ particle precipitation related to the main auroral oval. The black regions show the constraints of the wave cavity. The horizontal black bar is the ‘lid’ beyond which Rossby waves cannot propagate because the flow is strongly sheared westwards in the planet’s corotating reference frame. The vertical black bars are the locations where zonal winds are expected to be inhibited by relatively strong ion drag.
flowing gas at these altitudes would not have a $\sim 10.7$-hour period.

If the atmosphere is not rigidly rotating – i.e. if the zonal winds vary rapidly with latitude – then these attractive properties of Rossby waves break down. For example, within a strongly curved eastward jet Rossby waves may propagate with an eastwards phase velocity. We require a structure that slowly moves westwards, and therefore suitable $\sim 10.7$-hour Rossby waves must be located in regions where there are no strong jet curvatures and where the atmosphere is close to rigid rotation.

The troposphere and lower stratosphere are most certainly not suitable locations, with strongly curved jet structures observed at pressures higher than 100Pa (e.g. [Read et al., 2009a]). However, the altitudes of interest here, in the upper stratosphere and lower thermosphere, are at pressures around 0.01Pa or less, or $\sim 10$ pressure scale heights higher than the observed jets. We would not expect these jet structures to penetrate to such high altitudes. For example, [Conrath et al., 1990] calculated mean flows in the stratosphere using a simple model that was forced by tropospheric jets as a lower boundary condition. The magnitude of the jets decayed with altitude – roughly in proportion to the pressure – indicating that their magnitude will be negligible in the upper stratosphere.

Instead, we would expect the dominant process forming zonal winds in the polar regions of the lower thermosphere and upper stratosphere to be the steady westwards drag of the magnetosphere on the thermosphere. This causes a continuous input of westwards momentum that must be transferred downwards to the deep atmosphere. This implies a vertically sheared structure to the zonal flow, with the shear weakening with depth.
As a first approximation, we will assume that this shear is consistent with rigid rotation at each altitude. This means that at each altitude we treat the atmosphere as a rigidly rotating shell, with the westwards angular velocity of this shell decreasing with decreasing altitude. There are no direct measurements of neutral winds to support this model, however Doppler observations of the ion flows (e.g. [Stallard et al., 2004]) indicate approximately linear variation of the zonal ion flows as a function of latitude, consistent with rigid rotation. These rigidly rotating zonal ion flows then directly drive the zonal neutral winds, and so it is likely that they will also be close to rigid rotation.

There is expected to be some localised curvature of the zonal flows close to the main auroral oval ([Cowley et al., 2008]) that will certainly violate the assumption of rigid rotation in the thermosphere. However, deeper in the atmosphere as the shear weakens we can expect this to be less important. We will thus provisionally assume rigid rotation as a simple background condition, even though it is unlikely to be exactly true throughout our region of interest.

The natural location for suitable ∼10.7-hour Rossby waves is thus the region just below the thermosphere, in the altitude range 600-900km, as identified by [Smith (2014)]. This region is expected to exhibit weak shear in zonal velocity as a function of altitude, as stated above, but still to be close to corotation, and so a westwards propagating Rossby wave would also only slightly lag corotation. Indeed, the existence of a strong rotational shear immediately above these altitudes indicates that Rossby waves existing below the shear layer could not propagate significantly to higher altitudes. In order to do so and remain coherent, the waves would require an eastward
phase velocity to counteract the background westward flow. For westward
propagating Rossby waves such vertical propagation is impossible (unless
the Rossby waves can couple to an eastward propagating wave of a different
type). This suggests that the shear layer will act effectively as a ‘lid’ that
inhibits the propagation of Rossby waves into the thermosphere.

Another important structure in the polar regions is the main auroral oval,
which we take to lie at an approximate distance $r_0 = 1.5 \times 10^7$ m from the
pole, corresponding to a colatitude of $\sim 16^\circ$ and a polar radius of $\sim 54,000$
km, consistent with the UV and IR auroral locations determined by [Nichols
et al. (2009)] and [Badman et al. (2011)]. This region is subject to precipitation
by much more energetic electrons compared to the bulk of the polar cap. For
example, [Galand et al. (2011)] modelled electron energies of 500 eV for ‘diffuse’
auroral emissions and 10 keV for the ‘hard’ electron precipitation in the main
auroral oval. This difference is significant, because in our altitude range of
interest there is much greater electron density at the location of the main
auroral oval, since the ‘hard’ electrons in this region penetrate deeper into the
atmosphere, resulting in ionisation as deep as 700 km altitude ([Galand et al.
2011]). Thus we expect significantly greater ion drag at these latitudes. Since
the auroral oval lies very nearly along lines of constant latitude, one would
expect the zonal component of any wind structure to be more significantly
inhibited by ion drag than the meridional component: if the wind has a
dominant zonal component at these latitudes then any particular parcel of gas
will spend longer in the region of enhanced ion drag. Therefore the presence
of the main oval should inhibit wave modes with strong zonal components at
that latitude, producing a nodal line in the zonal wind perturbation at the
latitude of the main oval.

These considerations lead to the notion of an open ‘wave cavity’ in the polar regions, in which the thermospheric shear layer acts as the ‘lid’ and the main auroral oval as the ‘walls’. This situation is sketched in Fig. 2. The purpose of this paper will be to seek Rossby wave solutions in this cavity, and analyse whether they are suitable candidates for explaining the magnetospheric periodicities.

3. Details of model

3.1. Theory

The description of global-scale planetary waves is achieved using tidal theory (e.g. Lindzen and Chapman [1969]). However, full solutions of Laplace’s tidal equation are complicated. It is thus common to analyse specific situations using simplified geometries. At equatorial and mid-latitudes, wave modes can be analysed using a beta-plane approximation (in which the spherical geometry is neglected and the variation of the Coriolis parameter with latitude is approximated as linear). One such analysis was carried out by Lindzen [1967]. We are chiefly interested in waves close to the poles, for which a standard beta-plane approximation is poor. This situation was analysed by Haurwitz [1975] by approximating the polar regions as flat. We will apply this theoretical analysis to Saturn. The paper by Haurwitz uses somewhat archaic notation and so, for clarity, we repeat the derivation using more modern notation (closely similar to that employed by Lindzen [1967]), with as much detail as possible presented in the Appendix, reserving the main results and a discussion of the important assumptions for the main text.
We assume that in its basic, unperturbed state the polar upper atmosphere is isothermal, in hydrostatic equilibrium and rigidly rotating. The observed neutral temperature in the stratosphere does not vary significantly with altitude, lying approximately in the range 134-143K in the altitude range 350-850km \(\text{(Moses et al., 2000)}\). A constant value of \(\sim 140\)K thus seems appropriate. We choose to use a constant temperature of \(T_0 = 144\)K, because taking the gravitational field strength to be 12ms\(^{-2}\) and the composition to be pure H\(_2\), this implies a round atmospheric scale height of 50km.

Also related to the temperature and significant for the theory that follows is the value of \(\gamma = c_p/c_v\), the ratio of specific heats. For a diatomic gas at room temperature this is equal to 1.4. However, below a temperature of \(\sim 250\)K the rotational states of diatomic hydrogen are not fully populated and so the specific heat capacity falls, approaching that of a monatomic gas at \(\sim 50\)K, for which \(\gamma = 1.67\) \(\text{(e.g. Sears and Salinger, 1975)}\). Assuming an ortho:para ratio of 3:1, the appropriate intermediate value is close to \(\gamma = 1.5\) \(\text{(Leachman et al., 2009)}\) and we adopt this value.

The physical quantities describing the basic state are the unperturbed pressure, density and temperature \(p_0\), \(\rho_0\) and \(T_0\), which are linked by vertical force balance

\[
\frac{\partial p_0}{\partial z} = -\rho_0 g
\]  
\((1)\)

and by the equation of state of an ideal gas

\[
p_0 = \rho_0 R_m T_0/\mu = \rho_0 g H
\]  
\((2)\)

where \(R_m\) is the molar gas constant, \(\mu = 2.0 \times 10^{-3}\)kg is the molar mass of molecular hydrogen and \(H\) is the pressure scale height. These equations
taken together imply that, for our isothermal region, both $p_0$ and $\rho_0$ fall exponentially with altitude $z$ and with scale height $H$, i.e.

$$
p_0 = p_{00} e^{-\Delta z/H} \quad (3)
$$

$$
\rho_0 = \rho_{00} e^{-\Delta z/H} \quad (4)
$$

where $\Delta z = z - z_{00}$. We take $p_{00} = 1.7 \times 10^{-3}$ Pa, the approximate pressure at $z_{00} = 900$ km above the 1-bar level in the Moses et al. (2000) model of the neutral atmosphere. Note that because we assume a constant value of the background temperature, the pressures at altitudes other than 900 km do not correspond exactly to those in the Moses et al. (2000) model.

The planet’s 1-bar pressure surface may be approximated as an ellipsoid with polar radius $R_p \sim 54,000$ km and equatorial radius $R_e \sim 60,000$ km. In this situation the polar regions are well approximated as a spherical surface with radius of curvature $R_c = R_e^2/R_p \simeq 67,000$ km. The Coriolis parameter $f$ is then given by:

$$
f = 2\Omega \cos \frac{r}{R_c} \simeq 2\Omega \left(1 - \frac{r^2}{2R_c^2}\right) \quad (5)
$$

where $r$ is the radial distance from the pole along the curved surface of the planet. We take $\Omega \simeq 1.65 \times 10^{-4}$ rad s$^{-1}$ to be the deep planetary angular velocity. We have derived this value by averaging the two independent determinations of the rotation period by Anderson and Schubert (2007) and Read et al. (2009b).

To further simplify matters we can approximate the polar regions as flat. To do so we use the approximate expression for $f$ given above, but take $r$ to represent the radial coordinate in cylindrical polar coordinates. We
investigate a situation centred around the north pole, so that \( r \) is in the radial direction (equatorwards), \( \phi \) is eastwards (anti-clockwise viewed from above the north pole) and \( z \) is vertically upwards. An analysis of the significance of approximating the polar regions as flat was presented by Bridger and Stevens (1980). They found that while there were some small quantitative differences in modelling the polar regions as flat rather than curved, the same qualitative wave behaviour was observed. The approximation is thus clearly reasonable for this initial study.

We then introduce perturbations to the three-component neutral wind (\( u, v \) and \( w \) representing eastward, northward and upward winds respectively) and to the pressure and density (\( \delta \rho, \delta \rho \)). We do not explicitly denote the wind perturbations with a ‘\( \delta \)’ because the unperturbed wind is zero in our rigidly rotating frame of reference. These perturbations are assumed to be sufficiently small that second-order terms can be neglected. The horizontal momentum equation yields the first two equations, vertical force balance the third, continuity the fourth and energy conservation the fifth:

\[
\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0 r} \frac{\partial \delta p}{\partial \phi} \tag{6}
\]

\[
\frac{\partial v}{\partial t} + fu = \frac{1}{\rho_0} \frac{\partial \delta p}{\partial r} \tag{7}
\]

\[
\frac{\partial \delta p}{\partial z} = -g\delta \rho \tag{8}
\]

\[
\frac{\partial \delta \rho}{\partial t} + w \frac{\partial \rho_0}{\partial z} + \rho_0 \left( \frac{1}{r} \frac{\partial u}{\partial \phi} - \frac{1}{r} \frac{\partial vr}{\partial r} + \frac{\partial w}{\partial z} \right) = 0 \tag{9}
\]
\[ \frac{\partial \delta p}{\partial t} + w \frac{\partial p_0}{\partial z} = \gamma g H \left( \frac{\partial \delta \rho}{\partial t} + w \frac{\partial \rho_0}{\partial z} \right) + q \rho_0 (\gamma - 1) \] 

(10)

The symbol \( q \) represents the rate of external thermal energy input per unit mass. We have included this function to maintain the generality of our derivation, but in this study we will only consider free oscillations that are not continuously forced. In the third equation we have assumed hydrostatic equilibrium holds and thus neglected vertical accelerations of the neutral gas. It is noted that these equations are closely analogous to Eqns. 1-5 of\cite{Lindzen1967}.

We further assume that all of these perturbation variables, and the heating function \( q \), vary as

\[ e^{i(\omega t + k \phi)} \] 

(11)

Note that for positive \( \omega \) and positive \( k \) this indicates a wave with phase velocity

\[ c = -\omega/k \] 

(12)

so phase fronts propagate in the negative \( \phi \) direction, i.e. westwards. We will assume positive \( k \) throughout, so that eastward phase propagation is implied by negative values of \( \omega \).

The analysis then proceeds as described in the Appendix, by combining Eqns [6][10] and then separating variables. The following points from the derivation are worth restating here:

1. A separation constant \( h \) is introduced, commonly referred to as the ‘equivalent depth’. This parameter characterises each wave mode and links their horizontal and vertical structure.
2. To simplify the equations we require that $\epsilon^2 r^2 \ll 1$ where

$$\epsilon^2 = \frac{\omega^2}{ghk^2}$$  \hspace{1cm} (13)

This assumption will be justified further below.

3. To simplify the equations we assume a constant Coriolis parameter which we calculate at the location of the main auroral oval ($r_0 = 1.5 \times 10^7$ m):

$$f_0 = 2\Omega \left( 1 - \frac{r_0^2}{2R_c^2} \right)$$  \hspace{1cm} (14)

with the exception of a single term that depends upon the radial derivative of $f$, for which we use the approximate value:

$$\frac{\partial f}{\partial r} \simeq -\frac{2\Omega r}{R_c^2}$$  \hspace{1cm} (15)

This is a type of beta-plane approximation.

Once these approximations are made, the part of the solution that represents variation in the $r$ direction turns out to satisfy Bessel's equation, provided that a parameter $m$ is given by

$$m^2 = \frac{2\Omega k}{R_c^2 \omega} - \frac{(f_0^2 - \omega^2)}{gh} - \frac{2f_0 \omega}{ghk}$$  \hspace{1cm} (16)

This yields the following expressions for the perturbation variables (noting again that we are ignoring the forcing function $q$, such that these expressions
are appropriate for free oscillations only):


del{[u]} = -i u_0 J'_k(mr) F(z) e^{z/2H} e^{i(\omega t + k\phi)} \quad (17)

del{[v]} = v_0 J_k(mr) \frac{F(z)}{mr} e^{z/2H} e^{i(\omega t + k\phi)} \quad (18)

del{[\delta p]} = -i \delta p_0 K_k(mr) F(z) e^{-z/2H} e^{i(\omega t + k\phi)} \quad (19)

del{[\delta \rho]} = -i \delta \rho_0 K_k(mr) \left[ \frac{F(z)}{2} - HF'(z) \right] e^{-z/2H} e^{i(\omega t + k\phi)} \quad (20)

del{[w]} = w_0 K_k(mr) \left[ \left( \frac{1}{2} - \frac{1}{\gamma} \right) F - HF' \right] e^{z/2H} e^{i(\omega t + k\phi)} \quad (21)

The function $F$, to be discussed in Section 3.4, defines the vertical structure. The function $K_k$ is given by

\[ K_k(mr, \omega) = J_k(mr) - \frac{\omega mr}{f_0 k} J'_k(mr) \quad (22) \]

Provided that $\omega \ll f$ (true for all slowly propagating solutions that are relevant here) and if $mr$ and $k$ are of order unity (also true for all situations considered here), the second term in the equation for $K_k$ is much smaller than the first and thus $K_k \simeq J_k$.

The various constants $u_0$ etc., which describe the perturbation amplitudes, are related by defining characteristic horizontal and vertical speeds $u_{00}$, $v_{00}$ and $w_{00}$ given by:

\[ v_{00} = \frac{mkgH}{f_0} = ku_{00} \quad w_{00} = \frac{\omega H}{\kappa} \quad (23) \]

where $\kappa = (\gamma - 1)/\gamma$. This allows us to write the following simple expression:

\[ \frac{w_0}{w_{00}} = \frac{\delta p_0}{p_{00}} = \frac{\delta \rho_0}{\rho_{00}} = \frac{u_0}{u_{00}} = \frac{v_0}{v_{00}} \quad (24) \]
3.2. Horizontal structure

We now apply these solutions to Saturn. The equations above permit a continuous spectrum of wave modes with different values of $k$ and $m$. We are interested in wave modes with $k = 1$ (commonly referred to as ‘$m = 1$’ in the context of magnetospheric periodicities, because $m$ is the usual label for the longitudinal wave number when considering spherical harmonics). This restricts us to solutions involving the first order Bessel function $J_1$. Next, we note our proposed condition that the zonal wind is inhibited at the latitude of the main auroral oval, due to the increased ion drag at this latitude. Thus the wave mode with $u(r_0) = 0$ is preferred. This occurs when $J'_1(mr_0) = 0$. This is true if $mr_0 = j_0$, where $j_0 = 1.841$, and this gives us a unique value for $m = 1.23 \times 10^{-7} \text{m}^{-1}$. We can now calculate a value of $v_{00} \simeq 230 \text{ms}^{-1}$ for this situation.

Fig. 3 shows the variation of $J_1(x)$, $J'_1(x)$, and other functions that appear in our solutions, where in this case $x$ corresponds to the dimensionless parameter $mr$. The mapping to $r$ for our specific situation is shown on the upper axis. This shows the main auroral oval at $r = 1.5 \times 10^7 \text{m}$ corresponding to the first zero in $J'_1(x)$. This is the first nodal line in the zonal winds. The next significant radius is $r = 3.1 \times 10^7 \text{m}$, corresponding to the first zero in $J_1(x)$. This is the first nodal line in the meridional winds.

The triple-dot-dash line shows $K_1$ when $\omega/f_0k = 0.1$, corresponding to a wave speed of $\sim 20\%$ of the planetary rotation. This is not significantly different from the curve for $J_1$, showing that the second term of $K_1$ (Eqn 22) is relatively small. For realistic values of the wave speed that are at least $\sim 100$ times smaller, the curves for $J_1$ and $K_1$ are almost indistinguishable.
Figure 3: Horizontal structure functions. The solid and dotted lines show the Bessel function $J_1$ and its first derivative. The other lines show various combinations of these functions, as indicated in the key. The vertical grey lines show the locations of the first zeroes in $J_1$ and $J'_1$. 
We now plot the horizontal structure of the solutions. We take a value of $F(z) = 1$ so that the plots represent a snapshot at any altitude, and $v_0 = 1$ so that the plots can easily be scaled to more complex situations.

Fig. 4 then shows views from above the north pole, with the location of the first zeroes in $J_1$ and $J'_1$ shown with the circular dot-dash lines. Panel (a) shows the horizontal flow pattern using unscaled arrows. This represents precisely the type of twin-vortex flow that has been proposed to explain the 10.7-hour periodicities. Panel (b) shows the magnitude of the velocity at each location as a contour plot. The maximum velocity perturbation when $v_0 = 1 \text{ms}^{-1}$ is at the pole, and has a value just below $0.5 \text{ms}^{-1}$. Panel (c) shows the horizontal distribution of the pressure perturbation. This is proportional to $K_1$ but for all solutions discussed here it is dominated by $J_1$ so that the plot is indistinguishable from a plot involving the term in $J_1$ alone. Taken together, these plots indicate that the wind perturbation is essentially a two cell circulation around regions of high and low pressure which drifts slowly westwards.

We can further calculate the pattern of field-aligned currents. The currents generated will depend on the distribution of $F(z)$ with altitude. However, the overall pattern should be the same at all altitudes. We therefore take $F(z) = 1$ and $v_0 = 1 \text{ms}^{-1}$ again to perform a baseline calculation of the currents. As discussed by Smith (2014) the primary process for producing field-aligned currents in the upper stratosphere is the horizontal divergence of the Hall current. We calculate these currents for the altitude range 700-900km, assuming a vertically uniform electron density in this region, but we allow it to vary with latitude to represent the enhanced electron density in
the region of the main auroral oval:

\[ n = n_0 \left(1 + 100 \times \exp \left[ \frac{(r - r_0)^2}{2W^2} \right] \right) \]  

(25)

Here a background electron density \( n_0 = 5 \times 10^8 \text{ m}^{-3} \) is enhanced by a factor of \( \sim 100 \) in the region of the main auroral oval. The region of enhancement is represented by a gaussian of FWHM\( \sim 1000 \text{ km} \), defined by \( W = 400 \text{ km} \). This distribution is shown in Fig. 4d. The value of \( W \) has little effect on the total field-aligned current because the integrated divergence depends on the maximum value of the enhanced electron density, not on its horizontal distribution. The values of the background and enhanced electron densities are based on the results of Galand et al. (2011) and are discussed in Smith (2014).

Using this expression for \( n \) we calculate profiles of Pedersen and Hall conductivity using the expressions given by Smith (2013) and Smith (2014). We use \( B = 60000 \text{ nT} \) for the polar magnetic flux density. We then calculate the horizontal Pedersen and Hall currents and their height-integrated divergences. The resulting field-aligned currents are shown in Fig. 4e (for the background electron density only) and Fig. 4f (for the enhanced electron density). The background electron density produces two broad areas of relatively low field-aligned current. These are the currents predicted by the model of Southwood and Cowley (2014), which assumes a uniform background conductance. These are mostly due to divergence of the Pedersen current. In contrast, the enhanced electron density produces two pairs of upwards- and downwards-directed current sheets either side of the main auroral oval. These are mostly due to divergence of the Hall current.
If we integrate the field-aligned current across any of the four current sheets shown in Fig. 4, we find a total value of $\sim 0.06\text{MA}$. This is an order of magnitude lower than that required to explain the periodicities. The current scales linearly with wind speed, indicating that we will require higher wind speeds than $v_0 = 1\text{ms}^{-1}$ to produce the required currents. The assumption that $F(z) = 1$ also probably overestimates the total current, because $F(z)$ is likely to vary with altitude. More complete calculations will be presented in Section 4.

### 3.3. Horizontal propagation

Fig. 4 demonstrates that our proposed wave structure produces the correct circulation pattern and the correct general pattern of field-aligned currents required to explain the magnetospheric periodicities. We must now establish whether the pattern rotates with an appropriate value of $\omega$. Substituting $m = j_0/r_0$ into Eqn. 16 gives a cubic equation for $\omega$ in which the only free parameter is the separation constant $h$:

$$\omega^3 - 2f_0\omega^2 - \left( f_0^2 + \frac{j_0^2 gh}{r_0^2} \right) \omega + \frac{2\Omega gh}{R^2 c} = 0 \quad (26)$$

Fig. 5a shows how the possible values of $\omega$ vary for different values of $h$. Solid lines show real solutions for $\omega$. Each region of the real solutions is labelled with a letter from A to E. Complex solutions are shown by dashed lines (showing the real parts) and pairs of dotted lines (showing the two conjugate imaginary parts of the two complex solutions). The horizontal shaded region (only just visible as a thin line at $\omega = 0$) shows $\pm 0.01\Omega$, i.e. modes that propagate at $\sim 1\%$ of the planetary rotation velocity must lie within this region.
Figure 4: In all plots the dash-dot lines show the location of the first zero in $J'_1$ (at $r = 1.5 \times 10^7$ m) and the first zero in $J_1$ (at $r = 3.1 \times 10^7$ m). In panels (a)-(c) the data are for $v_0 = 1$ and $F(z) = 1$ at $z = 900$ km. In panels (e) and (f) the currents have been integrated across the range $z = 700 - 900$ km based on the same assumptions.

(a) Horizontal flow pattern. Arrows are not scaled. (b) Total flow velocity. Dashed contours show flow speeds less than or equal to 0.25 ms$^{-1}$ with a spacing of 0.025 ms$^{-1}$. Solid contours show greater flow speeds, with the same spacing. (c) Pressure perturbation. Solid and dashed lines show positive and negative values. Dotted lines show zero contours. The contours are spaced at intervals of $5 \times 10^{-8}$ Pa. The maximum pressure perturbation contour shown is $4 \times 10^{-7}$ Pa. (d) Electron density model. A uniform background density (light grey) with a narrow enhanced region at the location of the main oval (dark grey). (e) Field-aligned currents calculated using the background electron density only. Solid and dashed lines show positive and negative values. Dotted lines show zero contours. The contours are spaced at intervals of $5 \times 10^{-6}$ nAm$^{-2}$. The maximum contour value shown is $3 \times 10^{-5}$ nAm$^{-2}$. (f) Field-aligned currents calculated using the enhanced electron density. The plot has been expanded so that only the central section is visible, as indicated by the dashed box in panels (e) and (f). Line formats have the same meaning as panel (e), however contours are now spaced at intervals of 0.1 nAm$^{-2}$. The maximum contour value shown is 0.3 nAm$^{-2}$. 
Figure 5: (a) Values of angular speed $\omega$ as a fraction of the deep planetary angular velocity $\Omega$ for solutions with a range of values of $h$. Solid lines show real solutions. Dashed and dotted lines shows complex solutions, with the real parts plotted as a dashed line and the conjugate imaginary parts of the two solutions as dotted lines. (b) Calculated values of $\epsilon^2 r_0^2$ for the real solutions shown in panel (a). The shaded region shows values below 0.01. (c) Expanded version of panel (a) with only real solutions shown. Solid lines indicate solutions for which $\epsilon^2 r_0^2 < 0.01$. 

23
We are wish to find steady state solutions, and so we are interested in
frequencies that are real – so that they do not exponentially grow or decay
– with a very small positive value of $\omega$. There are two regions on the graph
where these conditions appear to be fulfilled – region A and the part of region
D for $h > 0$. It is worth noting that the choice of $h$, rather than $1/h$ as the
abscissa is arbitrary, and so these two branches of the graph are effectively
part of the same group of wave modes, and are connected at $h = \infty$. It is
further worth noting that this branch of wave modes arises from the final term
in Eq. [26], which in turn arises from the variation in latitude of the Coriolis
parameter: these are therefore Rossby waves as generally understood.

As already discussed above, these wave solutions are only valid if $\epsilon^2 r^2 \ll
1$. Figure 5b shows this quantity calculated at $r_0$, for real solutions only.
The region shaded in grey shows when it falls below 0.01. In this region the
approximation that neglects terms involving $\epsilon^2 r^2$ is certainly valid. It is clear
that it is valid for regions A and D. Fig. 5c shows an expanded version of
Fig. 5a, showing real solutions only, to more clearly show the low frequency
wave modes that interest us. The solid lines show wave modes for which
$\epsilon^2 r^2_0 < 0.01$. Again, this clearly demonstrates that our approximation is valid
for the low frequency sections of regions A and D. These slowly propagating
modes are therefore our candidates for explaining the 10.7-hour periodicities.

3.4. Vertical structure

We now investigate the vertical structure function $F(z)$. Taking the equa-
tion derived in the Appendix (Eqn. A.20), and setting the external heating
parameter $q$ to zero, as required for free oscillations, $F$ is described by:

$$F'' + a^2 F = 0$$  \hspace{1cm} (27)
where $a$ is given by:

$$a^2 = \frac{\kappa}{H h} - \frac{1}{4H^2} \quad (28)$$

We find that $a$ is real if $0 < h < h_{lim}$, where $h_{lim} = 4\kappa H$, and in these circumstances the solutions of Eqn. [27] are vertically propagating waves. Otherwise, $a = i\alpha$ is imaginary. In this case they are evanescent waves in the vertical direction and the energy is trapped. Fig. [6] shows the calculated values of $a$ and $\alpha$ showing, by the dashed line, the narrow range of $h$ for which waves propagate. The vertical shaded region in Fig. [5a] shows the same range of $h$ for which vertical propagation is possible. Although this shows a very narrow range of possible solutions in which energy can propagate vertically, these also correspond to small values of $\omega$ with the value at $h_{lim}$ given by $\omega_{lim} \approx 0.00308\Omega$. The possible values of $\omega$ that propagate vertically are therefore all less than $\sim 0.308\%$ of the planetary rotation velocity. These are therefore very good candidates for explaining the 10.7-hour periodicities.

### 3.5. Vertical propagation

As already discussed, a wave in the stratosphere can only propagate into the shear layer in the lower thermosphere if it experiences a change in phase velocity that exactly cancels the westwards background flow. This is a form of a Doppler shift. Consider a wave propagating upwards from the deep atmosphere, with some initial angular speed $\omega^*$. If the wave is to propagate vertically, then we must have $\omega^* \leq \omega_{lim}$. As it rises through the atmosphere, its total angular speed in the corotating planetary frame must remain equal to $\omega^*$. If we represent the westwards angular speed of the gas in each layer as $\omega_{shear}$, then the angular speed of the perturbation relative to the gas in
Figure 6: Calculated magnitudes of the vertical wavenumber $a$ for Rossby wave solutions. The dashed line shows the region where $a$ is real. The solid lines show regions where $a = i\alpha$ is imaginary.

Each layer must be equal to

$$\omega = \omega^* - \omega_{\text{shear}}$$

so that its westward angular speed relative to the local gas decreases as it rises. The wave modes in the higher layers thus correspond to smaller values of $h$, which also correspond to larger values of $a$, as illustrated in Fig. 6. As discussed by Lindzen (1967), this means that the vertical phase velocity $\omega/a$ tends to zero. It can therefore never reach the layer where $\omega$ equals zero, which is referred to as a ‘critical layer’. It must either be absorbed or reflected at this altitude. Working out what happens in this situation is difficult, because as $a$ grows the wavelength decreases, and therefore the vertical gradients of $u$, $v$ and $\delta p$ also increase. The values of $\delta \rho$ and $w$ directly depend on the vertical gradient of $\delta \rho$; this means that the values of
$\delta \rho$ and $w$ tend towards infinity as we approach the critical layer, and thus the linearisation of the equations breaks down.

Critical layers have been well studied in the context of horizontally propagating Rossby waves in the Earth’s atmosphere. For example, Killworth and McIntyre (1985) describe a model in which, after a sufficiently long period of time, critical layers reflect horizontally propagating Rossby waves. A more recent study (Potter et al., 2013) also studied horizontally propagating Rossby waves and found partial reflection at critical layers. The question of what occurs when a Rossby wave impinges vertically on a critical layer has been much less well studied.

However, a well-studied example of a situation in which waves vertically impinge upon a critical layer is the case of the quasi-biennial oscillation in the Earth’s equatorial stratosphere (Baldwin et al., 2001). In this case, waves tend to be absorbed, modifying the existing jet by transfer of momentum. However, this is a very different situation to the one studied here, in particular involving small-scale waves rather than planetary-scale waves. It does not seem reasonable to infer by analogy that the waves represented by our model are absorbed.

We will therefore investigate two situations: one in which the Rossby waves are completely absorbed by the critical layer and another in which complete reflection occurs.

To calculate the altitude of the critical layer we ideally require a model of the shear in Saturn’s lower thermosphere and upper stratosphere. Unfortunately, while the existence of flow shear seems inevitable, the degree of shear in this region is unconstrained by direct observations and there are many un-
certainties in calculating it theoretically. Smith (2014) estimated the relative shear below 1000km using a simple viscous transfer model. However, the absolute neutral velocity at 1000km is very uncertain, and thus the absolute shear is difficult to estimate. Furthermore, advective processes may also be important in these regions, rendering the viscous calculation an overestimate of the shear (Smith and Aylward, 2008).

We therefore show in Fig. 7a a highly simplified and somewhat arbitrary illustrative model of a possible flow shear, represented by a constant flow below 600km altitude and a constant vertical shear above this altitude. The vertical dotted line labelled $A$ shows an angular speed of $\omega^* = 0.0025\Omega$. This is smaller than $\omega_{lim}$ and so a wave with this angular speed can propagate vertically in the deep atmosphere. At an altitude of about 700km, $\omega_{shear} = \omega^*$ and thus the wave cannot propagate beyond this altitude. The curve labelled $A$ in Fig. 7b illustrates this further by showing the values of $a$ and $\alpha$ implied at each altitude by the value of $\omega$. The dashed line below 725km indicates that the wave can propagate vertically in this region. At 700km $a \to \infty$, and so this is a critical layer. Although evanescent solutions are possible above 700km, the critical layer is assumed to absorb or reflect incoming waves and so no wave is set up in this region. It should be emphasised that the decision to begin the shear at 600km is entirely arbitrary – the actual critical layer may lie higher in the atmosphere, such that Rossby waves can penetrate high enough to interact with the ionosphere and generate currents.

In this situation, we thus have a region in which Rossby waves can freely propagate, but this region extends continuously into the deep atmosphere. Therefore any locally generated Rossby waves – possibly due to asymmetries
in the auroral forcing, which penetrates as deep as 700km – could propagate away into the deep atmosphere, carrying away energy. A more promising situation would be a wave source deep in the atmosphere, perhaps a persistent tropospheric asymmetry such as the Great White Spot (Fischer et al., 2014), which might drive Rossby waves that could propagate upwards into the stratosphere. This is the ‘open wave cavity’ model sketched in Fig. 2.

An alternative possibility is illustrated by the angular speeds labelled $B$ and $C$. These both involve $\omega^* > \omega_{\lim}$, and so they cannot propagate in the deep atmosphere, but only above a certain altitude in the shear layer, at which point $\omega$ falls below $\omega_{\lim}$. At a higher altitude still, they can no longer propagate as $\omega$ falls to zero, and there is a critical layer. The dashed lines in Fig. 7b show the altitudes where propagation is possible.

These situations are intriguing, because there appears to be a closed wave cavity or waveguide at high altitudes within which propagation is possible. This is sketched in Fig. 8. This implies that Rossby waves could become trapped in the upper stratosphere, with the energy unable to radiate away into the deep atmosphere. Having set up an oscillation in this cavity, we would then simply need to occasionally force the upper stratosphere to counter the gradual dissipation of the trapped waves.

This ‘closed wave cavity’ model depends on the assumption that the critical layer reflects waves. If it is a perfect reflector then a standing wave may be set up. If it is a perfect absorber then the wave would presumably be damped very rapidly. The former possibility is the most intriguing, because the concept of a ‘resonant cavity’ is very attractive in explaining the uniqueness and persistence of the 10.7-hour signals.
Figure 7: Shear model of the upper stratosphere and lower thermosphere. (a) The solid line shows our illustrative shear model. The vertical dashed lines labelled A, B and C show three possible angular speeds for waves propagating against this background. To emphasise that the angular speeds shown represent subcorotation of the atmosphere (i.e. westwards flow), the arrow indicates the planetary rotation direction. (b) Vertical wavenumbers implied by the exponential shear model and the angular speeds plotted in panel (a). The solid lines show regions where the vertical wavenumber is imaginary and no propagation is possible. The dashed lines show regions where the vertical wavenumber is real and waves can propagate vertically.
Figure 8: Sketch of alternative ‘wave cavity’ model, similar to Fig. 2. The wave cavity now lies between the two horizontal black bars. Rossby waves are trapped in the shear layer itself, and are unable to propagate to lower or higher altitudes.
4. Solutions

We now consider solutions for $F(z)$ for open and closed wave cavity models.

4.1. Open wave cavity

In this case we adopt one of two scenarios: either the Rossby waves originate deeper in the atmosphere and propagate upwards into the region of interest, or they are generated locally and propagate downwards into the deep atmosphere. We assume that upwards propagating waves will be absorbed by the critical layer, and so we do not have to worry about interference between waves moving in opposite directions. On this basis, since the vertical wavenumber $a$ is real, we can represent the vertical structure simply as

$$F(z) = e^{iaz} \quad (30)$$

so that positive $a$ corresponds to downwards phase propagation.

Inserting this into the expressions for $u$ and $v$ and recalculating the predicted field-aligned currents, we find that we need $v_0 \simeq 150\text{ms}^{-1}$ to generate integrated currents of $\sim 1\text{MA}$, as required to explain the observations. This implies peak wind speeds at $z = 900\text{km}$ of about $70\text{ms}^{-1}$ and a peak fractional pressure perturbation at $z = 900\text{km}$ of only 0.035.

4.1.1. Energy

To assess whether these values are energetically plausible, we need to estimate the vertical energy flux associated with the wave. This is achieved for atmospheric waves by calculating the product $w\delta p$ (e.g. Mak 2011) and
averaging it over a full cycle. This yields the following expression:

\[
E = w \delta p = \frac{p_{00} \omega H^2 a}{2 \kappa} \left( \frac{v_0}{v_{00}} \right)^2 K_k^2(m r, \omega) \quad (31)
\]

This energy flux depends strongly on both \( \omega \) and \( a \), which in turn both depend on the equivalent depth \( h \). Integrating over the whole of the polar region (from the pole to \( r = 3.1 \times 10^7 \) m) allows us to calculate the total integrated flux. In Fig. 9 we show its dependence on the vertical wavelength \( \lambda = 2\pi/a \). This shows a very large integrated energy flux of \( \sim 70 \) TW for a vertical wavelength of \( \sim 10 \) scale heights. However, for shorter wavelength disturbances a much smaller energy flux may be required. For example for a vertical wavelength of \( H \) the energy flux is \( \sim 12 \) TW and for a vertical wavelength of \( 0.1H \) it is \( \sim 1.2 \) TW.

For the case in which the waves are generated locally and then propagate
downwards, we can compare these calculated powers to the energy available from particle precipitation which is a plausible energy source for generating the waves in the upper atmosphere. We can estimate this by calculating the total incident energy due to particle precipitation in the main oval. The total area of particle precipitation is an annulus of width $\sim 1000\text{km}$ and circumference $\sim 100,000\text{km}$, yielding a surface area of $\sim 1 \times 10^{14}\text{m}^2$. Taken together with a precipitating energy flux of $0.2\text{mWm}^{-2}$ in the form of 10keV electrons, peaking at $\sim 800\text{km}$ and so delivering most of that energy to our region of interest (Galand et al., 2011), this implies a total energy flux of $\sim 0.02\text{TW}$. The energy flux is probably even higher than this, perhaps peaking closer to $5\text{mWm}^{-2}$ (Cowley et al., 2008). This implies that $\sim 25$ times more energy may be available, i.e. a total energy of flux of $\sim 0.5\text{TW}$.

Thus the total energy available from particle precipitation is much smaller than the maximum possible energy flux. This restricts locally generated wave modes except those with very short vertical wavelengths. However, it places no restriction on waves generated deeper in the atmosphere propagating upwards into the upper stratosphere/lower thermosphere, for which the energy source is unknown.

4.1.2. Consistency with assumptions

We can use these solutions to assess consistency with our main simplifying assumptions.

First, our assumption of hydrostatic equilibrium is related to the vertical motion associated with the waves. If the vertical amplitude of the waves is comparable to the scale height of the atmosphere then it seems unreasonable to treat the waves as a perturbation to a hydrostatic equilibrium
state. The vertical amplitude is given approximately by $w_0/\omega$, which, using $v_0 = 150\text{ms}^{-1}$ and $v_{00} = 230\text{ms}^{-1}$ yields a value of $\sim 2H$. This means that oscillations of the required amplitude are probably not sufficiently small for hydrostatic equilibrium to hold.

Second, we assumed that the perturbations were small enough to allow linearisation of the equations. In practice, this amounts to neglecting advection of the perturbed quantities by the wind perturbations themselves. This is reasonable if the time for the perturbed wind to cross the polar cap is much smaller than the time period of the wave. This amounts to the condition:

$$v \ll \omega r_0 \simeq 7\text{ms}^{-1}$$

(32)

where we have used $\omega = 0.003\Omega_0$. The wind speeds predicted with $v_0 = 150\text{ms}^{-1}$ are considerably greater than this, indicating that linearisation is also not a valid assumption.

While neither of these assumptions are strictly valid for the conditions required to produce $\sim 1\text{MA}$ currents, this of course does not rule out the possibility that similar non-hydrostatic and non-linear structures may be present. However, it does mean that our results must be treated with greater caution.

4.2. Closed wave cavity

The alternative concept of a closed wave cavity in which energy is trapped is more attractive than the open wave cavity for two reasons:

1. If the energy is trapped then a smaller input of energy will be required to sustain the wave.
2. The existence of resonant states in the closed cavity is an attractive explanation of the uniqueness and persistence of the \( \sim 10.7 \) hour signal. A full analysis is beyond the scope of this paper because it would require a self-consistent treatment of the shear itself. In lieu of such an analysis, we will present an illustrative calculation, employing the following assumptions:

1. We assume that the critical layer acts as a rigid reflecting ‘lid’ at which the values of \( u, v \) and \( \delta p \) drop to zero. This means that \( w \) and \( \delta \rho \) will not necessarily be zero at this altitude.

2. We use a simplified three layer model of the flow shear. The lowest layer, representing the deep atmosphere, is in perfect corotation. The second layer, representing the lower regions of the shear layer, has a subcorotation velocity of \( \omega_{sh} \) and a width of \( \Delta \). The upper layer, above the critical layer at \( z = z_c \), has a subcorotation velocity which is considerably greater than \( \omega_{sh} \), and is effectively inaccessible to the waves.

3. At the discontinuity between the lower and middle layers, we assume continuity of \( F \) and \( F' \). These conditions guarantee continuity of \( u \) and \( v \) and their vertical derivatives. It is impossible to guarantee continuity of \( \delta p, \delta \rho \) and \( w \) because these variables depend on \( K_k \), which is a function of \( \omega \), which by necessity varies between the layers.

On the basis of these assumptions, a standing wave develops in the middle layer, and we find the following solution:

\[
F(z) = \begin{cases} 
0, & \text{if } z \geq z_c, \\
\sin a(z_c - z), & \text{if } z_c - \Delta \leq z \leq z_c, \\
\sin(a\Delta)e^{-\alpha(z_c-\Delta)}e^{\alpha z}, & \text{if } z \leq z_c - \Delta.
\end{cases}
\]
with the following condition:

\[ \tan(a\Delta) = -\frac{a}{\alpha} \]  

(34)

where \( a \) is the wavenumber in the middle layer and \( i\alpha \) is the wavenumber in the lower layer. The value of \( \Delta \) is fixed by the three layer shear model. To find a solution, we must adjust \( \omega^* \) (the angular speed of the wave relative to the corotating deep atmosphere), on which \( a \) and \( \alpha \) depend, until we find a value for which the condition is satisfied. This implies that only specific wave modes are allowed.

We now investigate a concrete example of this three-layer model. We use \( \Delta = 200\text{km} \) and \( z_c = 900\text{km} \). We then take \( \omega_{sh} = 0.0030\Omega \) within the middle layer. The angular speed of the shear in the upper layer is unimportant, provided it is large enough to inhibit wave propagation; for the purposes of illustration we take it to be \( 0.012\Omega \).

In this case there are three possible solutions, corresponding to total angular speeds relative to corotation (\( \omega^* \)) of 0.46\%, 0.35\% and 0.32\% of \( \Omega_S \) and relative angular speeds within the shear layer (\( \omega \)) of 0.16\%, 0.049\% and 0.021\%. These solutions are shown by the vertical solid, dashed and dot-dash lines in Fig. 10a. We have deliberately chosen a value of \( \omega_{sh} \) that gives exactly three solutions. The number of solutions increases as \( \omega_{sh} \) approaches \( \omega_{lim} \); for values of \( \omega_{sh} > \omega_{lim} \) there are an infinite number of possible solutions, most of which are very short-wavelength.

The profiles of \( F(z) \) for the three solutions are shown in Fig. 10b using the same line formats as Fig. 10a. This shows that for solutions with a smaller total angular speed, the vertical wavelength in the shear layer decreases, while the exponential scale length below the shear layer increases.

37
Figure 10: Solutions for the three layer model. Panel (a) shows the three layer shear model as a dotted line. The vertical solid, dashed and dot-dash lines show the total angular speed of the three possible wave solutions in the corotating planetary reference frame. To emphasise that the angular speeds shown represent subcorotation of the atmosphere (i.e. westwards flow), the arrow indicates the planetary rotation direction. Panel (b) shows the calculated values of $F$ for these three models, using the same line formats.
such that Solution 1 has a broad peak in the shear layer whose amplitude drops relatively rapidly to very small values at 200km, whereas Solution 3 has three much narrower peaks in the shear layer and a much more gradual drop in amplitude below the shear layer, which falls to only about one half of its peak value at 200km.

The discrete number of wave modes produced by this analysis provides a potentially elegant mechanism for selecting specific frequencies, explaining the uniqueness of the 10.7-hour structure. However, it is clear that further work is required to demonstrate that such wave modes can exist in real sheared flows.

5. Conclusions

We have applied the theory of Haurwitz [1975] to find planetary wave solutions for the polar upper atmosphere of Saturn. Some of the solutions have the properties necessary to explain the 10.7-hour periodicities:

- There exist solutions for slowly westwards-propagating Rossby waves whose total angular velocity is slightly below that of the deep atmosphere.
- A broad spectrum of waves with different angular velocities are possible, thus permitting the total angular velocity to vary on a timescale of months if the background conditions change.
- The flow pattern associated with the wave is of exactly the form proposed to explain the periodicities, and so the field-aligned currents generated by the wave are also of the correct form.
The existence of a spectrum of wave modes has directed us to propose mechanisms for restricting the possible wave modes or allowing one mode to become dominant, all of which require further investigation:

- The shear layer in the lower thermosphere is proposed to act either as a ‘lid’ to inhibit propagation of waves to higher altitudes, or as a ‘cavity’ that traps waves in a restricted altitude range.

- The enhanced ion drag at the latitude of the main auroral oval is proposed to inhibit zonal winds, prejudicing the growth of wave modes with a nodal line in the zonal winds at this latitude.

The principal limitations of our model are as follows:

- Wind speeds of $\sim 70\text{ms}^{-1}$ are required to generate currents of the order of 1MA, as required to explain the magnetospheric observations, but speeds of this magnitude violate the underlying assumptions of the model (hydrostatic equilibrium and linearisation).

- We do not have a good model of the neutral wind shear, and so cannot accurately estimate the altitude to which Rossby waves can propagate.

- We do not have a good understanding of the behaviour of Rossby waves impinging vertically on a critical layer, and thus do not know if they are absorbed, reflected, or partially reflected.

Other questions which remain to be answered include:

- Do the details of the field-aligned currents driven by the wave structure match the magnetospheric observations?
• To what extent do seasonal variations in the background conditions, for example temperature, affect the predicted values of $\omega$, and do these variations explain the observations?

• What are the effects of the various damping processes on the wave structure?

• Is it possible for perturbations to the main auroral oval, driven by the wave structure, to feed back and provide energy to maintain the wave structure?

Acknowledgements

LCR was supported by STFCs UCL Astrophysics Consolidated grant ST/J001511/1.

Appendix A.

The following derivation closely follows that of Haurwitz (1975).

To begin the process of solving Eqns. 6-10 we make the following standard substitutions which greatly simplify the manipulations that follow:

$$u' = \rho_0^{1/2} u$$  \hspace{1cm} (A.1)
$$v' = \rho_0^{1/2} v$$  \hspace{1cm} (A.2)
$$w' = \rho_0^{1/2} w$$  \hspace{1cm} (A.3)
$$\delta p' = \rho_0^{-1/2} \delta p$$  \hspace{1cm} (A.4)
$$\delta \rho' = \rho_0^{-1/2} \delta \rho$$  \hspace{1cm} (A.5)
Substituting all of the above into Eqns. (6-10) yields the following:

\[ i\omega u' - f v' = -\frac{ik}{r} \delta p' \]  
(A.6)

\[ i\omega v' + f u' = \frac{\partial \delta p'}{\partial r} \]  
(A.7)

\[ \frac{\partial \delta p'}{\partial z} - \frac{1}{2H} \delta p' = -g \delta p' \]  
(A.8)

\[ i\omega \delta p' - \frac{1}{2H} w' + \frac{ik}{r} u' - \frac{1}{r} \frac{\partial v'}{\partial r} + \frac{\partial w'}{\partial z} = 0 \]  
(A.9)

\[ i\omega \delta p' = i\omega \gamma gH \delta p' - g(\gamma - 1)w' + (\gamma - 1)q \rho_0^{1/2} \]  
(A.10)

We can combine the equations to above to eliminate \( w' \), \( \delta p' \) and \( u' \), yielding two equations involving \( v' \), \( \delta p' \) and the forcing function \( q \) alone:

\[ v'(f^2 - \omega^2) = \frac{ikf}{r} \delta p' + i\omega \frac{\partial \delta p'}{\partial r} \]  
(A.11)

\[ \frac{\omega^2 H}{\kappa} \left[ \frac{\partial^2}{\partial x^2} + \left( -\frac{1}{4H^2} + \frac{k^2 \kappa g}{H r^2 \omega^2} \right) \right] \delta p' + \]
\[ i\omega \left[ \frac{\partial}{\partial z} - \frac{1}{2H} \right] (\rho_0^{1/2} q) - i\omega g \left( \frac{1}{r} \frac{\partial v'}{\partial r} - \frac{f k}{r \omega} v' \right) = 0 \]  
(A.12)

Here \( \kappa = (\gamma - 1)/\gamma \). We then separate variables, defining:

\[ u' = F(z)U(r) \]  
(A.13)

\[ v' = F(z)\frac{V(r)}{r} \]  
(A.14)

\[ \delta p' = F(z)P(r) \]  
(A.15)

\[ q = iQ(z)P(r) \]  
(A.16)
where it is clear that we have assumed that $u'$, $v'$ and $\delta p'$ have the same $z$-dependence.

Substituting Eqs. A.13 into Eqn. A.11 we obtain:

$$V(f^2 - \omega^2) = ikfP + i\omega rP'$$  \hspace{1cm} (A.17)

and further substituting Eqs. A.13 into Eqn. A.12 we obtain two equations by defining a separation constant $h$:

$$-\frac{1}{r} \frac{V'}{P} + \frac{f k}{\omega r^2} \frac{V}{P} - \frac{ik^2}{\omega r^2} = -\frac{i\omega}{gh}$$  \hspace{1cm} (A.18)

$$\frac{i\omega H}{g\kappa} \left[ \frac{F''}{F} - \frac{1}{4H^2} \right] - \frac{i}{gF} \left[ \frac{\partial}{\partial z} - \frac{1}{2H} \right] Q\rho_0^{1/2} = -\frac{i\omega}{gh}$$  \hspace{1cm} (A.19)

In these equations a prime on $F$ indicates differentiation with respect to $z$ and primes on $V$ and $P$ imply differentiation with respect to $r$.

Equation A.19 can be rearranged to yield the following second order equation for $F$:

$$F'' + \left[ \frac{\kappa}{H\hbar} - \frac{1}{4H^2} \right] F = \frac{\kappa}{\omega H} \left[ \frac{\partial}{\partial z} - \frac{1}{2H} \right] Q\rho_0^{1/2}$$  \hspace{1cm} (A.20)

Setting $Q = 0$ in this expression gives Equation 27, appropriate for free oscillations, discussion of which is continued in Section 3.4.

Combining Equations A.17 and A.18 to eliminate $P$ yields a second order differential equation for $V$:

$$V'' + \frac{1 + \epsilon^2 r^2}{r} V' + \left[ -\frac{k}{\omega r} \frac{\partial f}{\partial r} - \frac{(f^2 - \omega^2)}{gh} - \frac{2f\omega}{ghk(1 - \epsilon^2 r^2)} - \frac{k^2}{r^2} \right] V = 0$$  \hspace{1cm} (A.21)
use of Equation [A.7] and [A.18] yields equations for $U$ and $P$ in terms of $V$:

$$(1 - \epsilon^2 r^2)U = \epsilon^2 r^2 \frac{ifV}{\omega r} - \frac{iV'}{k}$$

(A.22)

$$(1 - \epsilon^2 r^2)P = -\frac{ifV}{k} + \frac{i\omega r V'}{k^2}$$

(A.23)

where

$$\epsilon^2 = \frac{\omega^2}{ghk^2}$$

(A.24)

These equations are too complex to permit simple analytic solutions. We now make two approximations that simplify matters considerably. Firstly, we restrict ourselves to circumstances in which $\epsilon^2 r^2 \ll 1$, so that we can neglect terms involving this factor. We will only consider solutions that satisfy this condition; this is demonstrated in the main text.

Secondly, we consider the terms involving the Coriolis parameter $f$, given by Eqn. 5. Across the range of latitudes in which we are interested, there is little variation in this parameter. It is thus reasonable to insert a constant value $f_0$ to reduce the complexity of our expressions. We choose the value at the latitude of the main auroral oval, $r_0$:

$$f_0 = 2\Omega \left(1 - \frac{r_0^2}{2R_c^2}\right)$$

(A.25)

However, there is one term in Eqn [A.21] that explicitly depends on the radial derivative of $f$. The radial derivative is approximately:

$$\frac{\partial f}{\partial r} \approx -\frac{2\Omega r}{R_c^2}$$

(A.26)

and we insert this expression into Eqn. [A.21]
Making these approximations, the horizontal structure equations reduce to:

\[ V'' + \frac{1}{r} V' + \left[ m^2 - \frac{k^2}{r^2} \right] V = 0 \]  \hspace{1cm} (A.27)

\[ U = - \frac{iV'}{k} \]  \hspace{1cm} (A.28)

\[ P = - \frac{ifV_0}{k} + \frac{i\omega r V'}{k^2} \]  \hspace{1cm} (A.29)

where

\[ m^2 = \frac{2\Omega k}{R_c^2 \omega} - \frac{(f_0^2 - \omega^2)}{gh} - \frac{2f_0\omega}{ghk} \]  \hspace{1cm} (A.30)

The first equation is a form of Bessel’s equation, and solutions that are finite at the pole can thus be written in terms of Bessel functions of the first kind \( J_k \):

\[ V = V_0 J_k(mr) \]  \hspace{1cm} (A.31)

\[ U = - \frac{imV_0}{k} J'_k(mr) \]  \hspace{1cm} (A.32)

\[ P = - \frac{ifV_0}{k} K_k(mr, \omega) \]  \hspace{1cm} (A.33)

where to simplify these expressions we have defined an additional function \( K_k \):

\[ K_k(mr, \omega) = J_k(mr) - \frac{\omega mr}{f_0 k} J'_k(mr) \]  \hspace{1cm} (A.34)

Provided that \( \omega \ll f \) (true for all slowly propagating solutions that are relevant here) and if \( mr \) and \( k \) are of order unity (also true for all situations considered here), the second term in the equation for \( P \) is much smaller than the first and thus \( K_k \approx J_k \).
Dimensional considerations prompt us to relate $V_0$ to characteristic velocities $u_0$ and $v_0$ according to:

$$v_0 = ku_0 = \rho_0^{-1/2} V_0 m$$ (A.35)

Combining all of these expressions to eliminate $V$, $U$ and $P$ yields Equations 18-21, discussion of which continues in the main text.

References


46


C. G. A. Smith. A Saturnian cam current system driven by asymmet-


