Rings of polynomials with Artinian coefficients

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Abstract

We study the extent to which the weak Euclidean and stably free cancellation properties hold for rings of Laurent polynomials \( A[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \) with coefficients in an Artinian ring \( A \).

Keywords: Stably free cancellation; weakly Euclidean ring; Laurent polynomials.

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Recall that a module \( S \) over a ring \( \Lambda \) is said to be stably free when \( S \oplus \Lambda^a \cong \Lambda^b \) for some positive integers \( a, b \). We say that \( \Lambda \) has stably free cancellation (\( = \) SFC) when any stably free \( \Lambda \)-module is free. Elementary duality considerations show this property is left-right symmetric. We show that Artinian rings have the SFC property. More generally, we study the extent to which the SFC property holds for the rings 

\[
L_n(A) = A[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]
\]

of Laurent polynomials in \( n \) variables \( t_1, \ldots, t_n \) with coefficients in an Artinian ring \( A \). Here we do not assume that \( A \) is commutative but we do require that the variables \( t_i \) commute both amongst themselves and with the coefficients in \( A \). When \( A \) is Artinian the Jacobson radical \( \text{rad}(A) \) is nilpotent ([9], p.81) and the quotient \( A/\text{rad}(A) \) is isomorphic to a product of matrix rings

\[
(*) \quad A/\text{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)
\]

where \( D_1, \ldots, D_m \) are division rings and \( d_1, \ldots, d_m \) are integers \( \geq 1 \). The Artinian ring \( A \) is said to satisfy the Eichler condition (cf [11], pp. 174-175) when in the decomposition (*) above, \( D_i \) is commutative whenever \( d_i = 1 \). We strengthen this condition as follows; say that \( A \) is strongly Eichler when in (*) above each division algebra \( D_i \) is commutative; then we have:

**Theorem I:** If the Artinian ring \( A \) is strongly Eichler then \( L_n(A) \) has property SFC for all \( n \geq 1 \).

There is a corresponding property which has strong stability implications for automorphisms of free modules. A ring \( \Lambda \) is weakly Euclidean\(^{(1)} \) (cf [6] Chap.1) when for all \( d \geq 2 \), any \( X \in GL_d(\Lambda) \) can be written as a product

\[
X = E_1 \cdot \ldots \cdot E_n \cdot \Delta_d(\lambda)
\]

\(^{(1)}\) The terminology arises from the classical theorem of H.J.S. Smith [10] which we may state as saying that an integral domain with a Euclidean algorithm is weakly Euclidean.
where each $E_i$ is an elementary transvection and $\Delta_d(\lambda)$ is an elementary diagonal matrix with $\lambda \in \Lambda^*$. Here $\Lambda^*$ denotes the group of invertible elements in the ring $\Lambda$. We say that the Artinian ring $A$ is very strongly Eichler when in the decomposition (*) above each $D_i$ is commutative and in addition each $d_i \geq 2$.

**Theorem II:** If the Artinian ring $A$ is very strongly Eichler then $L_n(A)$ is weakly Euclidean for all $n \geq 1$.

Both Theorem I and Theorem II would seem to be best possible. In relation to Theorem I, a result of Ojanguren and Sridharan [8] shows that, for $n \geq 2$, $L_n(D)$ fails to have the SFC property whenever the division ring $D$ is noncommutative. As regards Theorem II, when $n \geq 2$ the so-called ‘Colm matrix’ (cf [2], p.26)

$$
\begin{pmatrix}
1 + t_1t_2 & -t_2^2 \\
t_1^2 & 1 - t_1t_2
\end{pmatrix} \in GL_2(L_2(F))
$$

fails to decompose as a product of elementary matrices over any field $F$. A direct proof of this result may be found on p.54 of Lam’s book [7]. When $n = 1$ we nevertheless obtain the following useful result.

**Theorem III:** If the ring $A$ is Artinian then $L_1(A)$ is weakly Euclidean; furthermore, if $A$ also satisfies the Eichler condition then $L_1(A)$ has property SFC.

Finite rings are Artinian and strongly Eichler; thus we have:

**Theorem IV:** If the ring $A$ is finite then

(i) $L_n(A)$ has property SFC for all $n \geq 1$; moreover

(ii) $L_1(A)$ is weakly Euclidean.

The results proved here all continue to hold if the rings $L_n(A)$ are replaced by the standard polynomial rings $P_n(A) = A[s_1, \ldots, s_m]$ or even by rings of mixed type $A[s_1, \ldots, s_m, t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. However, as rings of the form $L_n(A)$ occur naturally as group rings $F[\Phi \times C_\infty^n]$ when $\Phi$ is finite, the construction $L_n(A)$ seems more relevant to applications in non-simply connected homotopy theory (cf [6], Chap.11).

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§1 : The weak Euclidean property for $L_1(A)$:

Given a ring $\Lambda$ and integer $d \geq 2$ there is a canonical $\Lambda$-basis $\{e^{(d)}(r, s)\}_{1 \leq r, s \leq d}$ for the ring of $d \times d$ matrices $M_d(\Lambda)$ given by

$$e^{(d)}(r, s)_{tu} = \delta_{rt}\delta_{su};$$

that is, $e^{(d)}(r, s)$ is the $d \times d$ matrix with ‘1’ in the $(r, s)^{th}$ position and ‘0’ elsewhere. By an elementary matrix of Type I in $M_d(\Lambda)$ we mean one of the form

$$E(r, s; \lambda) = I_d + \lambda e^{(d)}(r, s) \quad (r \neq s, \lambda \in \Lambda).$$
By an elementary matrix of Type II in $M_d(\Lambda)$ we mean one of the form

$$\Delta_d(\lambda) = \begin{pmatrix} 
\lambda & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
& & \ddots & & \\
0 & 0 & \ldots & 0 & 1 
\end{pmatrix} \quad (\lambda \in \Lambda^*)$$

Formally we have $\Delta_d(\lambda) = I_d + (\lambda - 1)e^{(d)(1,1)}$ where $\lambda \in \Lambda^*$. We say that $\Lambda$ is weakly Euclidean when for $d \geq 2$ each invertible matrix $X \in \text{GL}_d(\Lambda)$ can be written in the form

$$X = E \cdot \Delta_d(\lambda)$$

where $E$ is a product of elementary matrices of type I over $\Lambda$ and $\lambda \in \Lambda^*$. A ring homomorphism $\varphi : A \to B$ has the lifting property for units when the induced map $\phi_* : A^* \to B^*$ is surjective. We say $\varphi$ has the strong lifting property for units when in addition the following holds for $\alpha \in A$;

$$\alpha \in A^* \iff \varphi(\alpha) \in B^*.$$

It is straightforward to see that:

(1.1) Let $\varphi : A \to B$ be a surjective ring homomorphism; if $\text{Ker}(\varphi)$ is nilpotent then $\varphi$ has the strong lifting property for units.

Elsewhere ([6], Prop. 2.43, p.21) we have shown:

(1.2) Let $\varphi : A \to B$ be a surjective ring homomorphism where $B$ is weakly Euclidean; if $\varphi$ has the strong lifting property for units then $A$ is also weakly Euclidean.

Thus we have:

(1.3) Let $\varphi : A \to B$ be a surjective ring homomorphism with nilpotent kernel; if $B$ is weakly Euclidean then $A$ is also weakly Euclidean.

**Proposition 1.4**: Let $D_1, \ldots, D_m$ be (possibly noncommutative) division rings; then $M_{d_1}(D_1[t, t^{-1}]) \times \ldots \times M_{d_m}(D_m[t, t^{-1}])$ is weakly Euclidean for any positive integers $d_1, \ldots, d_m$.

**Proof**: If $D_i$ is a division ring then $D_i[t, t^{-1}]$ is a (possibly noncommutative) integral domain which admits a Euclidean algorithm (cf [3]). It is now straightforward to see that matrix rings $M_{d_i}(D_i[t, t^{-1}])$ are also weakly Euclidean. (cf.[6] p.22). The required conclusion now follows as the class of weakly Euclidean rings is closed under finite direct products. \(\square\)

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(2) The referee points out that the strong lifting property for $\varphi$ may be re-stated as saying that $\varphi$ has the lifting property and is a local morphism in the sense of Camps and Dicks [1].
Theorem 1.5: Let $A$ be an Artinian ring; then $A[t, t^{-1}]$ is weakly Euclidean.

Proof: The radical $\text{rad}(A)$ of the Artinian ring $A$ is nilpotent (cf [9] p, 81). Consequently $\text{rad}(A)[t, t^{-1}]$ is a nilpotent ideal in $A[t, t^{-1}]$. Moreover

$$A/\text{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$$

for some division rings $D_1, \ldots, D_m$ so that

$$A[t, t^{-1}]/\text{rad}(A)[t, t^{-1}] \cong M_{d_1}(D_1[t, t^{-1}]) \times \ldots \times M_{d_m}(D_m[t, t^{-1}]).$$

The desired conclusion now follows from (1.3) and (1.4).

§2: Suslin’s Theorem and proof of Theorem II:

We shall use the following theorem of Suslin ([7], [12]):

Theorem 2.1: Let $F$ be a field and let $k \geq 3$; then any $X \in \text{GL}_k(L_n(F))$ can be written in the form

$$X = E_1 \cdots E_m \cdot \Delta_k(\lambda)$$

where $\lambda \in L_n(F)^*$ and each $E_i \in \text{GL}_k(L_n(F))$ is an elementary matrix of type I.

We note that the unit group $L_n(F)^*$ consists simply of elements of the form $\alpha \cdot e_i$ where $\alpha \in F^*$ and $e_i$ is an integer ([6], Appendix C).

Fixing a ring $\Lambda$ and an integer $q \geq 2$, we study elementary matrices over the rings $\Omega = M_d(M_q(\Lambda))$. Write

$$E(i, j; Z) = \tilde{I} + Z \cdot E(i, j)$$

where $\tilde{I}$ denotes the identity matrix in $M_d(M_q(\Lambda))$. When $M_q(\Lambda)$ is considered as the base ring we write ‘$\cdot$’ for matrix product over $M_q(\Lambda)$. Then elementary matrices of Type I in $GL_d(M_q(\Lambda))$ take the form

$$\Delta_d(Z) = \begin{pmatrix} Z & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

where $Z \in GL_q(\Lambda) = M_q(\Lambda)^*$. In the special case where $Z \in GL_q(\Lambda)$ is itself an elementary matrix of Type II over $\Lambda$

$$Z = \Delta_q(\lambda) = \begin{pmatrix} \lambda & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

4
with \( \lambda \in \Lambda^* \) we write \( \Delta_{d,q}(\lambda) = \Delta_d(\Delta_q(\lambda)) \in GL_d(M_q(\Lambda)) \).

When \( d \geq 2 \) there is a mapping, \( \nu : M_{dq}(\Lambda) \to M_d(M_q(\Lambda)) \) defined as follows: if \( X = (x_{rs})_{1 \leq r,s \leq dq} \in M_{dq}(\Lambda) \) and \( 1 \leq i,j \leq d \) then

\[
\nu(X) = (X(i,j))_{1 \leq i,j \leq d}
\]

where \( X(i,j) \in M_q(\Lambda) \) is given by

\[
x_{q(i-1)+k, q(j-1)+l};
\]

moreover:

\[\text{(2.2) For any ring } \Lambda, \nu : M_{dq}(\Lambda) \to M_d(M_q(\Lambda)) \text{ is a ring isomorphism.}\]

To record the relationship between the various elementary matrices under block decomposition we first observe that there are unique functions

\[
\upsilon : \{1, \ldots, dq\} \to \{1, \ldots, d\}; \quad \rho : \{1, \ldots, dq\} \to \{1, \ldots, q\}
\]

defined by the requirement \( t + q = q\upsilon(t) + \rho(t) \) for \( 1 \leq t \leq dq \). It is straightforward to verify that:

\[\text{(2.3) } \nu(\epsilon^{(dq)}(r,s)) = \epsilon^{(q)}(\rho(r), \rho(s)) \cdot E(\upsilon(r), \upsilon(s)).\]

The inverse relation is perhaps clearer, namely:

\[\text{(2.4) } \nu^{-1}(\epsilon^{(q)}(a,b) \cdot E(i,j)) = \epsilon^{(dq)}(q(i-1) + a, q(j-1) + b).\]

From (2.3) we note that:

\[\text{(2.5) } \nu(E(r,s; \lambda)) = E(r,v(s); \lambda\epsilon(\rho(r), \rho(s))) \quad (\lambda \in \Lambda).\]

Likewise we have:

\[\text{(2.6) } \nu(\Delta_{dq}(\lambda)) = \Delta_{d,q}(\lambda) \quad (\lambda \in \Lambda^*).\]

We first consider the rings \( L_n(F) = F[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \) where \( F \) is a field.

**Theorem 2.7:** Let \( d, q \geq 1 \) be integers such that \( dq \geq 3 \). If \( X \in GL_d(M_q(L_n(F))) \) then \( X \) can be expressed as a product

\[X = E_1 \cdots E_m \cdot \Delta_{d,q}(\delta)\]

where \( E_1, \ldots, E_m \in GL_d(M_q(L_n(F))) \) are elementary of Type I and \( \delta \in L_n(F)^* \).

**Proof:** Put \( \Lambda = L_n(F) \). If \( X \in GL_d(M_q(\Lambda)) \) put \( \hat{X} = \nu^{-1}(X) \in GL_{dq}(\Lambda) \). By Suslin’s Theorem, \( \hat{X} \) can be expressed as a product

\[\hat{X} = E_1 \cdots E_m \cdot \Delta(\lambda)\]

where \( \lambda \in L_n(F)^* \) and each \( E_i \in GL_{dq}(L_n(F)) \) is an elementary matrix of type I. Thus

\[\nu(\hat{X}) = \nu(E_1) \cdots \nu(E_m) \cdot \nu(\Delta(\lambda))\]
so that, writing \( E_i = \nu(E_i) \) we have \( X = E_1 \cdot \ldots \cdot E_m \cdot \Delta_{d,q}(\delta). \) \( \square \)

**Corollary 2.8**: If \( F \) is a field then \( M_q(L_n(F)) \) is weakly Euclidean for each \( q \geq 2. \)

The weak Euclidean property is preserved under finite direct products. Moreover the construction \( L_n \) commutes with both direct products and with the functor \( \Lambda \mapsto M_q(\Lambda); \) hence we have:

**Corollary 2.9**: \( L_n[M_{q_1}(F_1) \times \ldots \times M_{q_m}(F_m)] \) is weakly Euclidean whenever \( F_1, \ldots, F_m \) are fields and \( q_1, \ldots, q_m \geq 2. \)

**Theorem 2.10**: If the Artinian ring \( A \) is very strongly Eichler then \( L_n(A) \) is weakly Euclidean for \( n \geq 2. \)

**Proof**: Write \( A/\text{rad}(A) \cong M_{q_1}(F_1) \times \ldots \times M_{q_m}(F_m) \) for some fields \( F_1, \ldots, F_m \) and integers \( q_1, \ldots, q_m \geq 2. \) Then \( L_n(\text{rad}(A)) \) is a nilpotent ideal in \( L_n(A) \) and

\[
L_n(A)/L_n(\text{rad}(A)) \cong L_n[M_{q_1}(F_1) \times \ldots \times M_{q_m}(F_m)].
\]

The desired conclusion now follows from (1.3) and (2.9). \( \square \)

Theorem II is now the conjunction of (1.5) and (2.10).

§3: Proof of Theorems I, III and IV:

The following is a straightforward deduction from Nakayama’s Lemma (cf [6] pp. 170-171).

**Proposition 3.1** Let \( \varphi : \Lambda \to \Omega \) be a surjective ring homomorphism such that \( \text{Ker}(\varphi) \) is nilpotent; if \( \Omega \) satisfies \( SFC \) then so also does \( \Lambda. \)

Suppose that \( A \) is an Artinian ring such that

\[
A/\text{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)
\]

where \( D_1, \ldots, D_m \) are division rings. We shall apply (3.1) in the case \( \Lambda = L_n(A), \)
\( \Omega = L_n(A)/L_n(\text{rad}(A)) \) and \( \varphi \) is the natural mapping. Then

\[
\Omega \cong M_{d_1}(L_n(D_1)) \times \ldots \times M_{d_m}(L_n(D_m)).
\]

We showed in [5] that \( \Omega \) has property \( SFC \) provided each \( D_i \) is commutative; that is, provided \( A \) is strongly Eichler. Thus from (3.1) we obtain:

**Proposition 3.2**: If the ring \( A \) is Artinian and strongly Eichler then \( L_n(A) \) has property \( SFC. \)

As we observed in the Introduction, Ojanguran and Sridharan proved in [8] that \( L_n(D) \) fails the \( SFC \) property whenever \( n \geq 2 \) and the division ring \( D \) is noncommutative. However, in the case \( n = 1 \) one may show that \( L_1(D) = D[t, t^{-1}] \) has
regardless of whether the division ring $D$ is commutative or not. Indeed, in that case, $D[t, t^{-1}]$ is projective free (cf [4] or [5] Prop 2.9). Now the SFC property is preserved under finite direct products and passage to matrix rings ([6] p. 171-173). Thus $M_{d_i}(L_1(D_1)) \times \ldots \times M_{d_m}(L_1(D_m))$ has property $SFC$. From (3.1) we get:

**Proposition 3.3**: If the ring $A$ is Artinian then $L_1(A)$ has property $SFC$.

The conjunction of (3.2) and (3.3) is Theorem I of the Introduction.

Any finite ring $A$ is trivially Artinian so that $A/\text{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$ where $D_1, \ldots, D_m$ are finite division rings. However, a celebrated theorem of Wedderburn (cf [13] p.1) now shows that each $D_i$ is commutative; that is:

**3.4** Any finite ring is Artinian and strongly Eichler.

Thus from (1.5), (3.2) and (3.4) we have:

**Corollary 3.5**: Let $A$ be a finite ring; then

(i) $L_n(A)$ has property $SFC$ for all $n \geq 1$;

(ii) $L_1(A)$ is weakly Euclidean.

We may regard the coefficient ring $A$ as a degenerate case $A = L_0(A)$. Thus suppose that $A$ is Artinian and write $A/\text{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$ where $D_1, \ldots, D_m$ are division rings. Then each $M_{d_i}(D_i)$ is weakly Euclidean and has property $SFC$. As both these properties are closed under finite direct products then $A/\text{rad}(A)$ is weakly Euclidean and has property $SFC$. However, $\text{rad}(A)$ is nilpotent so that, from (1.3) and (3.1), we conclude the following which should be well known but is difficult to locate explicitly in the literature.

**3.6** Any Artinian ring is weakly Euclidean and has property $SFC$. 

7
REFERENCES


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