THEORY OF CHAOTIC HAMILTONIAN RATCHETS

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Yea verily, I say unto you a man must have chaos yet within him to birth a dancing star.

_Friedrich Wilhelm Nietzsche_
理论混沌哈密尔顿触发器

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混沌哈密尔顿触发器首次被使用一个不对称的双阱和一个由逐步增加的脉冲序列提供的时间不对称性来实现，这是实现定向传输的必要条件。该触发系统的特点是，经过一个特征时间，即“触发时间”，其经典电流饱和到一个有限值。有限的触发电流的产生归因于具有正和负动量的粒子的短时间扩散率的差值。该论文的第一部分详细研究了这个系统。具体来说，扩散率和平均电流被进行了分析，并得到了与数值模拟一致的公式。这些修正被证明了由于之前未被考虑的动量依赖性修正的准线性扩散造成的。这些修正通过考虑脉冲序列内相继脉冲的关联来找到的。经典触发电流的分析形式也从这些关联获得。

第二系统使用一个摇晃线性项来创建必要的空间不对称性，而通过一个逐步增加的脉冲序列再次引入时间不对称性。该系统表明...
to demonstrate a ratchet effect in a similar fashion to the double well system, illustrating the generic nature of the model. The ability to use this rocking ratchet to preferentially select atoms of a given initial momentum, thus creating a chaotic filter, is also introduced. The diffusion coefficient and average current are once again investigated analytically, and the resulting formulae shown to give excellent agreement with numerical results.

Finally, the possibility of performing chaotic Hamiltonian ratchet experiments in pulsed standing waves of light (optical lattices) is discussed, and recent results obtained by the Laser Cooling Group at UCL for the rocking ratchet are shown.
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Ever since the early days of atomic physics, scientists have constantly searched for different ways to control and manipulate the motion of atoms. In recent years this challenge has been approached from a new angle by the marriage of three disparate research fields - laser cooling of atoms, chaos theory and the study of microscopic ratchet systems.

As will be shown below the theoretical study of ratchet systems can traced back almost 100 years, however experimental work in the field has, until recently, been somewhat limited. The advent of laser cooling over the last three decades has provided an ideal testing ground for many ratchets that had previously only been investigated analytically. Furthermore, the possibility of using laser cooling techniques to prepare atoms in a well defined state has stimulated interest in a new type of ratchet - the clean, dissipation-free Hamiltonian ratchet, where directed motion is created by the action of deterministic chaos. It is the study of these chaotic Hamiltonian ratchets that is the subject of this thesis.
1.1 What is a Ratchet?

It is appropriate to begin by discussing exactly what one means by a ratchet, and why their study is of scientific importance. A brief summary of the various non-chaotic ratchet models that have been investigated will then be given. Some of the key concepts of chaos theory will be introduced before showing how one can use these ideas to create a chaotic Hamiltonian ratchet.

1.1 What is a Ratchet?

At its most basic, a ratchet can be described as a device which only allows motion in one direction. Macroscopic, everyday examples include ticket turnstiles, unidirectional winches, bicycle transmissions, ratchet screwdrivers and even football rattles. However, in each of these examples one must apply a directional force (i.e. push the turnstile barrier) to make the ratchet turn. The central question underpinning the study of ratchets is: can one obtain directed motion from unbiased, random fluctuations - can useful work be extracted from the system, if the acting force averages to zero? For macroscopic systems the answer is yes, with self-winding wrist watches (for which the fluctuations arise from the motion of the wearer's wrist) being a prime example. In the case of microscopic systems the answer is less clear, and it is this ambiguity that has sparked the interest in their study.

Much of the early work in the field was motivated by the need to understand how various biological processes occur. Ratchet mechanisms have, for example, been used to explain the action of ion pumps (proteins which push electrically charged particles through cell membranes), the 'molecular forklift' kinesin (which transports proteins within a cell), and also how muscle contraction occurs. In this guise, ratchets are often referred to as 'molecular motors' (see [1]).
More recently, the field has seen the advent of quantum ratchets, first proposed by Reimann et. al. [2]. Whilst this area is relatively new, experiments have been performed concerning electron transport in semiconductors that suggest that one can use quantum effects to control the particle current. These experiments generally fall into the so-called 'mesoscopic' regime, lying between the classical and the quantum regimes. It will be shown that the second system considered in this thesis provides the basis for a device which could control the particle current of atoms.

The vast majority of ratchet systems that have been studied operate under the influence of external noise and as such can be referred to as Brownian ratchets. The replacement of Brownian fluctuations by the action of deterministic chaos as detailed in this thesis is a completely novel approach, however it is appropriate to put this work in context. Therefore, a brief summary of the different ratchet systems that have so far been investigated will now be given, beginning with the seminal work of Smoluchowski and Feynman.

1.2 A Brief History of Ratchets

1.2.1 The Smoluchowski-Feynman Ratchet

The birth of the study of ratchets can be traced to a *gedanken* experiment performed by Smoluchowski in 1912 in which he considered the possibility of directed transport in spatially asymmetric systems in contact with a heat bath. Later, in his 1963 'Lectures on Physics' [3] Feynman expanded on this early work using the concept now known as the Smoluchowski-Feynman Ratchet.
1.2. A Brief History of Ratchets

The system comprises a ratchet and pawl mechanism attached to a paddle wheel by a solid rod, as shown schematically in Fig. 1.1. Suspended from this rod is a small weight which is raised as the ratchet turns and as such measures the amount of work done in the system. The whole arrangement is contained within a gas-filled box. Feynman suggested that if the pawl was missing, bombardment of the paddles by the gas molecules would cause the ratchet wheel to undergo rotational Brownian motion. With the pawl in place, preventing motion in one direction (backward), and the gas in equilibrium it is reasonable to believe that the ratchet would turn in the opposing (forward) direction. The weight would therefore be lifted and one would generate useful work from equilibrium fluctuations - a direct violation of the Second Law of Thermodynamics. Feynman showed that for the system to be...
affected by the Brownian motion of the gas, it would have to be very small. The pawl, which would have to be light enough to allow forward motion of the ratchet, would therefore also be affected by the molecular collisions - causing it to lift up and allow motion in the backwards manner. On average motion in the two directions cancels out, in keeping with the Second Law.

Feynman later showed that in the presence of non-equilibrium fluctuations (for example if the temperature of the gas surrounding the paddle wheel is higher than that surrounding the ratchet) directed motion does occur and the system does work.

1.2.2 General Properties of Ratchets

The Smoluchowski-Feynman system highlights a number of the key criteria needed to successfully create directed motion in a microscopic ratchet. Firstly, in keeping with the Second Law, the system must not be in thermal equilibrium. This is, in general, analogous to saying that one must break the temporal (time-inversion) symmetry of the system. In most, although not all, ratchet systems it is also necessary to break spatial symmetry - in the Smoluchowski-Feynman case, this is done by using asymmetric ratchet teeth. Spatial symmetry can be broken in a number of ways: in this thesis an asymmetric, double-well potential and a rocking, sinusoidal potential are used.

When investigating ratchets it is worth recalling Curie's Principle: "If a certain phenomenon is not ruled out by symmetries then it will occur". This suggests that one should observe directed motion in the Smoluchowski-Feynman ratchet - however in this case, thermal equilibrium can be expressed as a symmetry condition known as the detailed balance symmetry.
1.2. A Brief History of Ratchets

It should be noted that there are a class of systems which do not exhibit a ratchet effect, despite the presence of spatial asymmetry and broken thermal equilibrium [4, 5]. If, for a potential $V(x)$, there exists a $\Delta x$ and a $\Delta V$ such that $-V(x) = V(x + \Delta x) + \Delta V$ the potential is said to be supersymmetric. A ratchet system obeying the resulting supersymmetry criterion $-V(x) = V(x + L/2)$ (where $L$ is the fundamental period) will not display directed transport, despite being far from thermal equilibrium.

Since Feynman’s ground-breaking work, the study of ratchets has followed a distinct trend. Initially researchers concentrated on systems governed by Brownian motion, spurred on by the need to explain the various biophysical processes discussed earlier. The field then moved on to noise-free dissipative (inertia) ratchets, before making the recent radical step of removing dissipation altogether in favour of deterministic chaos.

This evolution will now be discussed in some detail, beginning with a summary of Brownian ratchets, followed by a discussion of the progression to deterministic ratchets.

1.2.3 Brownian Ratchets

In 2002, Peter Reimann published in Physics Reports a paper entitled "Brownian motors: noisy transport far from equilibrium" [5]. This extremely detailed work provides a thorough overview of the field, and as such it (and the references therein) forms essential background reading for the study of Brownian ratchet systems - see also [6]. Whilst in the context of this thesis a thorough summary of this report is superfluous, a brief discussion of some of the key points from this and other Brownian ratchet papers is appropriate. In particular, Reimann shows that ratchet systems can generally be divided
into three main categories: fluctuating state, tilted or rocking, and pulsed.

Note that whilst the first system studied in this thesis bears close resemblance to a pulsed ratchet, the second system investigated in this work is effectively a combination of a pulsed and a rocking ratchet.

For the Brownian ratchet schemes discussed below, one considers the over-damped (i.e. with inertial term: $m\ddot{x} = 0$) one dimensional stochastic case:

$$\eta\ddot{x}(t) = -V'(x(t), f(t)) + y(t) + F + \xi(t)$$

$$\langle \xi(t)\xi(s) \rangle = 2\eta k_B T \delta(t - s) \quad (1.1)$$

where $\eta$ is a viscous friction coefficient, $\xi(t)$ is a Gaussian white-noise term of zero average which is used to represent thermal fluctuations, $y(t)$ is a tilting process and $F$ is a constant load force which is not considered part of the system. The potential $V(x, f(t))$ is periodic and $y(t), f(t)$ are either periodic or stochastic functions of time. When considering symmetries of the system, one states that the potential of the system is symmetric if: $V(-x, f(t)) = V(x + \Delta x, f(t))$. Similarly, the tilting process is symmetric if $-y(t) = y(t - T/2)$, where $T$ is the fundamental time period.

**Pulsed Ratchets**

For a pulsating ratchet, one considers the case where $y(t) = 0$ and $F = 0$:

$$\eta\ddot{x}(t) = -V'(x(t), f(t)) + \xi(t) \quad (1.2)$$

There are in fact a number of different ratchet types that fall under the banner of pulsating ratchets. Most commonly, the potential is given by the form $V(x(t))[1 + f(t)]$, where $V(x)$ is a spatially periodic asymmetric potential. When $f(t) = \pm 1$ this case is, for obvious reasons, known as
1.2. A Brief History of Ratchets

the on-off ratchet (see Fig. 1.2). Such a ratchet has been demonstrated experimentally by Faucheux et al. [7] and by Gorre-Talini et al. [8] among others. In the latter reference, the authors showed how an on-off ratchet such as this can be used to separate different sized objects - in this case, two sizes of latex spheres diluted in water and acted on by an electrostatic ratchet potential. Theoretically, recent studies have concerned the collective motion of particles in an on-off ratchet [9], current reversals [10] and the mean velocity [11]. The on-off concept has also been extended to two dimensions in [12].

![Figure 1.2: Schematic of an On-Off Ratchet. Courtesy of [13]](image)

If $f(t)$ is no longer restricted to the values ±1, such that the amplitude of the potential may change over time, then the system is known as the fluctuating potential model. This case was first studied by Astumian and Bier [14], who were able to relate their findings to the motion of a kinesin molecule along a biopolymer. A further extension of the pulsating ratchet is
the travelling potential ratchet, governed by the equation:

\[ \eta \ddot{x}(t) = -V'(x(t)) - f(t) + \xi(t). \]  

(1.3)

Whilst these latter schemes are interesting in their own right, and have accumulated a large research following, the ratchet systems considered in this thesis are effectively a special case of the on-off ratchet - where the potential is switched on by the action of a \( \delta \)-pulse, or kick.

**Tilting Ratchets**

In the case of tilting ratchets, the governing equation is now:

\[ \eta \ddot{x}(t) = -V'(x(t)) + y(t) + \xi(t) \]  

(1.4)

where now \( f(t) = 0 \) and once again \( F = 0 \). Tilting ratchets can themselves be sub-divided into three main classes, dependent on the relationship between the potential \( V(x(t)) \) and the tilting term \( y(t) \). In the case where the potential is symmetric, whilst \( y(t) \) is not, one has an *asymmetrically tilting ratchet*. If \( V(x) \) is asymmetric, and \( y(t) \) is a stochastic process one has a *fluctuating force ratchet*. Finally if \( V(x) \) is asymmetric and \( y(t) \) is periodic in time then one obtains a *rocking ratchet* (see Fig. 1.3). Tilting ratchets of various kinds have been extensively studied by Magnasco [15], Doering et al. [16], Bartussek et al. [17] and Reimann et al. [18] among others.

It should be noted that when considering ratchet systems in general (Brownian and non-Brownian) the term "rocking ratchet" is often used to describe the entire class of tilting ratchets. Indeed, in this thesis (and in keeping with previous works on Hamiltonian ratchets) the term *rocking ratchet* is used where one might consider the system to be more accurately described as an *asymmetrically tilting ratchet*.
1.2. A Brief History of Ratchets

Fluctuating state ratchets

The third sub-group of ratchets relies on an altogether different mechanism to induce directed transport. In the case where \( f(t) = y(t) = 0 \), asymmetry is introduced to the system by inducing a change of state. Over the course of the system evolution, a particle makes a transition from one well-defined state to another, before returning back to the original state such that the equation of motion is now:

\[
\eta \dot{x}(t) = -V'(x(t)) + \xi_i(t)
\]  

(1.5)

where the subscript \( i \) refers to the considered state. This ratchet scheme has been successfully applied to protein motors and biological separators in [20, 21], as well as being considered more generally by Jülicher et. al. [22] and Doering [23].

A good alternative example is the two-potential atom-optics ratchet proposed by Robilliard et. al. [25, 24], see Fig. 1.4. An atom, initially in the lower state makes a transition to the upper state \((M \rightarrow N)\). It then remains
Figure 1.4: Representation of the fluctuating state ratchet scheme used in [24]

in the upper state until point $P$, at which point a spontaneous process brings
the atom back to the lower state at point $Q$. The atom has now moved one
well to the right, a clear example of a ratchet effect.

Other Brownian ratchets

There are a number of ratchet systems which broadly fall under the banner
of Brownian ratchets, but cannot be clearly assigned to one of the three main
categories. Seebeck ratchets, where a ratchet effect is induced by the action
of temperature fluctuations which have the same periodicity as the potential,
temperature ratchets, where the temperature variation is time-periodic and
Feynman ratchets can all be modelled using a fluctuating potential scheme.
However, there are discrepancies which are discussed in full in [5].

Reimann’s report also covers the introduction of friction to the ratchet
model and the concept of a drift ratchet. Here a net displacement of sus-
pended particles is produced as the liquid in which they are suspended is
1.2. A Brief History of Ratchets

Pumped back and forth through pores with a ratchet profile [26].

One interesting spin-off from this field was introduced by J. M. R. Parrondo et. al. in [27]. Parrondo's games, as they have become known, use a model based on the on-off ratchet to show that two losing games can combine to have a probability of winning.

1.2.4 Quantum Brownian Ratchets

In general, the ratchet systems discussed thus far have predominantly been studied in the classical regime. Whilst classical ratchet currents have been observed in systems where quantum effects are pronounced (such as in the DC-SQUID ratchet [28]), studies of true quantum ratchets are less common.

The first proposal for a quantum ratchet, where the ratchet current is directly influenced by quantum effects, was published by Reimann et. al. in [2, 18]. The group studied the quantum Brownian motion of particles in an adiabatically rocked potential and showed that whilst at low temperatures the particles would be classically trapped in the potential wells, the tilting of the potential dramatically increases the probability of quantum tunnelling - resulting in a net quantum ratchet current. One notes that this quantum tunnelling current flows in the opposite direction to the classical current.

This pioneering work was followed by that of Yukawa et. al. [29] later in the same year. In this latter paper, the authors used a tight binding formalism in conjunction with an on-off external field to model the quantum ratchet dynamics. Further work by the same group [30] modelled transport in the ratchet system using a path-integral technique (see also [31]). More recently, quantum Brownian ratchets in the limit of weak potentials and force fields have been studied by Scheidl et. al. [32].
1.2. A Brief History of Ratchets

Experimentally, quantum tunnelling ratchets have been successfully demonstrated by H. Linke et. al. in a semi-conductor ratchet for electrons [33, 13]. In [19], a possible rocked quantum-dot ratchet is proposed, where electron-wave interference is used to create a non-linear voltage response which in tern leads to a ratchet current. The possibility of using quantum tunnelling ratchets as heat pumps is also explored in the same paper. A further suggestion for an experimental quantum ratchet is made in [34], where the ratchet effect is demonstrated theoretically in a molecular wire.

1.2.5 Non-Chaotic Deterministic Ratchets

So far, the focus has been on ratchets which rely on the presence of genuine stochastic noise to create directed motion. In this section, the next step in the evolution of ratchets is introduced - the study of deterministic ratchets.

Early work in this field was performed by Dialynas et. al. [35], who replaced the stochastic noise associated with Brownian ratchets by a time-periodic "correlated" noise, given by $\xi(t) = A\sin(\omega t)$. By combining this with an asymmetric potential, they were able to demonstrate a ratchet effect in the absence of true noise.

In [36], Sarmiento and Larralde compare the transport properties of truly deterministic ratchets (where $\xi(t) = 0$) with transport in thermal ratchets (under the influence of Gaussian noise). Their analysis of a slowly rocked asymmetric ratchet potential showed that the current in thermal ratchets is generally deterministic in nature, with the coupling to the thermal bath only becoming important in regions where the deterministic current becomes small.

In general, when one considers deterministic ratchets one finds that the
resulting transport arises from the presence of chaotic dynamics which mimics the role of noise in Brownian ratchets. It is therefore appropriate at this point to break from the discussion of ratchet systems and introduce some of the key concepts of chaos theory, before returning to discuss recent progress in the study of chaotic ratchets.

1.3 A Brief Introduction to Chaos Theory

1.3.1 Classical Chaos

Classical chaos theory can, with some accuracy, be said to have begun with a question. At the end of the 19th century, a contest was organised in honour of King Oscar II of Sweden and Norway to find the best answer to whether the planets would maintain regular orbits indefinitely, or if the orbits were likely to change over time. The prevailing belief was that celestial motion was regular and ultimately predictable. Henry Poincaré's prize winning entry, in which he stated that the stability of the solar system could not be guaranteed, contained the basic concepts that would later form the basis of chaotic dynamics. Poincaré demonstrated that small differences in the initial conditions of a system can lead to large changes down the line.

Despite this early work, the development of chaos theory did not really progress until the work of Komolgorov, Arnol'd and Moser, who looked again at the stability of the solar system in the middle of the last century. The major concern of their work was to address the question of ergodicity. This concept, introduced by Boltzman in his theory on gases, states that molecular motion should be considered random, with the molecule eventually exploring all of the phase space available to it. Their KAM theory,
1.3. A Brief Introduction to Chaos Theory

as it became known, showed the presence of both stable and unstable trajectories in systems perturbed away from integrability, with the number of stable trajectories decreasing as the perturbations become larger. A phase space portrait would therefore show stochastic regions bounded by invariant surfaces. Thus, given certain starting conditions, the solar system could be considered stable. They further noted that motion in the stochastic regions of phase space, when bounded by invariant (KAM) curves, is not ergodic. Since this work the study of chaotic systems has grown rapidly, partially due to the advent of computers and more recently because of the ability to demonstrate chaotic phenomena experimentally.

When characterising a system as classically chaotic, one requires that the perturbation of the system be deterministic and exponential, rather than linear, in nature. This ensures that the future evolution of the motion cannot be predicted from considering the history of the system, as is possible for a linear perturbation. This condition is quantified using the so-called Lyapunov exponent and the related Komolgorov-Sinai entropy. The exponential sensitivity also means that the spectrum of motion is continuous, and therefore non-integrable.

1.3.2 Quantum Chaos, The Kicked Rotor and Dynamical Localisation

One notes that, in general, classical dynamics is regarded as the limiting case (as $1/h \to \infty$) of quantum mechanics, known as the correspondence principle. This implies that since chaotic dynamics exists at the classical level, there must be a quantum mechanical counterpart. However, by considering a general solution of the Schrodinger equation for a bound conservative
system \((H(p, q, t) \equiv H(p, q))\):

\[
\psi(x, t) = \sum_n A_n U_n(x) e^{-\frac{2\pi i E_n t}{\hbar}},
\]

one notes immediately that the energy and frequency spectrum of the quantum motion is discrete, and that the evolution of the wavefunction is quasi-periodic for large times - ruling out the possibility of chaos. Whilst this result seems at odds with the correspondence principle, quantal and classical behaviours agree up to a finite time, \(t_H\), the 'Heisenberg time', given by the mean level spacing \(t_H = \Delta E/\hbar\). Beyond the Heisenberg time (more commonly known as the quantum break-time), classical chaos is actually found to be suppressed in quantum systems.

Note that in Eq. (1.6), \(E_n\) refers to an energy eigenvalue of the time-independent Hamiltonian. However, in this thesis a time-periodic system is considered. It will be shown in chapter 2 that one can still use an expression like Eq. (1.6), but the \(E_n, U_n\) refer respectively to the eigenvalues and eigenstates of the Floquet operator, \(U\).

Most often, quantum and classical chaos are introduced in terms of the \(\delta\)-kicked rotor - a system that has become known as the paradigm of chaos. In the usual representation, the rotor simply consists of a bar rotating about a frictionless pivot and subject to a periodic impulse, or kick. The analysis of the kicked rotor will be covered in detail in the next chapter, at this stage one merely notes that its classical evolution in momentum and position is described by the so-called standard map [37]:

\[
\begin{align*}
p_{n+1} &= p_n + K \sin x_n, \\
x_{n+1} &= x_n + p_{n+1}
\end{align*}
\]

This mapping can in general, not be solved explicitly. One obtains infor-
Figure 1.5: Typical Poincaré phase space portrait for standard map

1.3. A Brief Introduction to Chaos Theory

Information about the system evolution by performing successive iterations and plotting each value of $p$ and $x$ to build up a picture known as a Poincaré surface of section. One notes that the transition from regular to chaotic dynamics is entirely governed by one variable, $K$ the kick strength. A typical phase space portrait is shown in Fig. 1.5 - note the presence of islands, denoting regular periodic trajectories, and stochastic regions in which the motion is chaotic. The classical energy growth in the system is, to first order, linear in time and has the form $\bar{E} = (K^2/4)t$.

Quantum mechanically, the energy growth mimics the classical case for a certain time until, at the break-time, the energy saturates to a constant value. This quantum suppression of classical chaotic diffusion is known as dynamical localisation and was first observed by Casati et al. in 1979 [38].
1.3. A Brief Introduction to Chaos Theory

Figure 1.6: Dynamical localisation in the kicked rotor. Note the characteristic triangular shape of the natural logarithm of the momentum distribution - the hallmark of dynamical localisation. In both cases $K = 5$.

Typical results for dynamical localisation in the kicked rotor are shown in Fig 1.6 - note the break-time, $t^*$, and the triangular profile of the logged quantum momentum distribution showing the exponential localisation. In a later paper [39], Shepelyansky showed that if a time reversal is performed the classical system does not return to its initial state, whereas the quantum system does - the Schrödinger equation can always be integrated back in
1.4. Chaotic Ratchets

1.4.1 More on Deterministic Ratchets

In the previous discussion of Brownian ratchets, the three main ratchet models were introduced in the over-damped limit, where the inertial term in Eq. (1.1), \( m\ddot{z}(t) \), was considered to be zero. Whilst this approximation is appropriate for describing, for example, the motion of molecular motors inside cells, one might expect that the finite inertia present in many systems would considerably affect the ratchet dynamics.

The situation of finite inertia was first investigated by Jung et. al. [40] in the case of a rocked deterministic (\( \xi(t) = 0 \)) ratchet. They found that with the presence of an inertial term, the particle motion became complex and for sufficiently large rocking amplitudes both regular and chaotic motion were observed. Current reversals were also observed as the rocking amplitude was varied, for given values of inertial mass, \( m \), and the friction coefficient \( \eta \). This work was further developed by Mateos [41] who showed that (for certain parameter ranges) the current reversals arise due to a bifurcation from a chaotic to a periodic regime. Furthermore, Mateos showed that transport in the system arose from deterministic anomalous diffusion. In [42, 43], it was shown that the addition of a small amount of disorder to the system induces
1.4. Chaotic Ratchets

strong diffusive motion and that this disorder can cause current reversals in both regular and chaotic trajectories. Most recently, Mateos [44] has shown that one can control the direction of the current in inertial ratchets by careful choice of the initial parameters, such that a periodic attractor transports particles in one direction and a chaotic attractor transports them in the opposite direction.

The next important contribution came from the research of Sergei Flach and coworkers concerning the symmetry considerations in deterministic ratchets. In their work [45, 46, 47, 48, 49] they consider an equation of motion of the form:

\[ m \ddot{x} + \gamma \dot{x} - f(x) - E(t) = 0 \quad (1.8) \]

where now \( f(x) = -V'(x) \), \( E(t) \) is a time-periodic external field of zero mean and, to preserve their notation, \( \gamma \) is used in place of \( \eta \).

Symmetries of the system are then characterised by considering whether transformations in \( x \) and \( t \) exist, such that one can generate trajectories of equal and opposite velocities to \( \dot{x} \). In general these transformations fall into two categories: firstly those that change the sign of \( x \) \( (x \rightarrow -x) \) and shift \( t \) \( (t \rightarrow t + t_0) \), or those that shift \( x \) and reflect time \( t \rightarrow -t \). Then if one defines \( \hat{f}_a \Rightarrow f(x) = -f(-x), \hat{E}_a \Rightarrow E(t) = E(-t) \) and \( \hat{E}_{sh} \Rightarrow E(t) = -E(t + T/2) \) the following symmetries are found:

\[
\hat{S}_a : \quad x \rightarrow -x, \quad t \rightarrow t + \frac{T}{2}, \quad \text{if}\{f_a, E_{sh}\} \\
\hat{S}_b : \quad x \rightarrow x, \quad t \rightarrow -t, \quad \text{if}\{E_s, \gamma = 0\} \quad (1.9)
\]

In [45, 48], the authors explain the presence (or absence) of a persistent current by noting that in the Hamiltonian limit \( (\gamma = 0) \) the resulting phase space from Eq. (1.8) is mixed, i.e. it contains both regular and chaotic areas. In particular, the stochastic layer contains regular islands and a trajectory
trapped in one of these islands can perform ballistic transport by a mechanism known as a Lévy flight. If the symmetries $\hat{S}_a$, $\hat{S}_b$ are present, the average velocity of these flights, the current, is zero. In the absence of these symmetries, the island structure also desymmetrises and a finite current persists.

### 1.4.2 Hamiltonian Ratchets in Mixed Phase Space

The possibility of creating a dissipationless Hamiltonian ratchet was first investigated in detail by Dittrich, Schanz and coworkers [50, 51], who examined the classical and quantum dynamics of a ratchet in the mixed phase space regime.

Using a ratchet system where the force is periodic in space and time (such that $V'(x + 1, t) = V'(x, t + 1) = V'(x, t)$), for the classical case they analyse the nature of transport in the invariant sets present in the phase space. In other words, they consider the transporting properties of regular islands in relation to motion in the stochastic, chaotic regions. This transport is defined in terms of the phase space volume and the average velocity to be:

$$
\tau_M = \int_0^1 dt \int_0^1 dx \int_{-\infty}^{+\infty} dp \chi_M(x, p, t) \frac{\partial H}{\partial p} \tag{1.10}
$$

where $\chi_M$ is the characteristic function of the invariant set $M$. Thus, for a two-dimensional Poincaré surface of section one finds for a given island:

$$
\tau_i = A_i v_i, \text{ where } A_i \text{ is the area of the island and } v_i \text{ the velocity.}
$$

If the chaotic region containing the islands is bounded in momentum by two KAM-tori, $p_a$ and $p_b$, then by treating this region as the invariant set $M$ one finds from Eq. (1.10) that: $\tau_M = \langle T \rangle_a - \langle T \rangle_b$. Now using the sum rule:

$$
\tau_M = \sum_i \tau_{M_i} \tag{1.11}
$$
Schanz et. al. showed that the transport in the chaotic region is described by:

\[ A_{ch} v_{ch} = \langle T \rangle_a - \langle T \rangle_b - \sum_i A_i w_i \]  

(1.12)

where \( \langle T \rangle \) is the average kinetic energy. Note that \( v_i = w_i = x_i / t_i \), where \( w_i \) is the so-called winding number of the stable fixed point at the centre of the island, and the island moves a distance \( x_i \) after \( t_i \) periods. Schanz et. al. use this result to state that a Hamiltonian ratchet must have mixed phase space to demonstrate directed motion. As will be shown in this thesis, there is a 'loophole' that allows a chaotic ratchet. They also note that whilst Lévy flights are present, they stress that because these flights are a result of trajectories passing through "leaky barriers" (cantori) between chaotic regions, they are wrapped up in the motion of the chaotic region. As such (and in contrast to the work of Flach detailed above) Schanz et. al. consider that Lévy flights are not responsible for transport in Hamiltonian ratchets.

This theoretical framework was then shown to apply to a pulsed Hamiltonian with symmetric potential \( V(x) = (x \mod 1 - 1/2)^2 / 2 \) and an asymmetric kinetic energy \( T(p) = |p| + 3 \sin(2\pi p)/(4\pi^2) \).

For the equivalent quantum analysis, the group consider the invariants to be the stationary states of the quantum evolution operator. Transport in the quantum system is then given by the expectation values for the stationary states of the velocity operator, \( \hat{v} = \hat{T}'(\hat{p}) \). The analogy with the classical case arises through considering the band structure of the system. Regular states (corresponding to the classical islands) appear as straight lines, and chaotic states as curves with avoided crossings (see Fig. 1.7). The velocities (winding numbers) are then simply the slopes of the bands: \( \bar{v}_{\alpha,k} = d\epsilon_{\alpha}(k)/dk \) where \( \epsilon \) is the quasienergy and \( k \) the quasimomentum. Thus, as in the classical...
1.4 Chaotic Ratchets

Figure 1.7: Band structure for mixed phase space Hamiltonian ratchet from [51].

- Left-hand panel shows the diabatic case where $h = 1/32$ and the right-hand panel
  shows the adiabatic case where $h = 1/4$.

In the diabatic case, a regular state has a finite winding number whilst the winding number
for the chaotic case vanishes. In the diabatic regime one has a mixture of
chaotic and regular states, whilst in the adiabatic case only chaotic states
exist. Thus the sum of the winding numbers in the adiabatic regime is zero.

Making the transition from adiabatic to diabatic simply means reconnecting
the bands and this, Schanz et al. show, demonstrates that the sum of the
winding numbers must be zero. Therefore one obtains a sum rule analogous
to the classical one:

$$\sum_{\alpha} \bar{w}_{\alpha}^{ch} + \sum_{\alpha} \bar{w}_{\alpha}^{reg} = 0$$ (1.13)

In 2002, Cheon et al. [52] performed a classical study of a Hamiltonian ratchet using a rocking ratchet scheme with temporal asymmetry. The
Hamiltonian for their system is given by:

\[ H = \frac{p^2}{2} + \sum_{n=-\infty}^{\infty} \left[ -K \cos(x) + xA s_n \right] \delta(t - nT \{1 + \epsilon s_n\}) \]  

(1.14)

where \( s_n = (-1)^n \) and \( A \) is the strength of the rocking linear term. Note that the temporal asymmetry is introduced by perturbing the period of the kicks from unity by the addition of a small parameter \( \epsilon \). This method of 'chirping' the period of the \( \delta \)-kicks is also used in both our ratchet systems - indeed, the rocking ratchet scheme proposed in this thesis is closely analogous to that used in [52].

1.5 Chaos and Laser Cooling: The Experimental Perspective

The advent of laser cooling and trapping of atoms in the past thirty years has revolutionised the field of quantum chaos. The possibility of experimentally demonstrating concepts that had previously only been studied theoretically has led to burgeoning interest in the field. Whilst the techniques used to cool and trap atoms are discussed in detail in chapter 7, it is worth stressing that by using these techniques one can effectively confine atoms in a standing wave of laser light (known as an optical lattice) with little or no dissipation. As such, the system provides an excellent testing ground for models such as the kicked rotor which was introduced earlier, as well as providing a framework in which to study other phenomena such as chaos assisted tunnelling.

One of the most noteworthy contributions to this field is the work of Mark Raizen and colleagues at the University of Texas, Austin. In 1994 [54], the group published the first experimental demonstration of dynamical
localisation in ultracold sodium atoms. Here Moore et al. in fact used a periodically driven rotor, with scaled Hamiltonian of the form: $H = \frac{p^2}{2} - k\cos(\phi - \lambda \sin t)$, where $\phi$ is the spatial coordinate. Further work on this
1.5. Chaos and Laser Cooling: The Experimental Perspective

The first experimental realisation of a quantum δ-kicked rotor (once again using Sodium atoms) was performed by the same group using a train of Gaussian pulses narrow enough to approximate the δ-function limit [53] (see Fig. 1.8). The departure of the experimental energy growth from the classical prediction, demonstrating the onset of dynamical localisation, is clearly evident. The inset shows the characteristic triangular shape of the localised momentum distribution when plotted on a logarithmic scale. Note that a good summary of these experiments on Sodium is contained in ref. [57].

In 1998, the group switched to using Cesium atoms and investigated the effects of noise on dynamical localisation [58, 59, 60, 61] and the transport properties of the kicked rotor [62]. In this latter work, the dependence of the average energy growth on the kick strength of the system is investigated. One sees clearly (see Fig 1.9) that the growth departs significantly from the quadratic dependence of the quasi-linear approximation introduced earlier - it in fact oscillates in accordance with the theoretical (albeit classical) predictions of Rechester and White [63, 64]. This subtle dependence of the classical diffusion on the kick strength has important consequences for the ratchet systems outlined in this thesis. It will be shown later that the ratchet effect can be directly attributed to Rechester and White style corrections to the diffusion coefficient. The effect of finite pulse widths on the dynamical localisation of cesium atoms is explored in [65].

The ability to demonstrate the kicked rotor experimentally has led to many related research efforts. Examples include the investigation of diffusion in the system [66, 67, 68, 69], the work of Schmuel Fishman and others on quantum resonances present in the kicked rotor [70, 71, 72, 73, 74, 75] and
the related studies on accelerator modes carried out by Summy, d’Arcy and coworkers at Oxford [76, 77, 78, 79, 80, 81, 82]. Whilst these studies are of great interest, the absence of resonances or accelerator modes in the ratchet systems described in later chapters puts these topics beyond the scope of this thesis.

### 1.6 Fully Chaotic Hamiltonian Ratchets

Having now given a brief overview of the history of ratchets, an introduction to chaos theory and a summary of recent experimental developments, one can clearly see how the overlap of these three topics leads naturally to the present study of chaotic Hamiltonian ratchets.

The main advantage of using laser cooled atoms in this work is the relative ease with which one can create and manipulate the potential which the atoms experience. The leap from the kicked rotor system to a ratchet model
1.6. Fully Chaotic Hamiltonian Ratchets

is straightforward: one can easily create an asymmetric ratchet potential by simple choice of the experimental conditions, and by breaking the periodicity of the (approximately) \( \delta \)-pulses one introduces the necessary temporal asymmetry. Furthermore, one has complete control over system parameters such as the kick strength, allowing the ratchet dynamics to be explored in both the mixed phase space and globally chaotic regimes.

It must be stressed that in [51], Schanz et. al. state that "...a necessary condition for directed chaotic transport in Hamiltonian ratchets is a mixed phase space". Reimann adds further weight to this assertion on p. 150 of [5], saying "... systems with strong (hyperbolic) chaos do not admit a ratchet effect". The work detailed in this thesis, and in associated publications, shows these views to be too restrictive. The conclusions of Schanz et. al. and Reimann do not apply to systems such as those considered here, where one has an unbounded phase space. The results show that for both a pulsed double-well (chapters 2-5) [83, 84] and a pulsed rocking ratchet (chapters 6 & 7) [85] a strong ratchet current is observed, even in the region of global chaos. Furthermore, the presence of directed transport is found to be a direct result of generic, chaotic diffusion processes in the system, and therefore does not rely on any specific features of the classical phase space.
Double-Well Ratchet: Theoretical Analysis

2.1 Introduction

The first of the two systems investigated during the course of this research bears many similarities to the δ-kicked rotor, introduced in the previous chapter as the paradigm of quantum chaos. In keeping with other ratchet models the spatial symmetry of the system is broken, in this case by the use of a double-well potential:

\[ V(a, x, \phi) = \sin x - a \sin(2x + \phi) \]

This spatially periodic system is subjected to a series of pulses, or kicks, similar to that used in the kicked rotor. However, in order to obtain directed motion in the system it is also necessary to break the temporal symmetry, which is done by perturbing the sequence of kicks from a uniform period.
2.2. Classical Analysis

In this chapter the theory used to investigate the dynamics of spatially periodic, pulsed systems is covered for both the classical and quantum case. In both cases, key concepts are introduced for the kicked rotor before being applied to the double well ratchet. Finally, the phenomenon of dynamical localisation is discussed in more detail using an analogy with Anderson localisation in solids, first proposed by Fishman et. al. [86, 87, 88].

2.2 Classical Analysis

2.2.1 The Delta Kicked Rotor

As mentioned previously, there is a large body of both theoretical and experimental research on the kicked rotor, which has led to a greater understanding of the phenomenon of dynamical localisation and the onset of chaos in dynamical systems. This system is noise and dissipation free, and for this reason forms the basis of this investigation into chaotic Hamiltonian ratchets.

The kicked rotor, in the usual realisation, consists of two particles connected by a bar that is allowed to turn in the plane of the particles, about it's midpoint, $M$. A useful physical example is to consider a dipole, with dipole moment $\vec{\mu}$, in an electric field (as shown in Fig. 2.1), where the effects of gravity and friction are neglected. The rotor is subjected to a series of $\delta$-pulses, or kicks, with amplitude $\varepsilon$ and time between pulses, $T$ such that the electric field is given by:

$$\vec{E}(t) = \varepsilon \delta(t/T - n)$$

For the model of a dipole system, the potential is calculated to be $V = -\vec{\mu} \cdot \vec{E}$.
2.2. Classical Analysis

such that the Hamiltonian in terms of the angular momentum, $L$, is then:

$$H(L, \theta, t) = \frac{L^2}{2I} + E_0 \cos(\theta) \sum_{n=-\infty}^{+\infty} \delta(t/T - n)$$  \hspace{1cm} (2.2)

where $\theta$ is as shown in the diagram, $I$ is the moment of inertia and $E_0 = \mu \varepsilon$.

Hamilton’s equations of motion for the system are therefore:

$$-\frac{\partial H(L, \theta, t)}{\partial \theta} = E_0 \sin \theta \sum_n \delta(t/T - n)$$

$$\frac{\partial H(L, \theta, t)}{\partial L} = \frac{L}{I}$$  \hspace{1cm} (2.3)

One notes that the evolution of the system with time can be divided into two distinct periods of motion - a ‘free-evolution’ part and a ‘kick’ part. Motion in the ‘free-evolution’ period is straightforward: angular momentum is conserved and the rotation angle increases in accordance with Hamilton’s equation given above. So, if the angular momentum before kick $n$ is defined to be $L_n$ and after kick $n$ to be $L'_n$ then in the following period of motion
2.2. Classical Analysis

to kick \( n + 1 \) one has: \( L_{n+1} = L_n' = L_n \). In the case of the rotation:
\[
\theta_{n+1} = \theta_n - TL_n'/I = \theta_j + TL_{n+1}/I,
\]
in accordance with Eq. (2.3).

For the kick part of the evolution, the angle is continuous \( (\theta_n = \theta_n') \), but the angular momentum changes discontinuously. If one integrates Hamilton's equation for the angular momentum over a very short time, \( \epsilon \), about kick \( n \) one finds the following:
\[
L_n' - L_n = \int_{nT-\epsilon}^{nT+\epsilon} E_0 \sin \theta \sum_n \delta(t/T - n) = K \sin \theta_n T \quad (2.4)
\]
By making the substitutions: \( K = E_0 T^2, L_n = I \theta_n/T \) the dimensionless standard map as obtained by Chirikov [37] is recovered:
\[
\begin{align*}
\theta_{n+1} & = \theta_n + K \sin \theta_n, \\
\theta_{n+1} & = \theta_n + \theta_{n+1} \\
\end{align*}
\] (2.5)

where \( K \) is known as the kick strength.

In Fig. 2.2 one sees how the phase space varies as the kick strength, \( K \), is increased. Key features to note at low kick strength are the presence of tori, which appear either as approximately horizontal lines in phase space or as elliptical curves, or islands. These tori (known as KAM-tori, after Kolmogorov, Arnold and Moser who performed ground-breaking work into non-integrable Hamiltonian systems such as the kicked rotor) are formed by regular trajectories. They can be rational tori (formed from families of periodic orbits) or irrational tori which represent regular but quasi-periodic motion. A trajectory with initial conditions on an invariant torus will trace out the closed curve corresponding to that torus and will remain confined to that curve.
At $K = 0$ the system is integrable, the tori are flat and each corresponds to a single value of momentum. It can clearly be seen from Fig. 2.2 that as $K$ is increased the tori deform, then break up. The KAM theorem shows that the rational tori break up first, into chains of islands. As $K$ increases further, an increasing fraction of the phase space becomes chaotic. The last torus breaks at $K_{\text{crit}} = 0.9716$. This last torus is the so called ‘golden-ratio’ torus; its winding angle corresponds to the golden ratio $R \simeq 1.618$ and in this sense may be considered the ‘most irrational’ of the tori. The last two tori can be seen in the second panel of Fig. 2.2 at momenta: $p \approx 2\pi \cdot (0.618)$ and $p \approx 2\pi \cdot (1 - 0.618)$

Of most significance to the work here is that tori (classically) represent

---

Figure 2.2: Poincaré surface of sections with increasing kick strength for the Standard Map.
an impenetrable barrier: a classical trajectory (even a chaotic one) bounded by a torus may not cross it. Hence while the golden ratio torus exists, an ensemble of classical particles initially prepared with small momenta $p \simeq 0$ cannot absorb energy indefinitely since the momenta remain restricted to values $p < \simeq 2\pi R$.

Once the final torus has been broken, the energy of the system can grow diffusively without limit. For smaller values of $K$, i.e. $K_{\text{crit}} < K < 2$ analytical diffusion rates generally give poor results since the diffusion is hindered by the presence of broken phase-space barriers (cantori) and small stable islands. For $K > 8$ essentially no stable islands are visible and the system may be considered globally chaotic. We note that the value $K = 5$ is frequently used to illustrate fully chaotic behaviour since in this regime there are only a few very small regular islands, which represent a negligible fraction of phase-space.

The diffusion rate is often approximated (to lowest order) by a quasi-linear diffusion rate, dependent solely on the kick strength:

$$D_{\text{ql}} = \frac{K^2}{4}$$  \hspace{1cm} (2.6)

where the energy is given by: $E_n = D_{\text{ql}} n$, for kick number $n$.

It should also be noted that for systems with higher dimensionality (where chaotic orbits are no longer enclosed by tori) a single connected chaotic region can be formed. Trajectories can then come arbitrarily close to any point in phase space, a phenomenon known as Arnold Diffusion - for a fuller explanation the reader is referred to [89].

In chapter 4 it will be shown that the quasi-linear form of diffusion is not necessarily a good approximation to the actual diffusion coefficient. In fact there are a significant number of correction terms, first calculated by
2.2. Classical Analysis

Rechester and White [63, 64], which die away as the kick strength becomes large. These correction terms are essential in describing the ratchet effect that is demonstrated in this thesis. However, it is often sufficient and convenient to consider only the quasi-linear diffusion when discussing general trends in the numerical results.

The Atom-Optics Kicked Rotor

The kicked rotor system is generic and amenable to simulation in many different experimental situations. In the previous chapter, cold-atom techniques were highlighted as providing an ideal test bed for periodic pulsed systems such as this. In chapter 7 it will be shown how one converts the dimensionless system of the following theoretical analysis to real quantities used in an experiment. For now, it is simply noted that a cold atom version of the kicked rotor uses linear coordinates, resulting in the substitution of $\theta \rightarrow x$ and linear momentum for angular momentum. As such the potential for the kicked rotor would be a simple cosine, $V(x) = K \cos(x)$, with the new dimensionless Hamiltonian becoming:

$$H(x, p, t) = \frac{p^2}{2} + K \cos(x) \sum \delta(t - nT)$$

2.2.2 The Double Well System

Symmetries of the System

One common factor for most ratchet systems is the necessity of breaking the spatial symmetry of the system, often by the use of a 'saw-tooth' ratcheting surface, or potential. In the case detailed here, an approximation to a saw-tooth is created by the addition of a second harmonic to the potential of the
kicked rotor:

\[ V(a, x, \phi) = \sin x + a \sin(2x + \phi). \]  

By varying the ratio, \( a \) of the two terms and the relative phase between them, \( \phi \), one has complete control of the potential surface.

It is appropriate at this point to define an effective kicking strength for the double well system. It will be shown in chapter 4 that the quasi-linear rate diffusion for the double well ratchet is given by:

\[ D_{ql}^{(DW)} = \frac{K^2(1 + 4a^2)}{4} \]

Therefore a new effective kick strength is defined to be:

\[ K_{eff} = K\sqrt{1 + 4a^2}, \]  

(2.9)

to recover the form obtained for the kicked rotor: \( D_{ql}^{(DW)} = K_{eff}^2/4 \).

In order to break the time symmetry in the system, a small perturbation, \( b \), is introduced into the sequence of equally spaced kicks found in the kicked rotor. Thus for a cycle of \( 2j + 1 \) kicks, where \( j > 1 \):

\[ 1 + jb, 1 + (j - 1)b, \ldots, 1 - (j - 1)b, 1 - jb. \]  

(2.10)

Without loss of generality, one can take the average time between kicks in one cycle to be unity: \( \langle T_i \rangle \). The time evolution of the system can then be treated in the same manner as for the Kicked Rotor.

The number of kicks in a cycle, \( N \), is given by \( N = 2j + 1 \) for \( N \) odd, and \( N = 2j \) for \( N \) even. The smallest kick-cycle for which a ratchet effect is produced is for \( N = 3 \), i.e.: \( T_1 = 1 + b, T_2 = 1, T_3 = 1 - b \) (see chapter 4).

The general form for the time dependence can now be written as:

\[ f(t) = \sum_{s=0}^{\infty} \sum_{M=1}^{N} \delta \left( t - \left( sT_{tot} + \sum_{i=1}^{M} T_i \right) \right), \]  

(2.11)

where \( s \) is the cycle number and the individual kick number is now \( n = s + M \).
2.2. Classical Analysis

The Ratchet Map and Phase Space Properties

As shown earlier for the $\delta$-kicked rotor, the dimensionless Hamiltonian for a kicked, spatially periodic system is:

$$H(p, x, t) = \frac{p^2}{2} + KV(x)f(t), \quad (2.12)$$

where the time dependence is given by $f(t) = \sum \delta(t - nT)$ for the kicked rotor and by Eq. (2.11) for the double-well.

If one considers Hamilton's equations for this general case:

$$\frac{\partial H}{\partial p} = \dot{x} = p \quad (2.13)$$
$$\frac{\partial H}{\partial x} = -\dot{p} = KV'(x)f(t) \quad (2.14)$$

and integrates with respect to $t$ as before, one obtains the discrete mapping:

$$p_{n+1} = p_n - KV'(x_n)$$
$$x_{n+1} = x_n + p_{n+1}T_i \quad (2.15)$$

where $V'(x) = \cos x + 2a\cos 2x$ and $T_i$ (for $i = 1 \ldots N$) is the appropriate time interval between kicks.

Whilst the effect of varying the number of kicks in a cycle, $N$, has been studied, the major part of the analysis for this system was carried out for a system where $N = 3$ (it will be shown later that a 3-kick cycle produces the
2.2. Classical Analysis

The map for this system is then:

\[
\begin{align*}
    p_{n+1} &= x_n - K \cos x_n - 2aK \cos x_n \\
    x_{n+1} &= x_n + p_{n+1}(1 + b) \\
    p_{n+2} &= x_{n+1} - K \cos x_{n+1} - 2aK \cos x_{n+1} \\
    x_{n+2} &= x_{n+1} + p_{n+2} \\
    p_{n+3} &= x_{n+2} - K \cos x_{n+2} - 2aK \cos x_{n+2} \\
    x_{n+3} &= x_{n+2} + p_{n+3}(1 - b)
\end{align*}
\]

(2.16) \hspace{1cm} (2.17) \hspace{1cm} (2.18) \hspace{1cm} (2.19) \hspace{1cm} (2.20) \hspace{1cm} (2.21)

By plotting the Poincaré surfaces of section (SOS) for the above 3-kick map one can predict a great deal about the classical evolution of the system. In Fig. 2.3 the transition to chaos as a result of increasing the kicking strength is shown. One can clearly see that as the kick strength, \( K \), is increased the invariant tori and islands begin to break up and disappear until at \( K = 2 \) the system is globally chaotic. A further important feature to note is that for this system, where both spatial and temporal asymmetry have been introduced, the phase space is also not symmetric. In the second and third panels it is clear to see that not every island has a matching partner at negative momentum, i.e. not every trajectory has a counter-propagating partner. This asymmetry in the classical phase space underlies the observed asymmetric growth of the momentum distributions which will be shown later.

Whilst it is essentially the kicking strength that governs the chaotic nature of the system, it is clear that variation of the other system parameters will also have an appreciable effect on the classical phase space. By varying the ratio of the two terms in the potential, \( a \) (whilst keeping the effective kicking strength, \( K_{\text{eff}} \) constant), one is effectively varying the symmetry of the system. So, as \( a \to 0 \) the potential becomes increasingly like the standard
Figure 2.3: Poincaré surface of sections with increasing kick strength for the double-well ratchet. Kick strengths are from left to right, 0.01, 0.1, 0.5, 2. Other parameters are: $b = 0.1$, $a = 0.5$ and $\phi = 0$.

map and phase space becomes more regular and symmetric. One also finds that as the number of kicks in the cycle is increased, the system becomes more chaotic for a given $K$. As $N$ is increased, longer periods between kicks are introduced which allow trajectories to explore more of the phase space than usual.

The effect of varying the period-one perturbation, $b$ has some important consequences for the system. One notes from Fig. 2.4 that as $b$ is increased from zero (where the kicking sequence is that of the standard map) to $b = 0.7$ the phase space becomes increasingly chaotic, for the same $K$. Furthermore, there are fewer islands with symmetric partners as $b$, and therefore the tem-
2.2. Classical Analysis

For each panel, $K_{\text{eff}} = 0.5$, $a = 0.5$ and $\phi = 0$

poral asymmetry, increases. A final key result is that by changing the sign of the perturbation (c.f. Fig. 2.4 panels 1 and 4) one effectively reflects the position of the phase space features about the $p = 0$ axis.

Classical Simulation

In order to perform the classical simulation, a gaussian distribution (in both momentum and position) of a large number of particles ($\sim 10^6$) is used in what may be termed a classical wavepacket. Evolution of this wavepacket yields three quantities which contain all the necessary information concerning the ratchet behaviour. The system is analysed by evolving $M$ trajectories (particles) for $n$ kicks, and at each point the average energy and momentum
of the ensemble are calculated:

\[
\langle E_n \rangle = \frac{1}{2M} \sum_{m=1}^{M} p_{n,m}^2 
\]  
(2.22)

\[
\langle p \rangle = \frac{1}{M} \sum_{m=1}^{M} p_{n,m}
\]  
(2.23)

Note that in general, the results in the following chapters refer to average energy as \( \langle p^2 \rangle \).

Finally, to demonstrate the asymmetric nature of the system’s evolution, the momentum distribution is plotted. Typical results are shown at the end of this chapter in Fig. 2.5 and Fig. 2.6.

2.3 Quantum analysis

2.3.1 The Heisenberg Picture of Quantum Mechanics

The quantum evolution is governed by the time-dependent Schrödinger equation:

\[
\frac{\hbar}{i} \frac{\partial \psi(t)}{\partial t} = \hat{H} \psi(t)
\]  
(2.24)

which has solutions:

\[
\psi(t) = e^{-i\hat{H}(t-t_0)/\hbar} \psi(t_0).
\]  
(2.25)

The Schrödinger equation is a first order differential equation in time, meaning that the state vector \( \psi(t) \) is determined at all time, \( t \), if it is known at time \( t_0 \). An evolution operator can therefore be defined in the following manner:

\[
|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle
\]  
(2.26)
2.3. Quantum analysis

This operator is unitary and for the time-independent Hamiltonian it takes the form:

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)}$$

(2.27)

whilst for the time dependent case (as in the kicked rotor and the double-well system) the operator takes the form:

$$\hat{U}(t, t_0) = \hat{T} e^{-i\int_{t_0}^{t} \hat{H} dt}$$

(2.28)

where $\hat{T}$ is a time ordering operator which ensures that time is evolved consecutively.

2.3.2 Quantum Observables

For a dynamical variable $x$ which obeys the eigenvalue equation:

$$x|x'\rangle = x'|x'\rangle$$

(2.29)

where $|x'\rangle$ is the appropriate eigenvector, the expectation value of $x$ is given by:

$$\langle x \rangle = \langle \psi | x | \psi \rangle$$

(2.30)

where $\psi$ is the state vector, which can be written:

$$|\psi\rangle = \int |x'\rangle dx' \langle x'| \psi \rangle$$

(2.31)

Multiplying on the left by $\langle \psi | x$ one obtains:

$$\langle \psi | x | \psi \rangle = \int \langle \psi | x | x' \rangle dx' \langle x'| \psi \rangle$$

$$= \int \langle \psi | x' \rangle \langle x'| x | x' \rangle dx' \langle x'| \psi \rangle$$

$$= \int \langle x' | x | x' \rangle dx' \langle x'| \psi \rangle^2$$

$$= \int |A_{x'}|^2 |x'\rangle dx' \approx \sum_{x'} |A_{x'}|^2 |x'\rangle$$

(2.32)
2.3. Quantum analysis

where the $A_{x'}$ are the expansion coefficients for the state vector: $|\psi\rangle = \sum_{x'} A_{x'} |x'\rangle$ and the integral has been discretised over $x'$.

2.3.3 Momentum basis

The simplest way of evolving the system in accordance with Eq. (2.26) is to expand the wavefunction into the complete set angular momentum eigenstates:

$$|\psi_0\rangle = \sum_l \langle l | \psi_0 \rangle |l\rangle$$  \hspace{1cm} (2.33)

where the momentum eigenstates are given in the $\theta$-representation of the kicked rotor to be:

$$\langle \theta | l \rangle = \frac{1}{\sqrt{2\pi}} \exp(\imath l \theta)$$  \hspace{1cm} (2.34)

One can now interrogate the system by simply evolving the probability amplitudes according to:

$$A_{l}^{(n+1)} = \sum_l U_{ml} A_{l}^{(n)}$$  \hspace{1cm} (2.35)

where $A_{l}^{(n)} = \langle l | \psi_n \rangle$.

2.3.4 The Quantum Delta Kicked rotor

Evolution Operator and the Quantum Map

The Hamiltonian for the quantum $\delta$-kicked rotor can be obtained by replacing the classical angular momentum with the quantum angular momentum operator:

$$L \rightarrow \hat{L} = -\imath \hbar \frac{\partial}{\partial \theta}$$  \hspace{1cm} (2.36)
2.3. Quantum analysis

One therefore obtains:

\[ \dot{H} = \frac{\tilde{l}^2}{2I} + E_0 \cos \theta \sum_n \delta(\frac{t}{T} - n) \]  

(2.37)

As with the classical analysis, the quantum evolution can be separated into a 'kick' part and a 'free' part:

\[ \hat{U}(t, t_0) = \hat{U}_{\text{free}} \hat{U}_{\text{kick}} \]  

(2.38)

The kick evolution operator is calculated by integrating across the kick for an infinitesimally small time \( \delta t \) (thus the effect of the kinetic energy term can be ignored):

\[ \hat{U}_{\text{kick}} = e^{-i \frac{\delta t}{\hbar} \int_{t_0}^{t_0+\delta t} V(\theta) \delta(\frac{t}{T} - n) dt} \]

\[ = e^{-iE_0T \cos \theta / \hbar} \]  

(2.39)

During the free evolution part, the kinetic energy is constant and so the free part of the evolution operator is given by:

\[ \hat{U}_{\text{free}} = e^{-i\frac{\tilde{l}^2}{2I} \int_{t_0}^{t_0+T_\theta} \frac{1}{2T} dt} \]

\[ = e^{-i \tilde{l}^2 T_\theta / 2I \hbar} \]  

(2.40)

The system can be made dimensionless using the parameters: \( \tau = \hbar T / I \), \( k = E_0 T / \hbar \) and using a dimensionless angular momentum operator, \( \tilde{L} = \hbar \tilde{l} \).

The total evolution operator in dimensionless form is then:

\[ \hat{U} = e^{-i \frac{\tilde{l}^2}{4} \hbar} e^{-i k \cos \theta} \]  

(2.41)

It should be noted that these scaling parameters can also be used (in conjunction with a rescaled angular momentum, \( \tilde{l} = \tau l \)) to obtain a map analogous to classical standard map:

\[ \tilde{l}_{n+1} = \tilde{l}_n + K \sin \theta_n \]  

(2.42)

\[ \theta_{n+1} = \theta_n + \tilde{l}_{n+1} \]  

(2.43)
2.3. Quantum analysis

where now $K = kr$.

Evolution by the matrix method

As mentioned earlier, information about the system parameters at time $t$ is obtained by evolving the expansion coefficients according to:

$$A_i^{(n+1)} = \sum_l U_{ml} A_l^{(n)}$$  \hspace{1cm} (2.44)

where the matrix elements are given by:

$$\langle m | \hat{U} | l \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-im\theta} \hat{U} e^{il\theta} d\theta$$  \hspace{1cm} (2.45)

Using the evolution operator obtained earlier it is found that:

$$\langle m | \hat{U} | l \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-im\theta} e^{-i\frac{z^2}{2}} e^{-ik \cos \theta} e^{il\theta} d\theta$$  \hspace{1cm} (2.46)

Now by using the Bessel identity:

$$e^{\pm iK \cos \theta} = \sum_{m=-\infty}^{\infty} i^m J_m(K) e^{\pm im\theta}$$  \hspace{1cm} (2.47)

the matrix elements for the kicked rotor become:

$$U_{ml} = i^{m-l} e^{-i\frac{z^2}{2}} J_{|m-l|}(K)$$  \hspace{1cm} (2.48)

Observables for the Quantum Kicked Rotor

For the kicked rotor one obtains the expectation values of energy and momentum by the method detailed earlier to be:

$$\langle E \rangle = \langle \Psi_n | \frac{\hat{p}^2}{2} | \Psi_n \rangle = \frac{1}{2} \sum_l l^2 |A_l^n|^2$$  \hspace{1cm} (2.49)

$$\langle p \rangle = \langle \Psi_n | \hat{p} | \Psi_n \rangle = \sum_l l |A_l^n|^2$$  \hspace{1cm} (2.50)
2.3. **Quantum analysis**

### 2.3.5 Double well quantum analysis

**Bloch Functions**

One major feature of systems with a spatially periodic potential \( V(x) = V(x + L) \), where \( L \) is the period) such as the \( \delta \)-kicked rotor and the double-well ratchet, is that one can use Bloch’s theorem, which states that, for a periodic system one can find solutions of the form:

\[
\psi_q(x) = e^{iqx}\phi_q(x)
\]  

(2.51)

where the \( q \) are quasi-momenta and \( \phi \) is a function periodic in \( x \) which can be expanded in the usual manner:

\[
\phi_q(x) = \sum_l A_l^q e^{ilx}
\]  

(2.52)

The wavefunction can now be written:

\[
\psi(x) = \int_{-1/2}^{1/2} a_q e^{iqx}\phi_q(x) dq
\]  

(2.53)

where the integral is over the first Brillouin zone. Quasi-momenta outside this zone are written \( q' = q + Q \), where since \( e^{2iQx} = 1 \) for \( V(x) = V(x + 2\pi) \) the Bloch form holds. Evolution of this wavefunction with time is then given by:

\[
\psi_{n+1}(x) = \hat{U}\psi_n(x) = \int_{-1/2}^{1/2} a_q e^{iqx}\hat{U} \sum_l A_l^q |l\rangle dq
\]  

(2.54)

Note that each plane wave is acted on individually by the evolution operator and then integrated over the Brillouin zone to obtain the complete wavefunction for the system.

In practical terms, the integral is discretised to become a sum over a finite number of quasi-momenta:

\[
\psi(x) = \Delta q \sum_{l,q} A_{l,q} e^{i(l+q)x}
\]  

(2.55)
where $\Delta q = \frac{1}{n_q}$, with $n_q$ the number of quasi-momenta.

In the case of the double-well, the quasi-momenta are conserved and one only needs to evolve the $\phi_q(x)$ functions for each quasi-momenta. The Schrödinger equation is therefore now:

$$-i\hbar \frac{\partial \phi_q(x)}{\partial t} = \left[ \frac{(\hat{p} + \hbar q)^2}{2} + KV(x) \right] \phi_q(x)$$  \hspace{1cm} (2.56)

**Minimum Uncertainty wavepacket**

In order to be able to maximise the amount of information available from the system as it evolves, the initial wavepacket must have the smallest possible uncertainty in both position and momentum. Consider initially a normalised Gaussian wavepacket, describing a free particle in position space at time $t = 0$:

$$\psi(x, 0) = \left( \frac{1}{\beta^2 \pi} \right)^{\frac{1}{4}} e^{-\frac{x^2}{2\beta^2}}$$  \hspace{1cm} (2.57)

The expectation value of $x$ is therefore:

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x, 0)|^2 dx = 0$$  \hspace{1cm} (2.58)

and:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x, 0)|^2 dx$$  \hspace{1cm} (2.59)

$$= \left( \frac{1}{\beta^2 \pi} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\beta^2}} dx$$  \hspace{1cm} (2.60)

$$= \frac{\beta^2}{2}$$  \hspace{1cm} (2.61)

where the following standard integral has been used:

$$\int_{-\infty}^{\infty} x^2 e^{-b^2 x^2} dx = \frac{\sqrt{\pi}}{2b^\frac{3}{2}}$$  \hspace{1cm} (2.62)
2.3. Quantum analysis

Once again the average momentum, \( \langle p \rangle \) is zero since the integrand is odd in \( x \):

\[
\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} \, dx
\]

(2.63)

It is also found that:

\[
\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \left( -i\hbar \frac{\partial}{\partial x} \right)^2 \psi \, dx
\]

\[
= \hbar^2 \int_{-\infty}^{\infty} \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \psi}{\partial x} \, dx
\]

\[
= \hbar^2 \int_{-\infty}^{\infty} \left( \frac{\partial^2 \psi^*}{\partial x^2} - \frac{\partial^2 \psi}{\partial x^2} \right) \, dx
\]

(2.64)

Thus one finds that the starting conditions have minimum uncertainty:

\[
\langle p^2 \rangle \langle x^2 \rangle = \frac{\hbar^2}{4}
\]

(2.66)

If one considers a wavefunction centred at \( (x_0, p_0) \) then the Gaussian wavepacket is written as:

\[
\psi_{GW\, P}(x_0, 0) = \left( \frac{1}{\beta \sqrt{\pi}} \right) e^{-\frac{(x-x_0)^2}{2\beta^2}} e^{i \frac{p_0(x-x_0)}{\hbar}}
\]

(2.67)

As mentioned earlier, in order to obtain information about the system from the amplitude coefficients in the Bloch basis, the initial amplitude coefficients must be calculated. This is done by evaluating the overlap integral between the initial Gaussian wavepacket and the Bloch functions given by Eq. (2.55), \( A_{p,q} = \langle q + p | \psi_{GW\, P} \rangle \):

\[
A_{p,q} = \left( \frac{1}{2\pi \beta \sqrt{\pi}} \right) \frac{1}{2} \int_{-\infty}^{\infty} e^{i(p+q)x} e^{-\frac{(x-x_0)^2}{2\beta^2}} e^{i \frac{p_0(x-x_0)}{\hbar}}
\]

\[
= \sqrt{\frac{\beta}{\sqrt{\pi}}} e^{-i(p+q)x_0} e^{-\frac{\beta^2}{4}(p+q-p_0/\hbar)^2}
\]

(2.68)

where the substitution \( p = \hbar \lambda \) has been made.
2.3. Quantum analysis

Evolution Operator for the Double-Well Ratchet

The evolution operator for the double-well ratchet is once again comprised of a 'free' and 'kick' part. The free-evolution is straightforward:

\[
\hat{U}_{\text{free}} = e^{-i \int_{n}^{n+\tau} \frac{(x+\Delta t)^2}{2} dt} = e^{-i \frac{\hbar T (t+\phi)^2}{2}}
\]

(2.69)

where the substitution \( p = \hbar \) has again been used.

Since the time dependence of the system is, as for the kicked rotor, of the form \( \delta(t - t') \) the evolution operator for the kick part is given as:

\[
\hat{U}_{\text{kick}} = e^{-i \hbar V(x)} = e^{-i \frac{K}{\hbar} (\sin x + \alpha \sin(2x + \phi))}
\]

(2.70)

The total evolution operator is therefore:

\[
\hat{U}_{\text{tot}} = \hat{U}_{\text{kick}} \hat{U}_{\text{free}} = e^{-i \frac{K}{\hbar} (\sin x + \alpha \sin(2x + \phi))} e^{-i \frac{\hbar T (t+\phi)^2}{2}}
\]

(2.71)

Matrix Elements

The evolution of the system proceeds as before:

\[
A_{l}^{(n+1)} = \sum_{l} U_{ml} A_{l}^{(n)}
\]

(2.72)

where the matrix elements are given by:

\[
\langle m | \hat{U} | l \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-im\theta} \hat{U} e^{il\theta} d\theta
\]

(2.73)

Note that, as with the kicked rotor, quasi-momentum \( q \) is conserved.

The Heisenberg matrix elements for the double-well are then obtained using (2.71):

\[
\langle m | \hat{U} | l \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-imx} e^{-i \frac{\hbar T (t+\phi)^2}{2}} e^{-i \frac{K}{\hbar} (\sin x + \alpha \sin(2x + \phi))} e^{ilx} dx
\]

(2.74)
The evolution operator is now split into two parts again, such that:

\[ \langle m | \hat{U} | l \rangle = \sum_k \langle m | \hat{U}_{\text{kick}} | k \rangle \langle k | \hat{U}_{\text{free}} | l \rangle \]  

(2.75)

The matrix elements for the free evolution become:

\[ \langle k | \hat{U}_{\text{free}} | l \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\pi} e^{-i\frac{\pi(I+2)}{2}} e^{i\pi x} dx \]

\[ = \frac{1}{2\pi} e^{-i\frac{\pi(I+2)}{2}} \int_0^{2\pi} e^{i(l-k)x} dx \]

\[ = e^{-i\frac{\pi(I+2)}{2}} \delta(l-k) \]  

(2.76)

In order to evaluate the matrix elements for the evolution across the kick, the sine version of the Bessel identity Eq. (2.47) is used:

\[ e^{\pm iK \sin nx} = \sum_m J_m(K) e^{\pm imnx} \]  

(2.77)

So for the kick one finds:

\[ \langle m | \hat{U}_{\text{kick}} | k \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\pi} e^{-i\frac{\pi (\sin x + a \sin (2x + \phi))}{\hbar}} e^{i\pi x} dx \]

\[ = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r \left( \frac{K}{\hbar} \right) J_s \left( \frac{aK}{\hbar} \right) e^{is\phi} \int_0^{2\pi} e^{i(k-m)x} e^{i(r+2s)x} e^{i(k-m)x} dx \]  

(2.78)

Now this integral is zero unless:

\[ k - m + 2s + r = 0 \]

\[ \Rightarrow r = m - k - 2s \]

giving:

\[ \langle m | \hat{U}_{\text{kick}} | k \rangle = \sum_{s=-\infty}^{\infty} J_{m-1-2s} \left( \frac{K}{\hbar} \right) J_s \left( \frac{aK}{\hbar} \right) e^{is\phi} \]  

(2.79)
So, the complete form for the matrix elements is:

\[ \langle m|\hat{U}|l\rangle = e^{-i\frac{K^2(l+q)^2}{2}} \sum_s J_{m-l-2s}(\frac{K}{\hbar}) J_s(\frac{aK}{\hbar}) e^{is\phi} \]  \hfill (2.80)

One notes that in practice it is unnecessary to perform the Bessel summation over a large number of orders, since the numerical value of the Bessel product drops rapidly towards zero as the order increases.

**Quantum observables and the Momentum Distribution**

The expectation values of momentum and energy for the double-well, for which detailed results are shown in the next chapter are given by:

\[ \langle E \rangle = \langle \Psi_n|\frac{p^2}{2}\Psi_n \rangle = \frac{1}{2} \sum_i (l + q)^2 \hbar^2 |A_{lq}^n|^2 \]  \hfill (2.81)

\[ \langle p \rangle = \langle \Psi_n|\hat{p}\Psi_n \rangle = \sum_i (l + q)\hbar |A_{lq}^n|^2 \]  \hfill (2.82)

\[ \langle \xi \rangle = \langle \Psi_n|\hat{\xi}\Psi_n \rangle = \sum_i (l + q)\hbar |A_{lq}^n|^2 \]  \hfill (2.83)

The final quantity which is used, particularly to highlight the asymmetry in the system and the onset of dynamical localisation, is the absolute square of the momentum amplitudes which shows the shape of the momentum distribution at a given kick, \( n \):

\[ N(p) = |A_{lq}^n|^2 \]  \hfill (2.84)

with \( p = (l + q)\hbar \).

Typical results for the classical and quantum average energy and momentum are shown in Fig. 2.5, for \( K = 2, b = 0.1, \phi = 0 \) and \( \hbar = 0.5 \) in the quantum case. One clearly sees the onset of dynamical localisation in the quantum energy growth, as it breaks away from the classical linear diffusion, given by: \( \langle p^2 \rangle = D_{ql} t \) where \( D_{ql} = K_{\text{eff}}^2 / 2 \). The quantum break-time, \( t^* \),
2.3. Quantum analysis

Figure 2.5: Typical results for the double-well ratchet. The top panel shows the saturation of the quantum energy resulting from the onset of Dynamical Localisation. The lower panel shows the saturation of the quantum and classical momentum. The quantum break-time, $t^*$, and the classical ratchet time, $t_r$, are also shown.

is taken at the crossing of an asymptote to the final quantum energy and the classical linear growth, as shown in the top panel of Fig. 2.5. The most striking and surprising result is the saturation of the classical momentum to a constant ‘current’ after a well defined time, as shown in the lower panel. This characteristic timescale for the classical evolution is quite new and is hereafter known as the ratchet time, $t_r$. In general, the ratchet time is taken to be when the momentum has reached 95% of it’s saturation value.

The momentum distributions corresponding to these results are shown in Fig. 2.6. The two panels of the figure show the first moment of the momentum distribution: $|p|N(p)$. This form is the clearest way to observe asymmetry in
2.4. Dynamical Localisation

Figure 2.6: The first moment of the classical and quantum momentum distributions are shown to highlight the asymmetry present. The dynamical localisation form of $|p|N(p) = \frac{|p|}{2\pi} \exp(-|p|/L)$ is shown in red. Parameters are $K = 2$, $a = 0.5$, $b = 0.1$, $\phi = 0$ and $\hbar = 0.5$ for the quantum.

the distribution and is used extensively throughout this thesis. The average current is then the difference in area under the two halves of the curve. In the case of the kicked rotor, it was shown in the previous chapter that a dynamically localised quantum momentum distribution can be fitted with the form $N(p) \sim \exp(-p/L)$ where $L$ is known as the localisation length. In the right-hand graph of Fig. 2.6 one sees this form overlaid in red, to aid comparison between the double-well and the kicked rotor. Clearly, for the double-well the localisation profile is more complex, only tending to the quantum kicked rotor case in the wings of the distribution.

2.4 Dynamical Localisation

In the previous chapter the phenomenon of dynamical localisation was introduced, in the context of the kicked rotor, simply as the “quantum suppression of classical chaotic diffusion”. As will be seen in the following chapters, this effect of ‘freezing’ the growth of the quantum momentum distribution plays
2.4. Dynamical Localisation

a central role in both ratchet systems discussed in this thesis. It is therefore worth reviewing the present understanding of Dynamical Localisation. In the process, some of the key parameters of the quantum system will be introduced.

Whilst there remains no classical or semi-classical explanation for dynamical localisation, a formal analogy with Anderson Localisation in disordered solids has been made by Fishman et al. [86, 87, 88] that provides some insight into how the phenomenon occurs. In the Anderson localisation case, the electronic wavefunction is exponentially localised in position space. Fishman et al. showed that for the Dynamical Localisation case, the eigenfunctions (Floquet states) are exponentially localised in momentum space:

$$u(p) \sim e^{-\frac{\text{Im} \cdot p}{L}}$$  \hspace{1cm} (2.85)

where $L$ is the localisation length. If one expands the initial momentum wavefunction (typically a narrow gaussian peaked about $p = p_0$) in a basis of these eigenfunctions, then the expected value of $(p - p_0)^2$ cannot then become much larger than $L^2$. Thus the limit of the quantum energy growth occurs at: $p^2 \sim L^2$.

In order to determine the timescale on which this effect occurs, Fishman used the following argument: for a strongly peaked initial momentum wavefunction, the most strongly excited modes are localised around $p = p_0$ and within $\sim L$. There are therefore $L/\hbar$ excited eigenfunctions each with an associated eigenvalue given by $\exp(-i\omega_j)$, where the $\omega_j$ lie between $0, 2\pi$. The typical spacing between values of $\omega_j$ is then:

$$\delta \omega \sim \frac{2\pi}{L/\hbar}$$  \hspace{1cm} (2.86)

When $t \approx 2\pi/\delta \omega$ the discrete nature of the superposition of eigenfunctions is felt and the quantum energy growth departs from its previous linear evolution.
Thus, one calls this timescale the 'break-time' which hence occurs at:

\[ t^* \sim \frac{2\pi}{\delta \omega} \sim \frac{L}{\hbar} \]  (2.87)

If one then assumes that, until the break-time, the quantum energy growth mimics the classical diffusion one finds:

\[ \langle p^2 \rangle \sim D t^* \sim L^2 \]  (2.88)

where \( D \) is the classical diffusion rate, (which one can estimate to lowest order to be the quasi-linear form: \( D_{qt} \approx K^2/4 \)). Therefore the localisation length is:

\[ L = \alpha \frac{D}{\hbar} \]  (2.89)

and hence the break-time is:

\[ t^* = \alpha^2 \frac{D}{\hbar^2} \]  (2.90)

The proportionality constant, \( \alpha \), was found by Shepelyansky to be \( \alpha = 1/2 \) for the kicked rotor.

2.5 Conclusions

In this chapter, much of the theory necessary to study chaotic Hamiltonian ratchets has been introduced. The classical system is studied by evolving an iterative map to obtain the average energy and momentum, whilst in the quantum regime one examines the evolution of the expansion coefficients using the Heisenberg matrix method.

It has been shown that in order to achieve directed transport in the system, one must break spatio-temporal symmetry. The ratchet system introduced in this chapter employs a double well potential to introduce spatial
2.5. Conclusions

asymmetry, and a chirped kicking sequence to lower the temporal symmetry. The key result is the accumulation of asymmetry in the classical momentum distribution and its subsequent saturation to a finite value after the so-called ratchet time, $t_r$. The quantum system is seen to demonstrate the phenomenon of localisation, which acts to 'freeze-in' asymmetry in the quantum momentum distribution after the break-time, $t^*$. In the next chapter the effect on the average momentum and energy growth in the ratchet as the system parameters are varied is investigated in detail. The general trends that emerge are then used to suggest optimal conditions for any experimental study.
3.1 Introduction

This chapter contains an extensive survey of the behaviour of the double-well ratchet as each of the system parameters is varied. As a result of this detailed work, and under comparison with a similar system (such as the rocking ratchet introduced in chapter 5), one is able to draw accurate conclusions concerning general trends particular to chaotic Hamiltonian ratchets. However, the reader more interested in the underlying mathematical processes may prefer to skip to the next chapter where analytical forms for the diffusion rate and average current are derived.

For the system to be considered a true ratchet, it must demonstrate a persistent reproducible transporting current in the long time limit. In other words, the hallmark effect of a ratcheting system is the existence of a finite average classical momentum (the ratchet current) reached after a finite time
3.2. Momentum Distributions

The numerical results detailed below show that the system does indeed produce this classical ratchet effect.

The quantum system is shown to demonstrate the phenomenon of dynamical localisation, first observed in the kicked rotor. The saturation of the quantum energy that occurs after the break-time, $t^*$, naturally means that the quantum average momentum also saturates to a finite value. In this chapter the numerical results for both the quantum and classical cases are presented, as a function of the system parameters. One finds that by careful parameter choice the quantum and classical currents can be enhanced, destroyed or reversed.

The two main timescales for the system, the classical ratchet time ($t_r$) and the quantum break-time ($t^*$), are also investigated as a function of the system parameters. It is shown that for the best experimental ratchet signature, one must make a careful choice of both $t_r$ and $t^*$ in order to ‘freeze in’ the maximum classical current.

The chapter begins by investigating the growth of momentum asymmetry in the system by considering the classical and quantum momentum distributions. The results for the classical and quantum average energy and momentum as a function of the system parameters are then presented, before discussing the system timescales and the best parameters for any experiment.

3.2 Momentum Distributions

As mentioned earlier, the signature of a ratchet effect is the presence of a non-zero finite average classical momentum. Physically this means that the distribution of particles in momentum space is no longer symmetric: there is a greater tendency for particles to move in one direction rather than the
3.2. Momentum Distributions

other. It is therefore instructive to begin the analysis of the double-well ratchet by looking in depth at the momentum distribution, both classically and quantum mechanically, and how its growth is affected by the various system parameters.

3.2.1 Effect of Temporal Asymmetry; varying $b$

One commonality between different Hamiltonian ratchet models is the presence of a spatially asymmetric potential. However, it has been stated earlier that for a successful chaotic Hamiltonian ratchet, temporal asymmetry must also be introduced in the form of chirped kicks. The effect of introducing this chirping (by the addition of a small perturbation $b$ to the period-one kicks of the kicked rotor) is shown in Fig. 3.1. In each of the three cases the first moment of the momentum distribution, $|p|N(p)$, is plotted in order to show the asymmetry more clearly. The difference in area under each half of the first moment plot is then proportional to the net current.

The left hand panel of the plot shows the behaviour when $b = 0, 0.1, 0.03$ for the classical system. Note that for $b = 0$, although spatial asymmetry is present in the form of the double well potential, the momentum distribution remains symmetric about $p = 0$. The remaining two graphs, where $b = 0.1$ and $b = 0.03$ respectively, show a definite imbalance between the positive and negative halves of the first moment curves. One also notes that, perhaps counterintuitively, the larger asymmetry appears to occur for $b = 0.03$. It will be shown later that this is due to the ratchet time, the time at which asymmetry stops accumulating, being longer for smaller $b$, in fact: $t_r \propto 1/b^2$.

The quantum plots in the right hand panel tell a similar story. Clearly for $b = 0$, the first momentum is once again symmetric about $p = 0$, with
3.2. **Momentum Distributions**

Figure 3.1: Classical and Quantum Momentum Distributions for varying chirp parameter $b$. Note that in both cases, when $b = 0$ the momentum distributions are symmetric. In the classical case greater asymmetry is shown for $b = 0.03$ than for $b = 0.1$ suggesting an inverse relationship between the current and $b$. The quantum graphs show similar asymmetry for $b = 0.1, 0.03$. The classical curves are for $K = 1.6$, $\alpha = 0.5$, $\phi = 0$, whereas the quantum plots are for $K = 1.7$, $\alpha = 0.5$, $\hbar = 0.5$, $\phi = 0.5$

definite asymmetry evident when $b = 0.1, 0.03$. However, these quantum results highlight a difficulty that occurs if one simply examines the momentum distributions. Whilst the shapes of the two chirped distributions appear quite different, the actual average quantum current (i.e. the difference in area under each half of the curve) is about the same. It will be shown later that whilst chirping the kicks creates asymmetry in the quantum momentum distribution, there is no clear trend as is found with the classical case - in
3.2. Momentum Distributions

In other words, the ratchet time is $b$-dependent, whilst the break-time is not.

3.2.2 Effect of Spatial Asymmetry; varying $a, \phi$

Having demonstrated that it is necessary to break time symmetry to create asymmetry in the momentum distribution, it remains to verify that spatial asymmetry is also necessary. In Fig. 3.2 one can clearly see how the first moment of the momentum changes with the ratio, $a$, of the two terms in the potential whilst the effective kick strength is kept constant at $K_{\text{eff}} = 1.7$.

For $a = 0.2, 0.4$ the asymmetry is strong, due to the fact that the potential is highly asymmetric. However, as $a$ increases the $\sin 2\alpha$ term begins to dominate and the potential becomes increasingly symmetric - as shown by the final plot for $a = 1.5$.

As one varies the relative phase, $\phi$, between the two terms in the potential, the spatial symmetry is clearly also changed. The effect on the momentum asymmetry is shown in Fig. 3.3. The maximum asymmetry occurs for $\phi = 0, \pi$, when the potential is at its most asymmetric. In changing the phase from $\phi = 0$ to $\phi = \pi$ the potential is effectively mirror-imaged, i.e. the asymmetry is reversed. This means that the momentum asymmetry also becomes reversed as $\phi$ is changed. For $\phi = \pi/2$ the potential, and hence the momentum distribution, is symmetric - as shown in the middle panel of the figure.

3.2.3 Variation of Kick Strength, $K$

In the previous chapters, it has been noted that the average energy growth in both the kicked rotor and the double-well ratchet can be approximated to a quasi-linear form: $E_{\text{ql}} = K^2_{\text{eff}} t/4$. Whilst this form is only an approximation
3.2. Momentum Distributions

Figure 3.2: Quantum and Classical momentum distributions for varying ratio between terms in the potential, $a$. In both cases, the asymmetry is greatest for $a \approx 0.4$, and decreases as $a$ increases. This is due to the increasing dominance of the $\sin 2x$ term, resulting in an increasingly symmetric potential. System parameters are, classical:

$$K = 2, b = 0.1, \phi = 0$$

Quantum:

$$K = 1.7, b = 0.1, \hbar = 0.5$$

(a more accurate form is derived in the next chapter), it is clear that the kick strength is the single most important factor in determining diffusion in the ratchet. One might then expect that this would also be the case for the average momentum, resulting in increasingly asymmetric distributions as $K$ is increased. In fact the $K$-dependence of the momentum is a much more subtle effect, and depends heavily on the other system parameters.

The reader will note from the plots shown below that as $K$ is increased, the apparent momentum asymmetry is small. This is due to the asymmetry effectively being masked by the expansion of the wavepacket. One can still
3.2. Momentum Distributions

Figure 3.3: Quantum and Classical momentum distributions for varying relative phase, \( \phi \). One can clearly see in each case that the maximum asymmetry (and therefore current) occurs for \( \phi = 0, \pi \) and that there is a current reversal at \( \phi = \pi/2 \). Parameters:

\[ K = 1.6, \ a = 0.5, \ b = 0.1, \ h = 0.25 \] for the quantum plots.

make valid inferences from the plots, although the true dependence of the ratchet current on the system parameters is perhaps better illustrated by the plots of average momentum shown in the next section. However, any experiment performed using cold atoms will attempt to assess asymmetry in the system using momentum distributions such as these. Showing them here highlights the difficulty facing an experimentalist, and the need for careful choice of the system parameters.

In Fig. 3.4, classical momentum distributions are shown for two different values of \( b \), and quantum momentum distributions for two different values of \( h \), all as a function of \( K \). For the classical case, there is a large asymmetry
3.2. Momentum Distributions

at low $K$ in both cases. However at higher values of the kicking strength the distributions for $b = 0.1$ are virtually symmetric, whilst those for $b = 0.01$ show significant asymmetry (albeit in the opposite sense to that at low $K$). This is a direct result of the ratchet times being different in each case. It will be shown later that in the case of $b = 0.1$ the ratchet time becomes too short at high $K$ to allow any asymmetry to accumulate.

Figure 3.4: Classical and Quantum Momentum distributions for varying kick strength, $K$. The classical distributions are shown for two values of the chirp: $b = 0.1$ and $b = 0.01$. Similarly, the quantum plots are shown for $\hbar = 0.5$ and $\hbar = 0.25$. In both cases, one notes that asymmetry in the momentum distributions at high $K$ is masked by the spread of the wavefunction. Values of kick strength are, from top to bottom, 1.6, 2.6, 3.2, 5.6. In the quantum plots, $b = 0.1$.

The quantum distributions show a similar effect: the $\hbar = 0.25$ plots show much greater asymmetry at low $K$ than those for $\hbar = 0.5$, whereas at higher
3.3. Average Energy and Momentum in the Double Well

$K$ there is little difference between them. In this case the effect is due to the variation of the quantum break-time with $\hbar$. When $\hbar$ is small, the break-time is long and greater asymmetry is allowed to build up before the system localises. At high kick strength, the break-time is long enough in both cases for the maximum amount of asymmetry to accumulate, resulting in very similar momentum distributions.

This balance between the timescales and the average classical and quantum momentum is very subtle, and is much more evident when one looks at $\langle p \rangle$ as a function of $K$ in each case, as will be done in the section concerning timescales below.

3.3 Average Energy and Momentum in the Double Well

From the momentum distribution results discussed above, it is clear that changing the system parameters by even small amounts can have a dramatic effect on the classical and quantum currents. In this section, the trends outlined above are looked at in detail by investigating the variation of both the average momentum (current) and the energy of the system as a function of each of the system variables. At each stage numerical results for the classical simulation are compared with the corresponding quantum results.

3.3.1 Variation of $b$

In Fig. 3.5 one sees the dependence on the chirp parameter, $b$. The top panel shows the evolution of the average classical momentum with kick number, or time, for various values of $b$. As expected, as $b$ increases the magnitude of the
3.3. Average Energy and Momentum in the Double Well

Figure 3.5: Variation of energy and momentum with $b$. The top panel shows the increasing average current (and ratchet time) with decreasing $b$. The middle panel shows the apparently random dependence of the quantum energy growth with $b$. The bottom panel shows the average classical and quantum current as a function of the chirp parameter. Note the classical current reversal at $b = 0$ and the apparent $b$-independence of the quantum current. The system parameters were $K = 1.7$, $a = 0.5$, $\phi = 0$ and $h = 0.25$ for the quantum results.

saturation current drops. The middle panel shows how the average quantum energy varies with the magnitude of the chirp. The saturation energy, and hence the break-time, varies only slightly with $b$ and no particular trend emerges. This is due to complicated averaging over terms in the current of the form $\sin npb$ and $\cos npb$ (see Chapter 4). In the third panel of the plot, the finite saturation value for the average momentum has been plotted against $b$ for both the classical and quantum case. The fit clearly shows the
3.3. Average Energy and Momentum in the Double Well

inverse $b$ dependence of the classical system, and one also notes that a current reversal occurs at $b = 0$. A further point to note, shown clearly in the top panel, is that as $b$ decreases, the saturation (ratchet) time increases. This will be discussed further later in this chapter.

3.3.2 Variation of $a$, $\phi$

It has already been noted that by changing the spatial symmetry of the double-well potential one can destroy or reverse the ratchet current. This is further demonstrated by Figs. 3.6 and 3.7.

![Graph showing variation of $a$, $\phi$](image)

Figure 3.6: Average Quantum and Classical Energy as a function of $a$. One notes that in general, the system exhibits maximum asymmetry when the potential is at its most asymmetric, i.e. at approximately $a = 0.4$. System parameters are; $K_{eff} = 1.7$, $a = 0.5$, $b = 0.05$ for the classical case and $\hbar = 0.5$ for the quantum.
3.3. Average Energy and Momentum in the Double Well

In the former one notes again the subtle dependence of the current on the ratio, \( a \), of the two terms in the potential (again, \( K_{\text{eff}} = 1.7 \) in each case). When \( a \) is small, the potential tends to the symmetric kicked rotor case, producing a relatively small current. For \( a \approx 0.4 \), the potential is at it’s most asymmetric, resulting in the maximum classical current. As \( a \) continues to increase, one sees as before that the current gradually decreases as the potential tends to a pure \( \sin 2x \) form. The effect is similar, although not as pronounced, for the quantum case due to the quantum system localising before significant asymmetry has accumulated.

![Figure 3.7: Average classical and quantum momentum as a function of phase \( \phi \). One can clearly see the expected \( \phi \)-dependence, with maximum current generated for \( \phi = 0, \pi \) and a current reversal for \( \phi = \pi/2 \). The ability to generate a current reversal in the system by simply varying the relative phase between the terms in the potential is shown clearly](image-url)
3.3. Average Energy and Momentum in the Double Well

by Fig. 3.7. As expected, the strongest current (for both the classical and quantum cases) occurs for \( \phi = 0, \pi \), with the current reversal occurring at \( \phi = \pi/2 \) when the potential is symmetric.

### 3.3.3 Variation of Kick Strength

It has been shown that the quantitative behaviour of the system when \( a, \phi \) are varied is independent of the value of other variables. However, this is not the case for the kick strength: the amount of asymmetry change in the momentum distribution as \( K \) is varied depends on the value of the chirp parameter, \( b \). The figures 3.8 and 3.9 further highlight this effect.

The top two panels of Fig. 3.8 show the average classical momentum as a function of time for various values of \( K \) and with chirp parameter \( b = 0.1 \) for the top panel and \( b = 0.01 \) for the middle panel. There are two main points to note from the curves. Firstly, both show significant negative currents for values of \( K < 2.4 \). However, as \( K \) is then increased to \( K = 2.6 \) and beyond, the results begin to differ quite considerably. In the case where \( b = 0.1 \), the average current drops to zero and remains there as \( K \) is increased. For \( b = 0.01 \) the current again drops to zero at about \( K = 2.6 \), but in contrast to the \( b = 0.1 \) results, an appreciable positive current persists for \( K = 4 \) and \( K = 5 \).

The second point to note is that the ratchet time (the time taken for saturation of the classical average momentum), is also clearly \( K \)-dependent - in addition to the \( b \)-dependence already highlighted earlier. The nature of this \( K, b \) dependence, and it's effect on the average current will be covered in more detail in the section on timescales below.

The lower panel of Fig. 3.8 shows the effect of changing \( K \) and \( b \) on the
Figure 3.8: Classical average momentum as a function of $K$. The top two panels of the figure show the variation of average momentum with time for different kick strengths. In the top panel, where $b = 0.1$, one notes that the current saturates relatively quickly; whereas in the middle panel, where $b = 0.01$ the ratchet time is significantly increased. The bottom panel shows the average classical momentum as a function of $K$ for three different values of $b$. Note the current reversal at $K \approx 2.6$ and the persistent positive current for $b = 0.03, 0.01$ which is absent when $b = 0.1$.

This highlights the dependence of the current on the ratchet time.

classical saturation current. One can clearly see the presence of the current reversal at $K \approx 2.6$ for $b = 0.01, 0.03$, which is absent when $b = 0.1$. The increase in the magnitude of the ratchet current as a function of $K$ is again clearly $b$-dependent.

In Fig. 3.9 one sees the dependence of the quantum system on varying $K$. The top panel shows a quantum-classical comparison of average momentum
3.3. Average Energy and Momentum in the Double Well

Figure 3.9: Average quantum momentum and energy as a function of kick strength, $K$. In the top panel one sees the dependence of the average current on $\hbar$. As $\hbar$ is decreased, the quantum current increasingly resembles the classical for $b = 0.1$. Note the presence of a peak in the quantum momentum at $K > 2.6$ which is absent for the classical case, due to the fact that the ratchet time is too small to allow significant accumulation of asymmetry.

As a function of $K$ for $b = 0.1$, and two values of $\hbar$. Immediately one sees that in contrast to the classical results, both quantum curves show similar large positive currents in the region $K = 2.6 \rightarrow 4$. For $K < 2.6$, the quantum current is clearly dependent on the value of $\hbar$ - as $\hbar$ is decreased, the magnitude and position of the negative peak shows greater agreement with the classical results. Once again, these features of the quantum current are a direct result of the system timescales, and will be discussed in more detail below.
3.3. Average Energy and Momentum in the Double Well

3.3.4 Variation with $\hbar$

At this stage it is natural to consider the effect that changing $\hbar$ has on the quantum results. In the previous chapter it was shown that for the kicked rotor there is a well studied relationship connecting the localisation length, $L$, and classical diffusion: $L \approx \frac{D}{\hbar}$. In general, this becomes $L \approx \frac{\alpha D}{\hbar}$. Taking the localisation length to be approximated by: $L^2 = D t^*$, one then finds that:

$$t^* \sim \frac{\alpha^2 D}{\hbar^2}$$  \hspace{1cm} (3.1)

The value of $\alpha$ for the double-well case will be addressed later.

In the top panel of Fig. 3.10, one sees clear evidence for this relationship: as $\hbar$ is decreased the break-time lengthens and the energy saturates to a higher value. The lower panel shows that (for this particular parameter set) the average current increases as $\hbar$ decreases, until at $\hbar \approx 0.25$ the quantum current is actually greater than for the classical case. Note that as $\hbar$ is further decreased, the quantum current saturates to a common value. It will be shown later that the optimum parameters for any experimental ratchet will have the quantum current comparable to the classical.

3.3.5 Variation with $N$

One parameter that has so far not been addressed is $N$, the number of kicks in a sequence. In the previous chapter it was noted that in order to achieve transport in a ratchet system, a sequence of at least 3 kicks must be used. Therefore, all the results quoted thus far have been for the 3-kick sequence: $T_i = 1 + b; T_{i+1} = 1; T_{i+2} = 1 - b$. In Fig. 3.11, the effect of varying $N$ on both the classical and quantum systems is demonstrated.

One notes from the top panel that there is only a small effect on the
3.3. Average Energy and Momentum in the Double Well

![Graph showing average quantum and classical momentum as a function of $\hbar$. The top panel shows how the break-time increases as $\hbar$ is decreased. The lower panel shows that the quantum current increases in magnitude as $\hbar$ is decreased until by $\hbar = 0.25$ it has passed the equivalent classical current. The curve for $\hbar = 0.125$ shows that there is no advantage in decreasing $\hbar$ further. System parameters are $K = 1.8$, $a = 0.5$, $b = 0.1$.](image)

Figure 3.10: Average quantum and classical momentum as a function of $\hbar$. The top panel shows how the break-time increases as $\hbar$ is decreased. The lower panel shows that the quantum current increases in magnitude as $\hbar$ is decreased until by $\hbar = 0.25$ it has passed the equivalent classical current. The curve for $\hbar = 0.125$ shows that there is no advantage in decreasing $\hbar$ further. System parameters are $K = 1.8$, $a = 0.5$, $b = 0.1$.

classical saturation current, and also that the maximum current is obtained for $N = 3$. The bottom panel shows that this minimal change in the classical current is mirrored by the quantum results.

The most striking effect of changing the kick sequence concerns the quantum energy growth. It can clearly be seen from the middle panel of Fig. 3.11 that the time taken for the system to localise, the quantum break-time $t^*$, dramatically increases with $N$. This a direct result of the fact that changing $N$ effectively changes the time periodicity of the system, since the kick sequence
3.3. Average Energy and Momentum in the Double Well

Figure 3.11: Average Energy and Momentum as a function of number of kicks in a cycle, $N$. The top panel shows that there is little change in the average classical current as $N$ is increased, and that the maximum current is achieved for $N = 3$. This is further illustrated for both the quantum and classical case in the bottom panel. The middle panel shows that the break-time significantly increases with $N$. For the classical case $K = 1.8$, $a = 0.5$, $b = 0.03$ and the quantum: $K = 2$, $a = 0.5$, $b = 0.03$.

is: $T = 2 + (N/2 - 1)b, \ldots, 1, \ldots, 1 - (N/2 - 1)b$. Recalling that the evolution operator for the free evolution is of the form: $U_{free} = \exp(-i(l + q)^2 \hbar T)$ one notes that increasing the period, $T$, has the same effect as halving the value of $\hbar$. 
3.4 Timescales for the Ratchet

It has already been noted that many of the system parameters have a bearing on either the ratchet time or the break-time. Empirically, it is found that $t_r$ and $t^*$ obey the following relationships:

$$t_r \approx \frac{2\pi}{(Kb)^2}$$  \hspace{1cm} (3.2)

$$t^* \approx \frac{\alpha^2 D}{\hbar^2}$$  \hspace{1cm} (3.3)

where $\alpha$ is the proportionality constant of Eq. (2.89).

Evidence for these forms is shown in Fig. 3.12 for the ratchet time and Fig. 3.13 for the break-time. The break-time has been the subject of con-
3.4. Timescales for the Ratchet

siderable study (see the section on Dynamical Localisation in section 2.4), however the ratchet time is quite new. An analytical form for \( t_r \) is derived in the next chapter that gives excellent agreement with Eq. (3.3). One point to note concerning the ratchet time is its asymptotic behaviour as \( b \to 0 \). It has already been shown that for \( b = 0 \), no current is observed in the system - a counter-intuitive result considering the empirical observation that current increases as \( b \) decreases. This can be understood by considering a plot of average current against time (as in Fig. 3.5). One can approximate the acceleration in the system by the gradient of the curve at the ratchet time to be \( dp/dt \approx \langle p \rangle / t_r \). Since the current grows as \(-1/b\) and the ratchet time is \( \propto 1/b^2 \), the acceleration is effectively proportional to \(-b\). Thus when \( b = 0 \) no current is observed as the acceleration in the system is zero.

Note that the ratchet time is defined here as the time at which the average momentum is 95% of its maximum value. Defining the break-time is somewhat more subjective. For the order of magnitude calculation contained here, it is taken as the crossing point of the classical quasi-linear diffusion rate with an asymptote to the maximum localised energy. For the two values of \( h \) shown in Fig 3.13 the break-time, with this definition is approximately:

\[
t^* \approx \frac{20D_{ql}}{\hbar^2}
\]  

(3.4)

Here \( D_{ql} = K^2 \) (for \( a = 1/2 \)) for the double well case where one is considering average energy growth as \( E = \langle p^2 \rangle \) which has been done throughout this work. This implies that \( \alpha \approx 4.5 \). If one considers the energy growth to be \( E = \langle p^2 \rangle / 2 \) (as in Shepelyansky’s work concerning localisation length) then \( \alpha \approx 6.3 \). These results contrast with the standard map where:

\[
t^* = \frac{D_{ql}}{2\hbar^2}
\]  

(3.5)
3.5 Dynamical Localisation in the Double Well Ratchet

Given the presence of perturbed kicks and spatial asymmetry in the ratchet system it is perhaps surprising that dynamical localisation actually occurs. It will be shown later that this localisation is not simply an interesting by-product of this work, but is essential in creating an experimentally detectable ratchet.

The discussion in section 2.4 concerning this unusual quantum interference effect introduced two key quantities which can be used to characterise

Figure 3.13: The figure shows the break-time as a function of $K$ for $\hbar = 0.25$ (top panel) and $\hbar = 0.5$ (bottom panel). Both curves show a good fit of the order of:

$$t^* \sim 20K^2/\hbar^2.$$
3.5. Dynamical Localisation in the Double Well Ratchet

Figure 3.14: Momentum distributions showing variation of localisation length, $L$, with system parameters. The first panel shows the increase in $L$ as $\hbar$ is decreased from 0.5 to 0.125. In the second panel, the localisation length increases as the kick strength $K$ is increased from 1.8 to 2.8. The final plot shows the dependence of the localisation length on the number of kicks in a cycle. The chirped, 3-kick cycle shows a larger localisation length than the time symmetric case.

the effect - the break-time and the localisation length. The former has already been discussed, and the two have been shown to be related by: $L^2 \approx D_{ql} t^*$. In Fig. 3.14 one sees how the localisation length is affected by varying the system parameters. Recalling that $\ln N(p) \propto p/L$ it is clear to see that $L$ increases with decreasing $\hbar$ and increasing kick strength, $K$. These findings are in keeping with Eq. (2.89) which asserts that:

$$L = \alpha \frac{D_{ql}}{\hbar}$$

Fig. 3.15 investigates this relationship for three different values of $\hbar$. A full study of the effects of each system parameter on the localisation length can be found in [90] where it is found that the constant of proportionality is: $\alpha \approx 6.4$. Note that this work was carried out assuming that the energy
Figure 3.15: Variation of localisation length with kick strength for various values of $h$. The figure clearly illustrates that the results show an excellent fit to the form

$$L = \alpha D_{ql}/\hbar,$$

where $\alpha \approx 6.4$

growth is given by: $E = \langle p^2 \rangle / 2$. This result shows excellent agreement with the value of $\alpha$ found earlier by considering the break-time. One key result from [90] is the dependence of $L$ on the number of kicks in a cycle resulting in a new form for the Shepelyansky relationship:

$$L \sim \frac{\alpha N D_{ql}}{\hbar} N$$

(3.7)

Further evidence for this result is shown in the third panel of Fig. 3.14. One notices that as chirping is introduced, the localisation length is increased - a direct result of the fact that a 3-kick cycle is being used.
3.6 Optimal Ratchet Parameters - What Makes the Best Ratchet?

The existence of the ratchet current is, as will be demonstrated in the next chapter, a classical effect arising from the asymmetric growth of the momentum distribution. The phenomenon of dynamical localisation is then used to 'freeze-in' this asymmetry, before it can be washed out by the continually expanding classical momentum distribution. It might seem reasonable to assume that one wishes to maximise the ratchet time, in order to maximise asymmetry. One would therefore also wish to make sure that the break-time is long enough to allow the maximum amount of classical asymmetry to build up. Naively, one would thus imagine that the best ratchet would maximise both $t_r$ and $t^*$. However, this argument ignores the subtleties inherent in the system which make the choice of parameters much more delicate.

It has been shown already that in the classical case, asymmetry in the momentum distribution grows until the ratchet time. After $t_r$, the wavepacket continues to expand, effectively diluting the asymmetry in the system. A similar effect occurs with the quantum momentum. In the previous section it was shown that $L^2 \approx D t^*$, meaning that as the break-time increases, the localisation length increasing dramatically. The quantum asymmetry can be quantified by using a scaled momentum: $p_L = \langle p \rangle_q / L$. Clearly, if $L$ is large, then the momentum asymmetry is small. Nevertheless it is important that dynamical localisation occurs after the maximum amount of classical asymmetry has accumulated. Thus it is found that the ideal ratchet will have: $t_r \approx t^*$.

The above argument also puts constraints on $\hbar$. In Fig. 3.16 the classical
3.6. Optimal Ratchet Parameters - What Makes the Best Ratchet?

Figure 3.16: Average current as a function of kick strength $K$. The figure shows the fine balance between the parameters in the system. For $K < 2.6$ increased quantum-classical correspondence is gained by decreasing $\hbar$ until $t^* \sim t_r$. However, for $K > 2.6$ the ratchet time is too short to allow any appreciable asymmetry build-up. Thus one must decrease $b$ in order to increase the ratchet time and recover $t^* \sim t_r$.

The current (for two different values of $b$) has been plotted with the quantum current (for three different values of $\hbar$ and $b = 0.1$) as a function of the kick strength. For the moment, consider only the classical curve for $b = 0.1$ and three quantum curves. For low values of $K$, the quantum curves show increasingly poor correspondence to the classical curve for increasing $\hbar$. This is due to the fact that in this region, the break-time (particularly for the $\hbar = 0.5$ case) is too short to allow sufficient asymmetry to accumulate before the onset of dynamical localisation. The agreement is much improved for $\hbar = 0.125$ where $t^*$ is longer. One would therefore wish to choose $\hbar$ in order
to give a break-time comparable to the ratchet time. However, making $\hbar$ too small would once again mean that the quantum asymmetry becomes diluted, since the localisation length is inversely proportional to $\hbar$: $L \propto \hbar^{-1}$.

Furthermore, Fig. 3.10 shows that with regards to the quantum current there is no particular advantage (for the parameters chosen) in decreasing $\hbar$ beyond a certain limit. A sensible choice of $\hbar$ would be in the region of $\hbar = 0.25$.

The appropriate choice of $\hbar$, and hence $t^*$, results in a much improved agreement between the classical and quantum currents in the region $1 < K < 2.6$. However, there is once again a large discrepancy for $K > 2.6$. In this case, one must consider the dependence of the classical ratchet time on the chirp parameter $b$. Recalling that $t_r \approx 2\pi/(Kb)^2$, one notes that as the kicking strength is increased the ratchet time decreases sharply, resulting in a lower value of average momentum. The plot also shows $t_r/200$ as a function of $K$, to aid visualisation. If one decreases $b$, the ratchet time is increased resulting in an increased average classical momentum for a given $K$, as shown by the curve for $b = 0.07$.

From the results shown in this chapter, one can suggest appropriate values for the parameters of an ideal chaotic double well ratchet. For the parameters concerning the spatial symmetry of the system, it was found that the maximum current was obtained for $a = 0.5$ and $\phi = 0, \pi$. Similarly, the maximum current was obtained for a three-kick cycle, i.e. $N = 3$. The requirement that $t_r \approx t^*$ puts limits on the other parameters. From Fig. 3.16, one can see that the maximum classical current occurs at about $K = 1.6$. Therefore:

$$t_r \approx t^*$$

$$\frac{2\pi}{(Kb)^2} \approx \frac{20K^2}{\hbar^2}$$

where it has been assumed that the break-time is calculated from $E = \langle p^2 \rangle$. 

**3.6. Optimal Ratchet Parameters - What Makes the Best Ratchet?**
3.7 Conclusions

Then one has:

\[ b \approx \sqrt{\frac{2\pi \hbar^2}{20K^4}} \]

(3.9)

which for the values given above means that \( b \approx 0.055 \), which is not inconsistent with the numerical results shown in Fig. 3.16.

3.7 Conclusions

The results shown in this chapter demonstrate that the chirped, double-well system introduced in chapter 2 does indeed admit a ratchet effect. The presence of a finite classical current, reached after a well defined ‘ratchet time’, unmistakably showing that this is the first Hamiltonian ratchet to operate in the fully chaotic regime.

The system has been studied extensively, both classically and quantum mechanically, and a thorough understanding obtained regarding the effects of changing each of the system parameters. The timescales of the system have also been investigated. Whilst the break-time is relatively well understood, the ratchet time is a completely new phenomenon arising from this work. The fine balance that must be achieved between these two timescales in order to perform a successful experiment has also been discussed.

In the next chapter, the ratchet effect will be shown to be a direct consequence of differential short time diffusion rates for particles with positive and negative momenta, arising from previously neglected corrections to the quasi-linear diffusion.
Diffusion and Average Current in the Double-Well Ratchet

4.1 Introduction

In the previous chapters it has been demonstrated that the hallmark of the Hamiltonian ratchet is the saturation of the classical current after a certain 'ratchet time', $t_r$. This novel effect, introduced for the first time in our work [83], will be shown in this chapter to be due to different short-time diffusion rates for particles with positive momenta to those with negative momenta. In order to explain analytically the surprising numerical results detailed below, one must investigate the classical diffusion rate beyond that given by the simple quasi-linear approximation, to include previously neglected correlations between successive kicks.

Corrections to the quasi-linear diffusion have been extensively studied for the standard map, using a variety of methods [63, 64, 91, 92].
4.2 Diffusion in the Double Well Ratchet

analysis detailed below for the double-well ratchet uses a modified form of the procedure used in [63] to obtain the adjusted diffusion coefficient. The method is also used to obtain an analytical form for the average classical current and the time dependence of the system, including a relationship for the ratchet time that shows excellent agreement with the numerical results of the previous chapter.

The chapter begins with a summary of the numerical results which highlight the correspondence between diffusion rates and the ratchet time. The method used to obtain the corrections to the diffusion coefficient is then introduced in the framework of the standard map, before showing how it can be adapted for the double well ratchet system and used to find analytical forms for the ratchet time and the average current. Finally, the expressions obtained are compared with numerical results.

4.2 Diffusion in the Double Well Ratchet

4.2.1 Asymmetric Diffusion - the Origin of the Ratchet Effect

As has been seen in chapters 2 and 3, the ratchet effect manifests itself as a net growth in the average classical momentum until the ratchet time, after which the current saturates to a finite value. This net current results from a build-up of asymmetry in the classical momentum distribution, as shown earlier in Figs. 3.1-3.4, which implies that particles with negative momenta diffuse at a different rate from those with positive momenta. One can show numerically that this is indeed the case by calculating independently the average energy growth for particles with negative momenta $\langle p^2 \rangle^(-)$ and positive momenta
4.2. Diffusion in the Double Well Ratchet

\( \langle p^2 \rangle^{(+)} \). Figure 4.1 shows these results for \( K = 1.6, a = 0.5 \) and two values of the time perturbation, \( b \). One can see clearly that \( \langle p^2 \rangle^{(-)} \) and \( \langle p^2 \rangle^{(+)} \) diverge from the classical expectation (shown in red) for a certain time, after which the diffusion rates equalise and run parallel to that predicted by the quasi-linear approximation.

![Classical Energy and Momentum Growth vs Time](image)

Figure 4.1: The top panel shows differential classical diffusion rates for \( K = 1.6, a = 0.5, D_0 \approx 2.5 \) and two different values of \( b \). Note that \( \langle p^2 \rangle \) is evaluated separately at each kick for particles with positive and negative momenta. We see that \( \langle p^2 \rangle^{\pm} \) diverge from linear growth by a quantity, which is similar in magnitude but opposite in sign for the negative and positive components. The + and − indicate \( \langle p^2 \rangle^{+}, \langle p^2 \rangle^{-} \) respectively. But once \( t > t_r \), we see linear growth: \( D^{+} \sim D^{-} \sim D \sim 2.5 \). The bottom panel shows saturation of the classical momentum for both cases shown in the upper graph.

By comparing the classical current (reproduced in the lower panel of
4.2. Diffusion in the Double Well Ratchet

Fig. 4.1) with this divergent energy growth one sees that the ratchet time, $t_r$, corresponds to the time at which the positive and negative diffusion rates revert to the quasi-linear form. One notes also that in keeping with the previous results, $t_r$ is clearly inversely proportional to the period-one chirp parameter, $b$. Therefore, whilst it is evident that the quasi-linear approximation accurately describes the long time classical energy growth in the system, its validity must be re-examined for short timescales in order to explain the observed behaviour.

In fact, the quasi-linear approximation for the diffusion coefficient was shown to be inaccurate for the standard map by Rechester and White [63, 64] who proceeded to obtain a series of corrections resulting from correlations between successive sequences of kicks in the evolution. However, the corrections terms they obtained were insensitive to the sign or magnitude of the momentum, in keeping with the non-transporting nature of the kicked rotor system. For the double-well ratchet, it will be shown that the corrections to the quasi-linear diffusion encompass the momentum-dependent and chirp dependent nature of the energy separation.

4.2.2 Obtaining the Quasi-Linear Diffusion for the Standard Map

To first order, the rate of average energy growth in ratchet and rotor systems in the chaotic regime can be approximated by the quasi-linear formula: $D_{ql} \approx K^2/2$. This relationship is obtained from a Fokker-Planck description of particle motion and as such is only valid under the assumption that there is phase randomisation after each iteration of the map. In other words one must ensure that the number of kicks, $n$, is much greater than the number
of kicks, \( n_c \), needed for a uniform phase distribution to occur. In reality, one must take into account the effect of correlations between successive kicks in the sequence. For a fuller discussion, the reader is referred to [93].

Rechester and White obtained the corrections to the quasi-linear diffusion using two different methods; firstly using a probabilistic method [63], and secondly by the use of Fourier-space paths [64]. Whilst for the standard map the latter method yields many more correction terms than the former, in the case of the ratchet systems it is mathematically unwieldy and, as will be shown, unnecessary to go beyond the leading higher-order terms. For this reason an altered version of the first, probabilistic method is used to obtain the corrections terms for the ratchet systems. One major difference between the method detailed here and that contained in [63] is the absence of a term which introduces external stochasticity. Rechester and White included this term to account for non-diffusive trajectories contained within stable islands in phase space. In practice, the noise term made negligible contribution to the diffusion coefficient but was included for mathematical rigour. In the chaotic Hamiltonian ratchet, it is assumed that the system is in the globally chaotic regime where no islands exist; therefore it is not necessary to include any extrinsic noise.

Before embarking on the derivation for the ratchet, it is instructive to introduce the method by using it to obtain the well-known corrections for the Standard Map:

\[
\begin{align*}
p_{n+1} &= p_n + K \sin(x_n) \\
x_{n+1} &= x_n + p_{n+1}
\end{align*}
\]

The diffusion constant is defined in terms of the average energy growth for the system starting with initial conditions \( p = p_0 \) at time \( t = 0 \), evolving
4.2. Diffusion in the Double Well Ratchet

with time $t$:

$$\left\langle \frac{(p - p_0)^2}{2} \right\rangle = Dt,$$

(4.3)

After $N$ kicks, the momentum of the system will take the form:

$$p_N = p_0 + K \sin(x_0) + K \sin(x_1) + \ldots + K \sin(x_{N-1})$$

$$= p_0 + \sum_{i=0}^{N-1} K \sin(x_i)$$

where for clarity one uses the substitution:

$$S_j = -\sum_{i=0}^{j} V'(x_i)$$

(4.4)

In terms of the conditional probability density $Q$ that the system evolves to a state $(x_N, p_N)$ at time $t = t_N$ one finds:

$$D(t_N) = \frac{1}{2t_N} \int Q(x_N, p_N, t_N | x_0, p_0, 0) P(x_0, p_0, 0)(p_N - p_0)^2 dx_0 dp_0 dx_N dp_N$$

(4.5)

where the initial probability distribution is given to be:

$$P(x_0, p_0, 0) = (2\pi)^{-1} \delta(p - p_0),$$

Note that, since only one value of $p_0$ is being considered, one can use the above $\delta$-function constraint to perform the integral over $dp_0$. The conditional probability density obeys the recursion property:

$$Q(x_N, p_N, t_N | x_0, p_0, 0) = \int Q(x_N, p_N, t_N | x_i, p_i, t_i) Q(x_i, p_i, t_i | x_0, p_0, 0) dx_i dp_i$$

(4.6)

Using the form:

$$Q(x_N, p_N, t_N | x_{N-1}, p_{N-1}, t_{N-1}) = \sum_{n_N=-\infty}^{\infty} \delta(p_N - p_{N-1} - K \sin(x_{N-1})) \cdot \delta(x_N - x_{N-1} - p_{N-1} - K \sin(x_{N-1}) + 2\pi n_N),$$

(4.7)
4.2. Diffusion in the Double Well Ratchet

one obtains:

\[ Q(x_N, p_N, t_N|x_0, p_0, 0) = \sum_{n_N=-\infty}^{+\infty} \ldots \sum_{n_1=-\infty}^{+\infty} \int_{0}^{2\pi} \frac{dx_1}{2\pi} \ldots \int_{0}^{2\pi} \frac{dx_{N-1}}{2\pi} \int_{-\infty}^{\infty} dp_1 \]

\[ \ldots \int_{-\infty}^{\infty} dp_{N-1} \delta(p_{N} - p_0 - S_{N-1})\delta(x_N - x_{N-1} - p_{N-1} - K \sin x_{N-1} + 2\pi n_N) \]

\[ \ldots \delta(p_i - p_{i-1} - S)\delta(x_i - x_0 - p_i - K \sin x_0 + 2\pi n_i) \] (4.8)

One notes that the sum over \( n_i \) occurs because of the periodic boundary condition for \( x_i \) (\( 0 \leq x_i \leq 2\pi \)). The above equation is now inserted into Eq. (4.5). Then using Eq. (4.4) and noting that the \( \delta \)-function constraints \( \delta(p_i - p_{i-1} - S) \) (for \( i = 1 \ldots N - 1 \)) take care of the \( p \)-integrals all the way to \( dp_N \), thus leaving only \( p_0 \), one finds:

\[ D = \frac{1}{2N} \sum_{n_N=-\infty}^{+\infty} \ldots \sum_{n_1=-\infty}^{+\infty} \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} S_N^2 \delta(x_N - x_{N-1} - p_0 - S_{N-1} + 2\pi n_N) \]

\[ \ldots \delta(x_1 - x_0 - p_0 - S_0 + 2\pi n_1) \] (4.9)

By making use of the Poisson Summation formula:

\[ \sum_{n=-\infty}^{+\infty} \delta(y + 2\pi n) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \exp[imy] \] (4.10)

Eq. (4.9) can be written as:

\[ D = \lim_{N \to \infty} \frac{1}{2N} \sum_{n_N=-\infty}^{+\infty} \ldots \sum_{n_1=-\infty}^{+\infty} \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} S_N^2 \]

\[ \exp \left( \sum_{j=1}^{N} \left[ im_j(x_j - x_{j-1} - p_0 - S_{j-1}) \right] \right) \] (4.11)

[Note that two dummy variables \( i \) and \( j \) are used to differentiate between the sum in the exponential and the product of integrals, although they refer to the same kick in the sequence] By taking the case where \( m_1 = m_2 \ldots = m_N = 0 \)
one recovers the quasi-linear diffusion:

\[
D = \lim_{N \to \infty} \frac{1}{2N} \sum_{m_N = -\infty}^{\infty} \cdots \sum_{m_1 = -\infty}^{\infty} \prod_{i=0}^{N-1} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2
\]

\[
= \lim_{N \to \infty} \frac{1}{2N} \prod_{i=0}^{N-1} \int_0^{2\pi} \frac{dx_i}{2\pi} K^2 \left[ \sin(x_0) + \sin(x_1) + \ldots + \sin(x_{N-1}) \right]^2
\]

\[
= \lim_{N \to \infty} \frac{1}{2N} K^2 \frac{N}{2}
\]

\[
= \frac{K^2}{4} \tag{4.12}
\]

### 4.2.3 Corrections to the Quasi-Linear Diffusion for the Standard Map

The effect of correlations on the diffusion coefficient for the standard map can be investigated by allowing \( m_j \neq 0 \) in Eq. (4.11). One therefore redefines Eq. (4.3) to be:

\[
\left< \frac{(p - p_0)^2}{2} \right> = Dt \equiv D_0 t + t \sum_{l=1}^{\infty} C(l), \tag{4.13}
\]

where \( D_0 \) is the quasi-linear diffusion obtained above and the \( C(l) \) are correction terms.

In general, the simplest correction term (denoted \( C(2) \)) occurs for two-kick correlations between kicks \( i \) and \( i + 2 \) as will be shown below. This term arises by setting: \( m_{i+2} = \pm 1, \ m_i = -m_{i+2} \) in Eq. (4.11). In the case of \( m_{i+2} = 1, \ m_i = -1 \) one finds:

\[
C(2) = \lim_{N \to \infty} \frac{1}{2N} \sum_{m_N = -\infty}^{\infty} \cdots \sum_{m_1 = -\infty}^{\infty} \prod_{i=0}^{N-1} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2} - x_{i+1})} e^{-S_{i+1}} e^{-i(x_{i+1} - x_i)} e^{iS_i}
\]

\[
= \frac{K^2}{4} \tag{4.14}
\]
4.2. Diffusion in the Double Well Ratchet

Note that the '\(S\)' terms in the exponent simplify considerably:

\[ S_{i+1} - S_i = K \sin(x_{i+1}) \]

Using the Bessel function identity:

\[ \exp[\pm iz \cos \theta] = \sum_n i^n J_n(z) \exp[\pm in\theta], \quad (4.15) \]

The integrand \( I \) becomes:

\[
I = \frac{S_N^2 e^{i(x_{i+2} - 2x_{i+1} + x_i)} e^{-iK \sin(x_{i+1})}}{\left[ \sum_{i=0}^{N-1} \sin(x_i) \right]^2} \sum_{n=-\infty}^{+\infty} J_n(K) e^{-inx_{i+1}} \quad (4.16)
\]

In order to have a non-zero integration, the exponential terms must be eliminated. This constrains the order of the Bessel function to \( n = 2 \) leaving:

\[
C(2) = \lim_{N \to \infty} \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} \int_0^{2\pi} \frac{dx_{i-2}}{2\pi} \left[ \sum_{i=0}^{N-1} \sin(x_i) \right]^2 e^{i(x_j + x_{j-2})} J_2(K) \quad (4.17)
\]

Thus one is left with the correlation between the remaining terms in \( x_j, x_j-2 \), which constrains the \( S_N^2 \) part of the equation:

\[
C(2) = \lim_{N \to \infty} \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} \int_0^{2\pi} \frac{dx_{i-2}}{2\pi} 2 \sin(x_{i+2}) \sin(x_i) e^{i(x_{i+2} + x_i)} J_2(K) \quad (4.18)
\]

Expanding the sine product gives:

\[
2 \sin(x_{i+2}) \sin(x_i) = \cos(x_{i+2} - x_i) - \cos(x_{i+2} + x_i)
\]

\[
= \frac{1}{2} [e^{i(x_{i+2} - x_i)} + e^{-i(x_{i+2} - x_i)} - e^{i(x_{i+2} + x_i)} - e^{-i(x_{i+2} + x_i)}] \quad (4.19)
\]
4.2. Diffusion in the Double Well Ratchet

The final term in this expansion will cancel the exponential in Eq. (4.18). Thus one obtains the simple result:

\[ C(2) = \frac{-1}{2N} K^2 J_2(K) \frac{1}{2} \]  

(4.20)

An identical result is obtained for the case \( m_j = -1, m_{j-1} = 1 \). There are also \( N - 1 \) terms of this kind for each \( m_j j = 2 \ldots N \); so finally:

\[ C(2) = -\frac{1}{2N} K^2 J_2(K)(N - 1) \]

\[ = -\frac{K^2}{2} J_2(K) \]  

(4.21)

This is the first correction term in the series found by Rechester and White:

\[ D = \frac{\sigma^2}{2} \left[ \frac{1}{2} - J_2(\epsilon)e^{-\sigma} - J_1(\epsilon)e^{-\sigma} + J_0(\epsilon)e^{-3\sigma} \ldots \right] \]  

(4.22)

where in the notation of this thesis, \( \epsilon \equiv K \) and the noise term is zero: \( \sigma = 0 \).

Other terms can be obtained by looking at correlations between increasing numbers of kicks - for example the \( J_1^2 \) correction term in Eq. 4.22 arises from 3-kick correlation terms with \( m_{i+2} = \pm 1, m_{i+1} = -m_{i+2} \) and \( m_i = -m_{i+2} \).

4.2.4 Derivation of the Double Well Ratchet Diffusion Coefficient

One can clearly see that for the standard map, the diffusion coefficient is momentum-independent. In other words, if the diffusion coefficient is obtained numerically as \( D = \langle p^2 \rangle /2 \); in the case of the kicked rotor, \( D(p^-) = D(p^+) \). However, as shown earlier, one requires the coefficient for the ratchet to be dependent on both momentum and \( b \), the period-one time perturbation, such that \( D(p^-) \neq D(p^+) \) - giving rise to asymmetry in the momentum distribution and hence transport in the system.
4.2. Diffusion in the Double Well Ratchet

In order to obtain the diffusion coefficient for the ratchet one must begin by defining the generalised map:

\[ p_{j+1} = p_j - V'(x_j) \]  
\[ x_{j+1} = x_j + p_{j+1}(t_{j+1} - t_j) \]

where \( t_{j+1} - t_j \) is the time between kicks. For the standard map, \( t_{j+1} - t_j = 1 \), always. However, for the chirped sequence this is not necessarily the case.

As before, the map gives:

\[ p_N = p_0 - V'(x_0) - V'(x_1) \ldots - V'(x_{N-1}) \]

\[ = p_0 - \sum_{i=0}^{N-1} V'(x_i) \]

and so one again defines:

\[ S_j = -\sum_{i=0}^{j} V'(x_i) \]

These considerations result in a modified form of Eq. (4.11):

\[ D = \lim_{N \to \infty} \frac{1}{2N} \sum_{m_N=-\infty}^{\infty} \ldots \sum_{m_1=-\infty}^{\infty} \prod_{i=0}^{N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 \]

\[ \exp \left( \sum_{j=1}^{N} \left[ \text{Im} \left( m_j(x_{j+1} - x_j - (t_{j+1} - t_j)(p_0 + S_j)) \right) \right] \right) \]

As with the standard map, setting \( m_j = 0 \) for all \( j \) one recovers the quasilinear diffusion. In the case of the double well potential this is:

\[ D_{ql} = \frac{K_{eff}^2}{4} \]

where

\[ K_{eff} = K \sqrt{1 + 4a^2} \]
4.2. Diffusion in the Double Well Ratchet

The main corrections to the diffusion coefficient for the ratchet arise from two-kick correlations of the form $C'(2,p) = \langle V'(x_i)V'(x_{i+2}) \rangle$ (this assertion will be proven numerically below). However, due to the nature of the chirped system, one must average over a 5 kick sequence in order to account for each different 2-kick combination. One therefore interrogates correlations over an arbitrary sequence of 5 kicks, beginning at kick $i$ - note that $i$ is necessarily not the first kick in the evolution; it is assumed that kick $i$ occurs after an arbitrary time.

It will be shown below that by setting $m_{i+2} = \pm 1$ and $m_{i+1} = -m_{i+2}$ one obtains the correction to the diffusion coefficient relating to the $K \sin x$ term in the double-well potential. The correction for the $Ka \sin 2x$ part is obtained by setting $m_{i+2} = \pm 2$ and $m_{i+1} = -m_{i+2}$. Each of the three 2-kick correlations mentioned above are now addressed individually.

A General Note Concerning the Choice of ‘m’ Values

Before deriving the correction terms for the double well diffusion coefficient it is appropriate to explain the rationale behind the choice of the $m_i$ values that appear in the exponent of equations such as Eq. (4.26). One must bear in mind that the purpose of this work is to obtain terms that contribute significantly to the transport in the system. In general, when choosing the $m_i$ values one must determine whether the resulting term will be asymmetric in momentum (i.e. contain terms of the form $\sin np\theta$) and if so, whether it makes an appreciable contribution to the overall transporting effect.

Most choices of $m_i$ can be eliminated immediately. For example one can immediately exclude $|m_i| \geq 3$ for the two kick case, since one will obtain terms of the form $V'(x_i)V'(x_{i+2}) \exp[3i(x_i - x_{i+2})]$ which will always integrate.
4.2. Diffusion in the Double Well Ratchet

to zero as a result of $V'(x_i)V'(x_{i+2})$ containing no $\sin 3x_i \sin 3x_{i+2}$ term. For 3-kick terms and higher, one can always eliminate $m_i$'s which will result in similar situations. Note that $|m_i| = 2$ can be eliminated for the $\sin x$ part of the potential (leaving only $|m_i| = 0, 1$), but this choice clearly picks out the $\sin 2x_i \sin 2x_{i+2}$ term from the $\sin 2x$ part.

The choices of $m_i$ given below are those which provide the majority contribution to the transport in the system. All other combinations either give terms that are even in momentum (i.e. dominated by $\cos n pb$) or contain products of greater than or equal to four sums of Bessel functions (i.e. $\sum_n J_n \sum_n J_n \sum_n J_n \sum_n J_n$) which give negligible contribution to the final result. Note that an example of this is shown in chapter 6, where a 3-kick correlation is derived for the rocking ratchet and shown not to contribute to the overall transport.

4.2.5 Correlation between kicks $i, i + 2$

The three kick map for this section is:

\begin{align*}
p_i &= x_{i-1} - V'(x_{i-1}) \quad (4.28) \\
x_i &= x_{i-1} + p_i (1 + b) \quad (4.29) \\
p_{i+1} &= x_i - V'(x_i) \quad (4.30) \\
x_{i+1} &= x_i + p_{i+1} \quad (4.31) \\
p_{i+2} &= x_{i+1} - V'(x_{i+1}) \quad (4.32) \\
x_{i+2} &= x_{i+1} + p_{i+2} (1 - b) \quad (4.33)
\end{align*}
The two-kick correction terms then take the form:

$$C(2, p)_{m_{i+2}} = \frac{1}{2N} \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} 2V'(x_i)V'(x_{i+2}) \cdot$$

$$e^{(\pm(x_{i+2} - x_{i+1} - (t_{i+2} - t_{i+1}) (p_0 + S_{i+1}))} e^{(x_{i+1} - x_i - (t_{i+1} - t_i) (p_0 + S_i))}$$

(4.34)

Note that examining specifically kicks $i$ and $i + 2$ picks out the term: $S_N^2 = V'(x_i)V'(x_{i+2})$. This assertion will be validated in the course of the derivation.

If one now looks specifically at the case $m_{i+2} = -1, m_{i+1} = 1$, one can simplify the above equation to:

$$C(2, p)_{m_{i+2}, m_{i+1}} = \frac{1}{2N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} 2V'(x_i)V'(x_{i+2}) \cdot$$

$$e^{-(x_{i+2} - x_{i+1} - (t_{i+2} - t_{i+1}) (p_0 + S_{i+1}))} e^{((x_{i+1} - x_i - (t_{i+1} - t_i) (p_0 + S_i))}

(4.35)

Now note that: $t_{i+2} - t_{i+1} = 1 - b$ and $t_{i+1} - t_i = 1$ giving,

$$C(2, p)_{m_{i+2}, m_{i+1}} = \frac{1}{2N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} 2V'(x_i)V'(x_{i+2}) e^{-(x_{i+2} - x_{i+1} - (1-b)(p_0 + S_{i+1}))} \cdot$$

$$e^{((1-b)S_{i+1} - S_i)}$$

(4.36)

Now one uses the fact that:

$$S_{i+1}(1-b) + S_i = -bS_{i+1} + (S_{i+1} - S_i)$$

$$= -bS_{i+1} - \left( \sum_{l=0}^{i+1} V'(x_l) - \sum_{i=0}^{i} V'(x_l) \right)$$

$$= -bS_{i+1} - V'(x_{i+1})$$

$$= -bS_i + bV'(x_{i+1}) - V'(x_{i+1})$$

$$= -(1-b)V'(x_{i+1}) - bS_i$$

(4.37)
4.2. Diffusion in the Double Well Ratchet

Therefore the diffusion coefficient becomes:

\[
C(2, p)_{m_{i+2}=-1}^{i+2} = \frac{1}{N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}-2x_{i+1}+x_i)}e^{-ip_0b} \\
e^{-(1-b)V'(x_{i+1})-bS_i} 
\]

(4.38)

One can further simplify this equation by using the Bessel function identity:

\[
\exp[\pm iz \cos \theta] = \sum_{n} i^{\pm n} J_n(z) \exp[\pm in\theta],
\]

(4.39)

to obtain:

\[
C(2, p)_{m_{i+2}=-1}^{i+2} = \frac{1}{N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}-2x_{i+1}+x_i)}e^{-ip_0b} \\
e^{-i((1-b)K \cos(x_{i+1})+2Ka \cos(2x_{i+1}))+bS_i} \\
\sum_{n} i^{-n} J_n((1-b)K)e^{-inx_{i+1}} \sum_{s} i^{-s} J_s((1-b)2K) e^{-i2sx_{i+1}} e^{-bs_{i+1}} 
\]

(4.40)

For a non-zero integration over \(x_{i+1}\) one requires:

\[-nx_2 - 2sx_2 + 2x_2 = 0\]

\[\Rightarrow -n - 2s + 2 = 0\]

\[\Rightarrow n = 2 - 2s\]

So finally:

\[
C(2, p)_{m_{i+2}=-1}^{i+2} = \frac{1}{N} \int_{0}^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}+x_i)}e^{-ip_0b} \\
\sum_{s} i^{-2+s} J_{2-2s}((1-b)K)J_s((1-b)2K) e^{-ibS_i} 
\]

(4.41)

It is appropriate at this point to address the remaining \(S\)-dependent expo-
4.2. Diffusion in the Double Well Ratchet

Expanding this term one obtains:

\[ C(2, p)_{m+2}^{i+2} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_i)(x_{i+2})e^{-i(x_{i+2}+x_i)}e^{-ip_0b} \]
\[ \sum_s i^{-2+s} J_{2-2s}((1-b)K)J_s((1-b)2Ka)e^{-ibS_{i-1}-V'(x_i)} \]

(4.42)

The presence of the derivative of the potential in the exponent gives rise to further Bessel functions:

\[ C(2, p)_{m+2}^{i+2} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}+x_i)}e^{-ip_0b} \]
\[ \sum_s i^{-2+s} J_{2-2s}((1-b)K)J_s((1-b)2Ka) \]
\[ \sum_p i^p J_p(bK)e^{ipx_i} \sum_r i^r J_r(2abK)e^{2r\pi i}e^{-ibS_{i-1}} \]

(4.43)

For the exponential terms in \( x_i \) one requires:

\[ -x_i + px_i + 2rx_i = -x_i, \]

For the next step of the derivation it is important to retain the \( V'(x_i)V'(x_{i+2}) \exp[-i(x_{i+2}+x_i)] \) part of the formula. Thus,

\[ p = -2r \]

Which implies:

\[ C(2, p)_{m+2}^{i+2} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}+x_i)}e^{-ip_0b} \]
\[ \sum_s i^{-2+s} J_{2-2s}((1-b)K)J_s((1-b)2Ka) \sum_r i^{-r} J_{-2r}(bK)J_r(2abK)e^{-ibS_{i-1}} \]

(4.44)
Now since $bK$ is small, the only non-zero term corresponds to $r = 0$:

$$C(2,p)^{i+2}_{mi+2=-1} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}+x_i)}e^{-ipob}$$

$$\sum_s i^{-2+s}J_{2-2s}((1 - b)K)J_s((1 - b)2Ka)J_0(bK)J_0(2abK)e^{-ibS_i-1}$$

(4.45)

The exponent $S_i-1$ has the form:

$$S_{i-1} = -\sum_{i=0}^{i-1} V'(x_i)$$

Which will give rise to a product of $i-1$ Bessel functions and their associated exponentials in accordance with Eq. (4.39). In order for these terms to be non-zero after integration the exponentials must be eliminated, forcing the Bessel function order to be zero in each case. Thus, including the $J_0$ terms already present, the correlation term becomes:

$$C(2,p)^{i+2}_{mi+2=-1} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}+x_i)}e^{-ipob}$$

$$\sum_s i^{-2+s}J_{2-2s}((1 - b)K)J_s((1 - b)2Ka)\left[J_0(bK)J_0(2abK)\right]^i$$

(4.46)

Assuming that $Kb \ll 1$, one can use the small argument approximation for $J_0$:

$$J_0(Kb) \approx 1 - \left(\frac{Kb}{2}\right)^2$$

Thus for short times, one can neglect the contribution from this term. However, this term effectively governs the longer time dependence of the system, and (as will be shown later) it is possible to extract an analytical form for the ratchet time from it.
4.2. Diffusion in the Double Well Ratchet

The remaining $x$-dependent exponential terms in Eq. (4.46) are dealt with by combining them with the $V'(x_i)V'(x_{i+2})$ term. This latter term can be expanded in the following way:

$$V'(x_i)V'(x_{i+2}) \equiv (K \cos x_{i+2} + 2Ka \cos 2x_{i+2}) \cdot (K \cos x_i + 2Ka \cos 2x_i)$$

$$= K^2 \cos x_{i+2} \cos x_i + 2K^2 a \cos x_{i+2} \cos 2x_i$$

$$+ 2K^2 a \cos 2x_{i+2} \cos x_i + 4(Ka)^2 \cos 2x_{i+2} \cos 2x_i$$

(4.47)

The first term, which arises from the $K \sin x$ part of the potential, is the appropriate one to use:

$$K^2 \cos x_{i+2} \cos x_i = \frac{K^2}{2} \{\cos(x_{i+2} - x_i) + \cos(x_{i+2} + x_i)\}$$

$$= \frac{K^2}{4} \{e^{i(x_{i+2} - x_i)} + e^{-i(x_{i+2} - x_i)} + e^{i(x_{i+2} + x_i)} + e^{-i(x_{i+2} + x_i)}\}$$

(4.48)

Now substituting the third term into Eq. (4.42):

$$C(2, p)_{m_{i+2}=-1}^{i+2} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_i)V'(x_{i+2})e^{-i(x_{i+2}+x_i)} \cdot e^{i(x_{i+2}+x_i)} e^{-ip\theta}$$

$$\sum_s i^{-2+s} J_{2-2s}((1-b)K)J_s((1-b)2Ka)$$

$$= \frac{1}{4N} e^{-ip\theta} \sum_s i^{-2+s} J_{2-2s}((1-b)K)J_s((1-b)2Ka)$$

(4.49)

Performing this analysis for the case: $m_{j+2} = 1$, $m_{j+1} = -1$ one obtains a similar result:

$$C(2, p)_{m_{i+2}=1}^{i+2} = \frac{1}{4N} e^{ip\theta} \sum_s i^{2-s} J_{2-2s}((1-b)K)J_s((1-b)2Ka)$$

(4.50)
The equations (4.49) and (4.50) are combined in the following fashion:

\[ C(2, p)_{K \sin x}^{i+2} = \frac{1}{4N} \sum_{s} J_{2-2s}((1-b)K)J_{s}((1-b)2Ka)\{i^{i+2}e^{-ipob} + i^{2-s}e^{ipob}\} \]
\[ = \frac{1}{4N} \sum_{s} J_{2-2s}((1-b)K)J_{s}((1-b)2Ka)\{e^{i\frac{\pi}{2}(2-s)}e^{-ipob} + e^{i\frac{\pi}{2}s}e^{ipob}\} \]
\[ = \frac{1}{4N} \sum_{s} J_{2-2s}((1-b)K)J_{s}((1-b)2Ka)\{e^{i\frac{\pi}{2}s}e^{-ipob} + e^{-i\frac{\pi}{2}s}e^{ipob}\} \]
\[ = -\frac{1}{4N} \sum_{s} J_{2-2s}((1-b)K)J_{s}((1-b)2Ka)\{e^{i\frac{\pi}{2}s}e^{-ipob} + e^{-i\frac{\pi}{2}s}e^{ipob}\} \]
\[ = -\frac{1}{2N} \sum_{s} J_{2-2s}((1-b)K)J_{s}((1-b)2Ka)\cos(p_{0}b - \frac{\pi}{2}s) \]

(4.51)

Since there are \(N/3\) terms of this kind (this particular correlation applies to only a third of the kicks in a given sequence) one obtains the final form of the kick \(i : i+2\) correction to the quasi-linear diffusion resulting from the \(\sin x\) part of the potential as:

\[ D_{K \sin x}^{i+2} = -\frac{1}{6} \sum_{s} J_{2-2s}((1-b)K)J_{s}((1-b)2Ka)(\cos p_{0}b \cos \frac{\pi}{2}s + \sin p_{0}b \sin \frac{\pi}{2}s) \]

(4.52)

To find the correction due to the \(\sin 2x\) part of the potential one uses \(m_{j} = \pm 2\) and \(m_{j-1} = -m_{j}\) in Eq. (4.26):

\[ C(2, p)_{K\sin 2x}^{i+2} = \frac{1}{2N} \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dz_{i}}{2\pi} s_{N}e^{(\sum_{j=1}^{N} \pm 2i(z_{i+2-x_{i+1}-(t_{i+2}-t_{i+1})(p_{0}+S_{i+1}))})} e^{(\sum_{j=1}^{N} \pm \frac{2i}{2}(z_{i+1}-x_{i}-(t_{i+1}-t_{i})(p_{0}+S_{i}))}) \]

(4.53)

and following the same working one finds the correction to the diffusion coefficient due to the \(K\sin 2x\) part of the potential, to be:

\[ C(2, p)_{K\sin 2x}^{i+2} = \frac{1}{6} \sum_{s} J_{4-2s}(2(1-b)K)J_{s}(4(1-b)Ka) \]
\[ \cos(2p_{0}b \cos \frac{\pi}{2}s + \sin 2p_{0}b \sin \frac{\pi}{2}s) \]

(4.54)
4.2.6 Correlation between kicks \( i + 1, i + 3 \)

The analysis for the correlation between kicks \( i+1 \) and \( i+3 \) of the cycle follows the same pattern as that given above, so what follows is much abbreviated.

The new 3-kick map is:

\[
\begin{align*}
p_{i+1} &= x_i - V'(x_i) \\
x_{i+1} &= x_i + p_{i+1} \\
p_{i+2} &= x_{i+1} - V'(x_{i+1}) \\
x_{i+2} &= x_{i+1} + p_{i+2}(1 - b) \\
p_{i+3} &= x_{i+2} - V'(x_{i+2}) \\
x_{i+3} &= x_{i+2} + p_{i+3}(1 + b)
\end{align*}
\]

Setting \( m_{i+3} = -1, m_{i+2} = 1 \) in Eq. (4.26) and noting that \( t_{i+3} - t_{i+2} = 1 + b \) and \( t_{i+2} - t_{i+1} = 1 - b \) one finds:

\[
C(2, p)^{i+1:i+3}_{m_{i+3}=-1} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_{i+1})V'(x_{i+3})e^{-i(x_{i+3}-x_{i+2}-(1+b)(p_0+S_{i+1}))}e^{i(x_{i+3}-x_{i+1}-(1-b)(p_0+S_{i+1}))} \\
= \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_{i+1})V'(x_{i+3})e^{-i(x_{i+3}-2x_{i+2}+x_{i+1})}e^{i2p_0}e^{i((1+b)S_{i+2}-(1-b)S_{i+1})}
\]

Simplification of the \( S \)-dependent exponent gives:

\[
S_{i+2}(1+b) - S_{i+1}(1-b) = (S_{i+2} - S_{i+1}) + b(S_{i+2} + S_{i+1}) \\
= -V'(x_{i+2}) + b(2S_{i+1} - V'(x_{i+2}) \\
= -(1 + b)V'(x_{i+2}) - 2bS_{i+1}
\]

which implies:

\[
C(2, p)^{i+1:i+3}_{m_{i+3}=-1} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_{i+1})V'(x_{i+3})e^{-i(x_{i+3}-2x_{i+2}+x_{i+1})}e^{i2p_0}e^{-i((1+b)V'(x_{i+2})-2bS_{i+1})}
\]
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which can be represented using Bessel functions as:

\[ C(2, p)_{m_{i+3}=-1}^{i+1;i+3} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} V'(x_{i+1}) V'(x_{i+3}) e^{-i(x_{i+3} + x_{i+1})} e^{ip_0 b} \]
\[ \sum_s i^{2-s} J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) e^{2b S_{i+1}} \]

(4.63)

As with the correlation between kicks \( i : i + 2 \), the exponential in \( S_{i+1} \)
gives rise to a product of \( J_0 \) Bessel functions, which can be approximated to
unity for short time and small \( Kb \). Once again, the need to eliminate the x-
dependent exponential terms picks out the \( K^2 \cos(x_{i+3}) \cos(x_{i+1}) \) term from
the \( V'(x_{i+1})V'(x_{i+3}) \) product. Thus the correction term simplifies to:

\[ C(2, p)_{m_{i+3}=-1}^{i+1;i+3} = \frac{1}{4N} e^{i2p_0 b} \sum_s i^{2-s} J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \]

(4.64)

with the corresponding term for \( m_{i+3} = 1, \ m_{i+2} = -1 \) being:

\[ C(2, p)_{m_{i+3}=1}^{i+1;i+3} = \frac{1}{4N} e^{-i2p_0 b} \sum_s i^{2-s} J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \]

(4.65)

The two terms are combined as before:

\[ C(2, p)_{K_{\min x}}^{i+1;i+3} = \frac{1}{4N} \sum_s J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \{ i^{-2+s} e^{i2p_0 b} + i^{2-s} e^{-i2p_0 b} \} \]
\[ = \frac{1}{4N} \sum_s J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \{ e^{\frac{i\pi}{2}(2-s)} e^{i2p_0 b} + e^{\frac{i\pi}{2}(2+s)} e^{-i2p_0 b} \} \]
\[ = \frac{1}{4N} \sum_s J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \{ e^{\frac{i\pi}{2} s} e^{i2p_0 b} + e^{-\frac{i\pi}{2} s} e^{-i2p_0 b} \} \]
\[ = -\frac{1}{4N} \sum_s J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \{ e^{\frac{i\pi}{2} s} e^{i2p_0 b} + e^{-\frac{i\pi}{2} s} e^{-i2p_0 b} \} \]
\[ = -\frac{1}{2N} \sum_s J_{2-2s}((1 + b)K) J_s((1 + b)2Ka) \cos(2p_0 b \frac{\pi}{2}) \]

(4.66)
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So, finally the diffusion correction due to $K \sin x$ for kicks $i + 1 : i + 3$ is:

$$C(2, p)^{i+1:i+3}_{K\sin x} = \frac{1}{6} \sum_s J_{2-2s}(1 + b)K)J_s(1 + b)2Ka)$$

$$\left(\cos 2p_0b \cos \frac{\pi}{2}s - \sin 2p_0b \sin \frac{\pi}{2}s\right)$$ (4.67)

with the $Ka \sin 2x$ correction being:

$$C(2, p)^{i+1:i+3}_{Ka \sin 2x} = \frac{1}{6} \sum_s J_{4-2s}(2(1 + b)K)J_s(4(1 + b)Ka)$$

$$\left(\cos 4p_0b \cos \frac{\pi}{2}s - \sin 4p_0b \sin \frac{\pi}{2}s\right)$$ (4.68)

4.2.7 Correlation between kicks $i + 2$, $i + 4$

The new 3-kick map is:

$$p_{i+2} = x_{i+1} - V'(x_{i+1})$$

(4.69)

$$x_{i+2} = x_{i+1} + p_{i+2}(1 - b)$$

(4.70)

$$p_{i+3} = x_{i+2} - V'(x_{i+2})$$

(4.71)

$$x_{i+3} = x_{i+2} + p_{i+3}(1 + b)$$

(4.72)

$$p_{i+4} = x_{i+3} - V'(x_{i+3})$$

(4.73)

$$x_{i+4} = x_{i+3} + p_{i+4}$$

(4.74)

The analysis then follows precisely the same procedure given above, resulting in the $K \sin x$ correction:

$$C(2, p)^{i+2:i+4}_{K\sin x} = \frac{1}{6} \sum_s J_{2-2s}(K)J_s(2Ka)(\cos p_0b \cos \frac{\pi}{2}s + \sin p_0b \sin \frac{\pi}{2}s$$

(4.75)

and the $Ka \sin 2x$ correction:

$$C(2, p)^{i+2:i+4}_{Ka \sin 2x} = \frac{1}{6} \sum_s J_{4-2s}(2K)J_s(4Ka)(\cos 2p_0b \cos \frac{\pi}{2}s + \sin 2p_0b \sin \frac{\pi}{2}s$$

(4.76)
4.2.8 Total Corrected Diffusion Coefficient

Combining these contributions together one obtains:

\[
D(p, b) = \frac{1}{6} \left( 3K_{eff}^2 - K^2 \sum_s J_{2-2s}(1-b)K J_s((1-b)2Ka)(\cos p_0b \cos \frac{\pi}{2}s + \sin p_0b \sin \frac{\pi}{2}s) 
\right.
\]
\[
+ (4Ka)^2 \sum_s J_{4-2s}(2(1-b)K)J_s((1-b)2Ka)(\cos 2p_0b \cos \frac{\pi}{2}s + \sin 2p_0b \sin \frac{\pi}{2}s) 
\]
\[
- K^2 \sum_s J_{2-2s}((1+b)K)J_s((1+b)2Ka)(\cos 2p_0b \cos \frac{\pi}{2}s - \sin 2p_0b \sin \frac{\pi}{2}s) 
\]
\[
+ (4Ka)^2 \sum_s J_{4-2s}(2(1+b)K)J_s((1+b)2Ka)(\cos 4p_0b \cos \frac{\pi}{2}s - \sin 4p_0b \sin \frac{\pi}{2}s) 
\]
\[
- K^2 \sum_s J_{2-2s}(K)J_s((2Ka)(\cos p_0b \cos \frac{\pi}{2}s + \sin p_0b \sin \frac{\pi}{2}s) 
\]
\[
+ (4Ka)^2 \sum_s J_{4-2s}(2K)J_s((4Ka)(\cos 2p_0b \cos \frac{\pi}{2}s + \sin 2p_0b \sin \frac{\pi}{2}s) \right) 
\]

(4.77)

As the build-up of asymmetry in the system is a short-time effect, terms that are symmetric with respect to momentum over this period can be neglected. The onset of transport is therefore solely attributed to the \sin p_0b dependent terms in the diffusion coefficient. Eq. (4.77) can then be simpli-
4.2. Diffusion in the Double Well Ratchet

\[ D(p, b)_{\text{asym}} = \]
\[ = \frac{K^2}{6} \left( - \sum_{s} J_{2s}((1 - b)K)J_s((1 - b)2Ka) \sin p_0 b \sin \frac{\pi}{2} s \right) \]
\[ + (4a)^2 \sum_{s} J_{2s}((1 + b)K)J_s((1 + b)2Ka) \sin 2p_0 b \sin \frac{\pi}{2} s \]
\[ - (4a)^2 \sum_{s} J_{4s}((1 + b)K)J_s((1 + b)2Ka) \sin 4p_0 b \sin \frac{\pi}{2} s \]
\[ - \sum_{s} J_{2s}((1 + b)K)J_s((1 - b)2Ka) \sin 2p_0 b \sin \frac{\pi}{2} s \]
\[ + (4a)^2 \sum_{s} J_{4s}((1 + b)K)J_s((1 - b)2Ka) \sin 4p_0 b \sin \frac{\pi}{2} s \right) \]  
(4.78)

This formula can now be rearranged to give the total correction to the diffusion coefficient as a function of the three \( \sin pb \) terms present:

\[ D(p, b) = - \frac{K^2}{6} \left[ \sin p_0 b \left\{ \sum_{s} (J_{2s}((1 - b)K)J_s((1 - b)2Ka) \right) \right. \]
\[ + J_{2s}((1 - b)2Ka(1 - b)) \sin \frac{\pi}{2} s \left\{ \sum_{s} (J_{4s}((2K(1 + b))J_s((2K(1 + b))) \right) \right. \]
\[ - \sin 2p_0 b \left\{ \sum_{s} (J_{4s}((2K(1 - b))J_s((2K(1 - b)))) \sin \frac{\pi}{2} s \right\} \right. \]
\[ - \sin 4p_0 b \left\{ \sum_{s} (J_{4s}((2K(1 + b))J_s((2K(1 + b)))) \sin \frac{\pi}{2} s \right\} \right. \]  
(4.79)

The validity of this expression will be demonstrated numerically below.
4.2.9 A Note Concerning Two-Kick Cycles \( (N_{\text{cyc}} = 2) \)

For the double-well ratchet, it has already been noted in the previous chapter that a ratchet effect only occurs when the number of kicks in the cycle is greater than 2. By following through the appropriate analysis, it can easily be shown that this is a direct result of the two correction terms for an \( N_{\text{cyc}} = 2 \) being equal and opposite, hence the net contribution will be zero.

4.3 Derivation of Ratchet Time \( t_r \) in double-well ratchet

The ratchet time is, by definition, the time at which the average current saturates and at which the diffusion rates \( D^+ \) and \( D^- \) equalise. One can therefore either obtain an estimate for the ratchet time from the derivation of the average current or the diffusion coefficient - here the latter approach is taken.

As stated in the derivation above, it is possible to obtain an approximation for the ratchet time by examining the longer time behaviour of the term \( (J_0(bK)J_0(2abK))^4 \) in Eq. (4.46). This is done by performing a sum to a finite time \( N - 2 \) (recall that the above derivation is performed to \( N = i + 2 \)).

One notes however, that each two-kick correlation contains such a term and so the sum to \( N - 2 \) will only address every third kick. This means that the sum effectively couples terms of the form \( (J_0(bK)J_0(2abK))^3 \). Therefore:

\[
\sum_{i=0}^{N-2} \left[ (J_0(bK)J_0(2abK))^3 \right]^i = \frac{1 - [(J_0(bK)J_0(2abK))^3]^{N-1}}{1 - [(J_0(bK)J_0(2abK))^3]} \tag{4.80}
\]

One notes that for short times (low kick number, \( N \)) this sum increases linearly with \( N \), but saturates as \( N \to \infty \) to a constant value. In general, if
4.3. Derivation of Ratchet Time \( t_r \) in double-well ratchet

one defines the term:

\[
\Phi(N, bk, n) = \frac{3}{N} \frac{1 - \left[ (J_0(bnK)J_0(2nabK))^3 \right]^{N-1}}{1 - \left[ (J_0(bnK)J_0(2nabK))^3 \right]}
\]

(4.81)

then the transport in the ratchet is now governed by the form:

\[
C_{\text{asym}}(2, p) = -\frac{K^2}{6} \left[ A\Phi(N, bk, 1) \sin p_0 b + B\Phi(N, bk, 1) \sin 2p_0 b + C\Phi(N, bk, 4) \sin 4p_0 b \right]
\]

(4.82)

where for small \( N \) each \( \Phi(N, nk, n) \) term approximates to unity, leading to a linear correction to the energy growth.

The ratchet time for each individual term is estimated by calculating the time at which \( \Phi(N, bk, n) \) is 95% of its value at \( N = \infty \):

\[
1 - \left[ \frac{(J_0(bK)J_0(2abK))^3}{(J_0(bK)J_0(2abK))^3} \right]^{\frac{95}{100}} = 1 - \frac{5}{100} = \frac{1}{[\Phi(N, bk, n)]^{\frac{95}{100}}}
\]

\[
\Rightarrow t_r - 1 \approx t_r = \frac{\ln(5/100)}{3 \ln((J_0(bK)J_0(2abK)))}
\]

(4.83)

Now using the small argument form of \( J_0 \) one obtains:

\[
J_0(bK)J_0(2abK) = \left( 1 - \frac{(Kb)^2}{4} \right) \left( 1 - \frac{(2abK)^2}{4} \right)
\]

\[
\approx 1 - \frac{(Kb)^2}{4}.(1 + 4a^2)
\]

\[
\Rightarrow \ln \left( 1 - \frac{(Kb)^2}{4}.(1 + 4a^2) \right) \approx -\frac{(Kb)^2}{4}.(1 + 4a^2)
\]

(4.84)

So finally the 'one-kick' ratchet time is given by:

\[
t_r^{(1\text{-kick})} = \frac{\ln(20)}{3\frac{(Kb)^2}{4}.(1 + 4a^2)}
\]

(4.85)
4.4. Derivation of the Average Current in the Double Well

Note that there are three contributions to the ratchet time; one for each of the two-kick correlations. Thus $t_r = 3t_r^{(1-kick)}$.

For $a = 1/2$ (in keeping with the results shown in Fig. 3.12), this simplifies to:

$$t_r = \frac{2 \ln(2)}{(Kb)^2}$$

$$\approx \frac{6}{(Kb)^2}$$

Note that this is the ratchet time for the dominant $\sin pb$ contribution to the diffusion coefficient. The corresponding ratchet times for the $\sin 2pb$ and $\sin 4pb$ terms are as follows:

$$t_r^{(\sin 2pb)} = \frac{2 \ln(20)}{(2Kb)^2} \approx \frac{1.5}{(Kb)^2}$$

$$t_r^{(\sin 4pb)} = \frac{2 \ln(20)}{(4Kb)^2} \approx \frac{0.375}{(Kb)^2}$$

4.4 Derivation of the Average Current in the Double Well

4.4.1 The Zero Order Term

The method used above to obtain the diffusion coefficient can also be used to derive an analytical form for the ratchet current. The average current is once again defined in terms of a conditional probability density:

$$\langle p(t_N) \rangle = \int Q(x_N, p_N, t_N | x_0, p_0, 0) P(x_0, p_0, 0)(p_N - p_0) dx_N dp_N$$

(4.89)
4.4. Derivation of the Average Current in the Double Well

As before:

\[ P(x_0, p_0, 0) = (2\pi)^{-1} \delta(p - p_0), \]

\[ Q(x_N, p_N, N|x_0, p_0, 0) = \sum_{n_{N}=-\infty}^{+\infty} \cdots \sum_{n_{1}=-\infty}^{+\infty} \int_{0}^{2\pi} \frac{dx_{0}}{2\pi} \int_{0}^{2\pi} dx_{1} \cdots \int_{0}^{2\pi} dx_{N-1} \]

\[ \delta(p - p_0) \delta(p - p_0 - S_{N-1}) \delta(x_N - x_{N-1} - p_{N-1} - K \sin x_{N-1} + 2\pi n_{N}) \]

\[ \cdots \delta(p_{1} - p_{0} - S_{0}) \delta(x_{1} - x_{0} - p_{1} - K \sin x_{0} + 2\pi n_{1}) \] (4.90)

By using the Poisson Summation relation one obtains the form:

\[ \langle p \rangle = \sum_{m_{N}=-\infty}^{+\infty} \cdots \sum_{m_{1}=-\infty}^{+\infty} \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dx_{i}}{(2\pi)} S_{N} \]

\[ \exp \left( \sum_{j=1}^{N} \left[ i \imath m_{j}(x_{j+1} - x_{j} - (t_{j+1} - t_{j})(p_{0} + S_{j})) \right] \right) \] (4.91)

where the substitution of \( S \) for the potential terms has been made.

One notes that in this case, when \( m_{j} = 0 \) for all \( j \) the integral vanishes:

\[ \langle p \rangle = \sum_{m_{N}=-\infty}^{+\infty} \cdots \sum_{m_{1}=-\infty}^{+\infty} \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dx_{i}}{(2\pi)} S_{N} \]

\[ = \prod_{i=0}^{N} \int_{0}^{2\pi} \frac{dx_{i}}{(2\pi)} K[\cos(x_{0}) + 2\cos(2x_{0}) + \cdots \cos(x_{N-1}) + 2\cos(2x_{N-1})] \]

\[ = 0 \] (4.92)

Therefore, one deduces that the presence of an average momentum, or current, in the ratchet system is a direct result of the existence of non-zero short time correlations between successive kicks in a sequence. The two-kick contributions are investigated below.

4.4.2 Correlation Between Kicks \( i : i + 2 \)

The derivation of the correlations for the current closely follows that used in the diffusion derivation, and as a result it is unnecessary to derive each
4.4. *Derivation of the Average Current in the Double Well*

individual two-kick term. Thus the following derivation is simply for the
\(i : i + 2\) correlation.

The map is as before:

\[
\begin{align*}
    x_i &= x_{i-1} + p_i(1 + b) \\
    x_{i+1} &= x_i + p_{i+1} \\
    x_{i+2} &= x_{i+1} + p_{i+2}(1 - b)
\end{align*}
\]

\(p_i = x_{i-1} - V'(x_{i-1})\) \hspace{1cm} (4.96)

\(p_{i+1} = x_i - V'(x_i)\) \hspace{1cm} (4.97)

\(p_{i+2} = x_{i+1} - V'(x_{i+1})\) \hspace{1cm} (4.98)

The contribution from the \(\sin x\) part of the potential is found by setting
\(m_{i+2} = \pm 1\) and \(m_{i+1} = -m_{i+2}\) in Eq. (4.91). This also picks out the term:

\(S_N = -K \cos(x_{i+2})\), giving

\[
\langle p \rangle_{m_{i+2}}^{i+2} = -\prod_{i=0}^{N} \int_0^{2\pi} \frac{dx_i}{2\pi} K \cos(x_{i+2}) \cos(z_{i+2}) e^{i(\pm(x_{i+2} - x_{i+1} - (1 - b)(p_0 + S_{i+1}))} .
\]

Taking the case where \(m_{i+2} = -1\):

\[
\langle p \rangle_{m_{i+2}}^{i+2} = -\int_0^{2\pi} \frac{dx_i}{2\pi} K \cos(x_{i+2}) e^{-i(x_{i+2} - 2x_{i+1} + x_i)} e^{-ip_0 e^{-i(1 - b)S_{i+1} + S_i}}
\]

\(4.100\)

The \(S\)-dependent exponentials are treated in the same manner, such that:

\[
S_{i+1}(1 - b) + S_i = -(1 - b)V'(x_{i+1}) - bS_i
\]

\(4.101\)

giving:

\[
\langle p \rangle_{m_{i+2}}^{i+2} = -\int_0^{2\pi} \frac{dx_i}{2\pi} K \cos(x_{i+2}) e^{-i(x_{i+2} - 2x_{i+1} + x_i)} e^{-ip_0 e^{-i(1 - b)V'(x_{i+1})} e^{-iS_i}}
\]

\(4.102\)
4.4. Derivation of the Average Current in the Double Well

One now addresses each $x$-integral in turn. So, for $x_{i+2}$:

$$
\int_0^{2\pi} \frac{dx_{i+2}}{2\pi} \cos(x_{i+2}) e^{-iz_{i+2}} = K \int_0^{2\pi} \frac{dx_{i+2}}{2\pi} \frac{e^{ix_{i+2}} + e^{-ix_{i+2}}}{2} \cdot e^{-iz_{i+2}} = \frac{K}{2} \tag{4.103}
$$

For the $x_{i+1}$ part:

$$
\int_0^{2\pi} \frac{dx_{i+1}}{2\pi} e^{2ix_{i+1}} e^{-i(1-b)K \cos x - i(1-b)2Ka \cos 2x}
= \int_0^{2\pi} \frac{dx_{i+1}}{2\pi} \sum_s i^{-s} J_s(K(1-b)) e^{-ix_{i+1}} \sum_n i^n J_n(2aK(1-b)) e^{-2nx_{i+1}} \tag{4.104}
$$

Now the only non-zero terms occur when:

$$
2x_{i+1} - sx_{i+1} - 2nx_{i+1} = 0
\Rightarrow s = 2 - 2n \tag{4.105}
$$

Therefore the $x_{i+1}$ integral gives a contribution:

$$
I_{x_{i+1}} = \sum_n J_{2-2n}(K(1-b)) J_n(2aK(1-b)) (i^n - 2)^{-2} \tag{4.106}
$$

The integral over $x_i$ gives:

$$
\int_0^{2\pi} \frac{dx_i}{2\pi} e^{-i2\sigma K \cos(x_i)} e^{i2\pi K \sin(x_i)}
= \int_0^{2\pi} \frac{dx_i}{2\pi} e^{-i\sum_s i^n J_n(2abK) e^{2nx_i}} \tag{4.107}
$$

Once again, the exponentials provide a selection rule for the Bessel function orders:

$$
-1 + s + 2n = 0
\Rightarrow s = 1 - 2n \tag{4.108}
$$
4.4. Derivation of the Average Current in the Double Well

Therefore:

\[ I_{x_i} = \sum_n J_{1-2n}(bK)J_n(2abK)i^{n-1} \]  \hspace{1cm} (4.109)

Now, since \(bK \ll 1\) the only term in the sum which is non-vanishing at small argument is one containing \(J_0\) (all other Bessel functions tend to zero as their argument tends to zero). So, the contribution due to \(x_i\) is:

\[ I_{x_i} = J_1(bK)J_0(2abK)i^{-1} \]  \hspace{1cm} (4.110)

The remaining terms in the current integral encompass the \(\exp[-ibS_{i-1}]\) part, each one being of the form:

\[
\int_0^{2\pi} \frac{dx_{i-1}}{2\pi} e^{ibK \cos(x_{i-1})} e^{ib2aK \cos(x_{i-1})} = \int_0^{2\pi} \frac{dx_{i-1}}{2\pi} \sum_s i^s J_s(bK) e^{ix_{i-1}} \sum_n i^n J_n(2abK) e^{i2nx_{i-1}} \]  \hspace{1cm} (4.111)

The selection rule is now simply \(s = -2n = 0\), and using the \(bK \ll 1\) argument one finds that there are effectively \((i-1)/3\) terms of the kind: \(J_0(bK)J_0(2abK)\). Note that as with the diffusion coefficient, this correlation between kicks \(i : i + 2\) addresses only \(1/3\) of the total number of kicks. So the total contribution from the \(J_0\) term is:

\[
\sum_3^\infty \left[ (J_0(bK)J_0(2abK))^3 \right]^{i-1} = \frac{1}{(J_0(bK)J_0(2abK))^3} \sum_3^\infty \left[ J_0(bK)J_0(2abK) \right]^{3i} = \frac{1}{1 - (J_0(bK)J_0(2abK))^3} \]  \hspace{1cm} (4.112)

So the whole average current term for \(m_{i+2} = -1\) is:

\[
\langle p \rangle_{m_{i+2} = -1} = -\frac{K}{2} \frac{J_1(bK)J_0(2abK)}{1 - (J_0(bK)J_0(2abK))^3} \sum_n J_{2-2n}(K(1-b))J_n(2Ka(1-b))i^{n-3} \]  \hspace{1cm} (4.113)
4.4. Derivation of the Average Current in the Double Well

Combining this with the term for \( m_{i+2} = 1 \) one finds that the current due to the \( \sin x \) term resulting from correlations between kicks \( i : i + 2 \) is (noting that this term only affects \( N/3 \) of the \( N \) kicks):

\[
\langle p \rangle_{m_{i+2} = -1}^{i+2} = \frac{K}{6} \frac{J_1(bK)J_0(2abK)}{1 - (J_0(bK)J_0(2abK))^3} \sum_n J_{2-2n}(K(1 - b))J_n(2Ka(1 - b)) \times \{i^{n-3}e^{-ipob} + i^{3-n}e^{ipob} \}
= \frac{K}{6i} \frac{J_1(bK)J_0(2abK)}{1 - (J_0(bK)J_0(2abK))^3} \sum_n J_{2-2n}(K(1 - b))J_n(2Ka(1 - b)) \times \{-i^{n-2}e^{-ipob} + i^{2-n}e^{ipob} \}
= \frac{K}{3} \frac{J_1(bK)J_0(2abK)}{1 - (J_0(bK)J_0(2abK))^3} \sum_n J_{2-2n}(K(1 - b))J_n(2Ka(1 - b)) \times \sin(p_0b - \frac{\pi}{2}n) \quad (4.114)
\]

A similar analysis for the \( K \sin 2x \) part of the potential, using \( m_{i+2} = \pm 2 \) yields the term:

\[
\langle p \rangle_{K \sin 2x}^{i+2} = -\frac{K}{3} \frac{J_0(2bK)J_1(4abK)}{1 - (J_0(bK)J_0(4abK))^3} \sum_n J_{4-2n}(2K(1 - b))J_n(4Ka(1 - b)) \times \sin(2p_0b - \frac{\pi}{2}n) \quad (4.115)
\]

4.4.3 Total Current

As already mentioned, one obtains the other correlation terms, for \( i+1 : i+3 \) and \( i + 2 : i + 4 \), using the same method as above and thus their derivation is not included here. In common with the diffusion coefficient, the main
Comparison with Numerics

4.5.1 Numerical Verification of Adjusted Diffusion Coefficient and Ratchet Time Formula

In order to demonstrate numerically the validity of the formulae derived above, the final average energy after a given number of kicks (less than the ratchet time) is plotted against $p_0$, the average initial momentum. Typical results are shown in Fig. 4.2. The top panel of the figure shows the total energy gain which results from both the non-transporting terms of the diffusion coefficient (i.e., those that are symmetric in momentum) and the transporting inducing terms (those that are asymmetric in momentum). However, one is
Figure 4.2: Average classical energy gain as a function of initial momentum for an ensemble of 1 million particles with $K = 14$, $a = 1/2$, $b=0.005$. The top panel shows the total average energy spread after 100 kicks. In the lower two panels, the momentum-independent contributions have been removed and the energy gain shown after 20 kicks (middle panel - scaled to 100 kicks) and after 100 kicks (bottom panel). A Fourier component analysis of both curves reveals that they can be closely approximated by a series in $\sin p_0 b$. This fit is illustrated by the dashed curves, which have been shifted vertically for clarity.
interested solely in the terms which give rise to an accumulation of asymmetry in the system, about which no information can be gained from the top panel of the figure.

In the remaining two panels, the energy growth which results from the symmetric terms (the quasi-linear term and those dependent on \( \cos pb \)) has

![Graph showing amplitudes of each Fourier component contributing to momentum asymmetry in the ratchet against kick strength.](image)

Figure 4.3: Amplitudes of each Fourier component contributing to momentum asymmetry in the ratchet are plotted against kick strength \( K \). Analytical predictions are compared with numerical results for varying numbers of kicks (all scaled to 100 kicks). One can see in the top panel that the numerical results for the \( \sin(pb) \) term show excellent agreement with the analytical prediction for all \( K \). In the middle panel one notes that after approximately \( K=10 \) the 100 kick curve begins to depart markedly from the analytical prediction. This effect is even more noticeable for \( \sin 4pb \) as shown in the lower panel. These plots show that for increasing kick strength, the time scale over which each term contributes to the final current changes.
been removed and a Fourier series fitted to what remains. An excellent fit is obtained to the form:

\[
\langle E_{\text{asym}} \rangle = A \sin p_0 b + B \sin 2p_0 b + C \sin 4p_0 b \quad (4.117)
\]

where the \( \sin p_0 b \) weighting coefficients in the two panels (approximately 4.7 and 3.8 respectively) compare favourably with that obtained from Eq. (4.79), where \( A = 4.1 \). Such close agreement between numerics and the analytical form confirms that the two-kick correlation is indeed the main cause of asymmetry in the double-well ratchet. Indeed, examining higher order terms one does find higher order Fourier components (\( \sin 8p_0 b, \sin 16p_0 b \) etc.). However the amplitudes are small enough to make negligible difference to the diffusion coefficient.

The graphs also illustrate the time dependence of the various terms in the sine-expansion of Eq. (4.117). In the middle panel the asymmetric energy spread is shown after 20 kicks, whereas in the bottom panel the system has evolved for 100 kicks. Note that in order to aid comparison with the result for 100 kicks, the middle panel has been scaled in energy by a factor of 5. One sees quite clearly that the \( \sin 2p_0 b \) contribution is quite strong after only 20 kicks, but has been damped by an order of magnitude at 100 kicks to leave almost a pure \( \sin p_0 b \) curve.

If one plots the dependence of the relative amplitudes of each Fourier component on kicking strength \( K \) after 20, 40 and 100 kicks, further evidence is seen for the varying timescales associated with each term. The upper graph of Fig. 4.3 demonstrates that the dominant \( \sin p_0 b \) term closely follows the analytical prediction beyond 100 kicks for high kicking strength. In contrast, the \( \sin 2p_0 b \) contribution is still significant at 40 kicks for all \( K \), but has been significantly damped by 100 kicks for \( K > 10 \). As expected,
the ratchet time for the $\sin 2p\theta$ term decreases faster with increasing kick strength than that for $\sin p\theta$ - as predicted by Eq. (4.87). The $\sin 4p\theta$ departs from the prediction even sooner, and at a smaller kicking strength; again as predicted by the theory. In this case, good agreement between numerics and the analytical formulae exists only up to approximately $K=5$ with the 100 kick curve becoming heavily damped soon after. One can clearly see the 40 kick curve departing from the analytical result more quickly and completely than the 20 kick result.

It should be noted that the ratchet times as predicted by Eqs. (4.86), (4.87) and (4.88) for $K = 14$, $a = 1/2$ and $b = 0.005$ (the parameters used in Figs. 4.2 and 4.3) are of the order of 1200 kicks, 300 kicks and 76 kicks respectively. The ratchet times are defined as being the time just before momentum saturation, but the system begins to depart from linear growth much earlier than $t_r$ and it is this effect that is being witnessed in Figs. 4.2 and 4.3.

When considering the ratchet time for the system, it is therefore appropriate to only consider the leading order term, as given by Eq. (4.86). Numerically, the fit to the ratchet time shown in Fig. 3.12 shows excellent agreement with the relationship derived here.

4.5.2 Numerical Verification of Average Current Formula

The analytical form for the average current in the double well ratchet, as given by Eq. (4.116), also shows good agreement with the numerical results detailed in chapter 3. In Fig. 4.4, the numerical results and analytical form are compared as a function of the kick strength, $K$. 

4.5. Comparison with Numerics
Figure 4.4: The top panel shows the full form of Eq. (4.116) fitted against numerical simulation. The middle panel shows the contribution to the fit from the $K \sin x$ term. The bottom panel shows the contribution from $K \sin 2x$. All results are for $b = 0.01$ and $a = 0.5$.

The top panel of the figure shows numerical results for $b = 0.01$, $a = 0.5$ and $p_0 = 0$ fitted against the analytical form for the average current as given by Eq. (4.116). In general the fit is encouraging, demonstrating the appropriate current reversals and overall trends. However there are discrepancies which must be addressed. By examining the individual contributions from each term of Eq. (4.116), one can gain some insight into the relative prominence of each at a given kick strength. The middle panel shows the first term of Eq. (4.116) fitted against the numerics and one can immediately see that the overall shape of the two curves (including the current reversals) is the same. This shows that, for these parameters, the average current is domi-
4.5. *Comparison with Numerics*

In the region where $K > 7$, much of the fine detail of the curve shape is due to the $K \sin x$ term. However, much of the fine detail of the curve shape is due to the $K \sin 2x$ term. In the region $2.6 < K < 7$ competition between the two terms results in the characteristic dips evident in the numerical curve. Also, the two terms combine to produce an excellent fit to the numerics for the region $2 < K < 2.6$.

One major disagreement between the analytical form and the observed current occurs as $K$ is further decreased. The numerical current decreases to zero much more rapidly than the prediction. A possible explanation for this is that the increasing regularity of the phase space as $K$ is decreased inhibits the chaotic diffusion and hence the resulting accumulation of asymmetry in the momentum distribution. It should also be noted that once the kick strength increases beyond $K \approx 2.6$, the numerical current never quite reaches the predicted by Eq. (4.116). This is possibly due to the ever decreasing ratchet time meaning that asymmetry in the system is washed out by the expansion of the classical ensemble of particles. One might expect that the agreement improves as $b \to 0$.

The comparison of the analytical form and numerical results for the current as a function of $b$ and the ratio $a$ are shown in Fig. 4.5. It can clearly be seen from the top panel that Eq. (4.116) predicts exactly the $b$-dependent behaviour demonstrated by the numerical simulation. In the lower panel (where this time $K$ is kept constant and $K_{\text{eff}}$ is allowed to vary with $a$), the fit is less good but still gives confidence in the formula for the current.
4.6. Conclusion

In this chapter, the existence of classical momentum saturation in the double-well ratchet was shown to be due to differential, early-time diffusion rates for particles with positive and negative momenta. These differential rates were shown to be a direct result of previously neglected momentum-dependent corrections to the quasi-linear diffusion rate, arising from correlations between successive kicks in the evolution. These correction terms were successfully obtained for the double well ratchet and the resulting adjusted diffusion coefficient was shown to accurately reproduce classical calculations.

Figure 4.5: The top panel shows excellent agreement between numerical results and analytical prediction for the variation of average current as a function of $b$, where $K = 1.6$ and $a = 0.5$. In the lower panel, numerical and analytical results are compared as the ratio, $a$, is varied whilst $K = 2$ and $b = 0.01$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4_5}
\caption{The top panel shows excellent agreement between numerical results and analytical prediction for the variation of average current as a function of $b$, where $K = 1.6$ and $a = 0.5$. In the lower panel, numerical and analytical results are compared as the ratio, $a$, is varied whilst $K = 2$ and $b = 0.01$.} 
\end{figure}
4.6. Conclusion

Furthermore, analytical forms for the ratchet time and average ratchet current were also derived and shown to give excellent agreement with numerical results.
The Rocking Ratchet

5.1 Introduction

Over the course of the next two chapters, a new type of chaotic Hamiltonian ratchet shall be introduced. This new system bears many similarities to the double-well ratchet studied in the previous chapters, in particular the need to break spatio-temporal symmetry in order to create directed transport. However, whilst the time symmetry of the system is broken in a similar way to the double well case, by once again using a chirped sequence of kicks, spatial asymmetry is introduced in an altogether different manner. The potential used comprises a basic cosine term, in addition to a linear term that changes sign with each successive kick:

\[ V(x) = -(K \cos x + Ax(-1)^j), \quad (5.1) \]

where \( j \) is the kick number.
This rocking ratchet is, as will be demonstrated, analytically and experimentally simpler to study and implement than the double-well case, and one may wonder as to the chronology of the work. In fact, this system was introduced by Cheon et. al. [52], who investigated the classical dynamics in the regular regime, soon after the submission of the now published work on double-well ratchets. Cheon et. al. found that as the system makes the transition from regularity, the resonant tori break asymmetrically - a feature which could be used to create a mixed-phase space ratchet. The motivation for the study of this system was to investigate not only whether it would admit a ratchet effect in the fully chaotic regime (and hence show that the results for the double-well are more generic), but also to see if any further light could be shed on the role of correlations in determining diffusion rates and average currents in chaotic ratchet systems.

The chapter begins by briefly covering the classical and quantum dynamics before showing that the system does indeed demonstrate a ratchet effect. Representative numerical results are used to show that the system obeys the principles of chaotic Hamiltonian ratchet dynamics (as found for the double-well), removing the need for a full parameter survey.

The most striking result to emerge from the system occurs when one considers the case where the initial classical ensemble has non-zero average momentum. The rocking ratchet can be used to preferentially select atoms moving in one particular direction from those moving in the opposite direction, creating a type of 'chaotic filter'. By considering the average energy spread as a function of initial momentum it will be shown that the filtering effect is a direct result of the diffusion coefficient, and how by careful selection of the system parameters one can tune the filter to select atoms of any given momentum.
5.2 Theoretical Analysis

In general, the theoretical framework used to analyse the rocking ratchet is the same as that for the double well as detailed in chapter 3. There are, of course, some differences which must be addressed and these are outlined below for both the classical and quantum dynamics.

5.2.1 Classical Dynamics

Symmetries of the System

The rocking ratchet, in keeping with the double-well system, requires that both spatial and temporal symmetries must be broken. Spatial symmetry is broken by the use of the following potential:

\[ V(x) = -(K \cos x + Ax(-1)^j), \]  

where \( A \) is effectively the amplitude of the rocking linear term, and \( j \) is the kick number.

Temporal asymmetry is once again introduced by using a chirped sequence of kicks, which can be represented by the general form:

\[ f(t) = \sum_{s=0}^{\infty} \sum_{M=1}^{N} \delta \left( t - \left( sT_{\text{tot}} + \sum_{i=1}^{M} T_i \right) \right), \]  

However, in contrast to the double-well case, a ratchet effect can be observed for a two-kick \((N = 2)\) sequence: \( T_i = 1 + b, T_{i+1} = 1 - b \). One recalls that for the double-well a two-kick sequence yields vanishing corrections to the diffusion coefficient, however the presence of the alternating linear term means that this is no longer the case for the rocking ratchet. Since the motivation for the study of this system is its relative simplicity, the analysis is carried out exclusively for the case of \( N = 2 \).
5.2. Theoretical Analysis

The Classical Map

Given the potential of Eq. (5.2), the dimensionless Hamiltonian for the rocking system is (for $N = 2$):

$$H = \frac{p^2}{2} - \sum_{j=0}^{\infty} \left[ K \cos x + A x (-1)^j \right] \sum_{s=0}^{\infty} \sum_{M=1}^{2} \delta \left( t - \left( s T_{\text{tot}} + \sum_{i=1}^{M} T_i \right) \right), \quad (5.4)$$

where as before $b$ is a small perturbation from period one kicks.

![Figure 5.1: Poincaré Surfaces of Section for the rocking ratchet as the kicking strength, $K$, is increased.](image)

Hamilton’s equations of motion for the system are therefore:

$$\frac{\partial H}{\partial p} = \dot{x} = p,$$

$$\frac{\partial H}{\partial x} = -\dot{p} = V'(x) f(\tau).$$
5.2. Theoretical Analysis

By integrating these equations in the same manner as for the double well case, one obtains the following two-kick map:

\[
\begin{align*}
    p_j &= p_{j-1} + K \sin x_j + A \\
    x_j &= x_{j-1} + p_{j-1}(1 + b) \\
    p_{j+1} &= p_j + K \sin x_{j+1} - A \\
    x_{j+1} &= x_j + p_j(1 - b)
\end{align*}
\]  

Figures 5.1 and 5.2 show how the resultant phase spaces changes as the system parameters are varied. Note that for \( A = 0 \), the system effectively becomes a ‘chirped’ kicked rotor, and the phase space bears many similarities with that shown in chapter 2.

\[\text{Figure 5.2: Poincaré Surfaces of Section} \]

As expected, the system becomes increasingly chaotic as the kick strength
5.2. Theoretical Analysis

is increased. One notes that for non-zero \( A \) the phase space appears asymmetric in each case, however when \( A = \pi \) there is in fact a form of symmetry. If one reflects the islands present about the \( p = 0 \) axis and then performs a translation in \( x \), one recovers an SOS picture with islands at positions identical to those in the figure. One can therefore deduce that no asymmetry in momentum will accumulate for \( A = \pi \) and indeed this will be shown to be the case below.

Classical Simulation

The classical system is once again simulated by the use of a classical gaussian ensemble, comprising of the order of 1 million particles. The average momentum and energy for the system are then obtained as before:

\[
\langle E_n \rangle = \frac{1}{2M} \sum_{m=1}^{M} p_{n,m}^2 \\
\langle p \rangle = \frac{1}{M} \sum_{m=1}^{M} p_{n,m}
\] (5.6) (5.7)

5.2.2 Quantum Dynamics

Evolution Operator for the Rocking Ratchet

The quantum dynamics of the system are once again investigated by the action of a matrix representation of the time evolution operator, \( U \), on the amplitude coefficients according to:

\[
A_i^{(n+1)} = \sum_l U_{ml} A_l^{(n)}
\] (5.8)

where \( m, l \) refer to the usual plane wave basis \( |l> = \frac{1}{\sqrt{2\pi}} e^{itx} \). As before, one uses a split-operator of the form:

\[
\hat{U} = e^{-\frac{i}{\hbar} V(x)} e^{-i\frac{p^2}{2\hbar} \tau}
\] (5.9)
5.2. Theoretical Analysis

It then remains to find the Heisenberg matrix elements, $U_{lm}$, for the rocking ratchet.

**Matrix Elements**

In the case of the rocking ratchet the evolution operator contains two distinct parts, one corresponding to each kick in two-kick cycle:

$$U^{\text{tot}}(T, 0) = U^{\tau_2} \cdot U^{\tau_1}$$  \hspace{1cm} (5.10)

where $T = \tau_1 + \tau_2$.

The wavefunction can be expanded in the plane-wave basis:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} C(p) e^{ipx} dp$$  \hspace{1cm} (5.11)

which can be discretised to:

$$|\psi\rangle = \sum_i C(p_i)|p_i\rangle \equiv \sum_{l,q_i} C_{qi}|l + q_i\rangle$$  \hspace{1cm} (5.12)

where $l$ is an integer ($l = 0, \pm 1, \pm 2, \ldots$) and $q$ is a fractional number, the 'quasi-momentum', which lies in the range $-0.5 < q < 0.5$.

The orthogonality of the plane-wave basis ($\langle p|p'\rangle = \delta(p - p')$) in this notation becomes:

$$\langle l + q|l' + q'\rangle = \delta(l - l')\delta(q - q')$$  \hspace{1cm} (5.13)

in other words integer and fractional parts must vanish separately.

In order to obtain the matrix elements for the rocking ratchet, one must evaluate:

$$\langle n + q''|U(T, 0)|l + q\rangle = \sum_{\alpha} \langle n + q''|U^{\tau_2}|\alpha\rangle \langle \alpha|U^{\tau_1}|l + q\rangle$$  \hspace{1cm} (5.14)
5.2. Theoretical Analysis

where \( |\alpha\rangle = |j + q\rangle \). One begins by considering the action of \( U^n \). The operator is split in the usual manner:

\[
U^n |l + q\rangle = e^{-i \frac{K}{\hbar} \cos z - \frac{1}{2} x} e^{-i \frac{q^2}{\hbar} n} |l + q\rangle
\]  

(5.15)

The kinetic energy part then gives:

\[
e^{-i \frac{q^2}{\hbar} n} |l + q\rangle = e^{-i \frac{\alpha + q^2}{2} n} |l + q\rangle
\]  

(5.16)

For the potential energy part one recalls that:

\[
|l + q\rangle = \frac{1}{\sqrt{2\pi}} e^{i(l+q)x}
\]  

(5.17)

and that:

\[
e^{-i \frac{K}{\hbar} \cos z} = \sum_{m=-\infty}^{\infty} i^{-m} J_m(K/\hbar) e^{-imx}
\]  

(5.18)

Thus in total one obtains:

\[
U^n |l + q\rangle = e^{-i \frac{\alpha + q^2}{2} n} \sum_m i^{-m} J_m(\tilde{K}/\hbar) \frac{e^{i(l+q-m-\tilde{\alpha})x}}{\sqrt{2\pi}}
\]  

(5.19)

where \( \tilde{\alpha} = A/\hbar \) and \( \tilde{K} = K/\hbar \). The matrix elements of \( U^n \) are now:

\[
\langle j + q' | U^n | l + q\rangle = \sum_m e^{-i \frac{\alpha + q^2}{2} n} i^{-m} J_m(\tilde{K}) \langle j + q' | l + q - m - \tilde{\alpha}\rangle
\]  

(5.20)

The overlap integrals are now split into integer and fractional parts as follows. One writes:

\[
q - \tilde{\alpha} = I_a + q_a
\]  

(5.21)

where \( I_a \) represents the integer part and \( q_a \) is the fractional part, with \( |q_a| \leq 0.5 \). Using the selection rule from Eq. (5.13) one can then write:

\[
j = l + I_a - m \quad \Rightarrow \quad m = l - j + I_a
\]  

\[
q' = q_a
\]  

(5.22)
5.2. Theoretical Analysis

resulting in:

$$\langle j + q'|U_{r_2}|l + q \rangle = e^{-i\frac{(j+q)^2}{2}\tau} e^{-iL_0} J_{l-j+L_0}(\tilde{K})$$  \hspace{1cm} (5.23)$$

Similarly, for the second part of the evolution operator, \(U_{r_2}\), one finds:

$$U_{r_2}|j + q'\rangle = e^{-i\frac{(j+q')^2}{2}\tau} \sum_s e^{-i\tau(j+q' - s + \tilde{A})} J_s(\tilde{K})$$

and therefore:

$$\langle n + q''|U_{r_2}|j + q'^\rangle = \sum_s e^{-i\frac{(j+q'^s)^2}{2}\tau} \sum_{s'} e^{-i\tau(j+q' - s + \tilde{A})} J_s(\tilde{K}) \langle n + q''|j + q' - s + \tilde{A}\rangle$$  \hspace{1cm} (5.25)$$

Since the matrix product is being evaluated as in Eq. (5.14), the second
selection rule from Eq. (5.22) can be used such that:

$$\langle n + q''|U_{r_2}|j + q'^\rangle \delta_{q'q_a} = \sum_s e^{-i\frac{(j+q'^s)^2}{2}\tau} \sum_{s'} e^{-i\tau(j+q' - s + \tilde{A})} J_s(\tilde{K}) \langle n + q''|j + q_a - s + \tilde{A}\rangle$$  \hspace{1cm} (5.26)$$

Now one requires:

$$\tilde{A} + q_a = \tilde{A} + (q - \tilde{A} - I_a)$$

$$= q - I_a$$  \hspace{1cm} (5.27)$$

therefore:

$$\langle n + q''|j - s + \tilde{A} + q_a\rangle = \langle n + q''|j - s - I_a + q\rangle$$  \hspace{1cm} (5.28)$$

Thus one finds:

$$q'' = q$$

$$n = j - s - I_a \Rightarrow \ i^{-s} = \tau^{n-j+L_0}$$  \hspace{1cm} (5.29)$$

The first selection rule above shows that quasimomentum is conserved over
a two-kick sequence for the rocking ratchet.
5.3. Numerical Results for zero initial momentum

Finally one obtains the matrix elements for the full time evolution over two kicks to be:

\[
\langle n + q|U(T,0)|l + q \rangle = \sum_{j} \langle (n + q|U_{\tau_2}|j + q_a)(j + q_a|U_{\tau_1}|l + q) \\
= e^{i(e_{\tau_2} + \frac{1}{2}g_{\tau_1}^2)} \sum_{j} a_{n-l-j+1a}(K)J_{j-n-l_a}(K)e^{-i(e_{\tau_1} + \frac{1}{2}g_{\tau_2}^2)}
\]

(5.30)

Quantum Wavepacket and Observables

The quantum system is once again evolved in the angular momentum basis using a minimum uncertainty wavepacket, to obtain the expectation values at the n-th kick for the average energy and momentum:

\[
\langle E(n) \rangle = \langle \Psi_n | \frac{\hat{p}^2}{2} | \Psi_n \rangle = \frac{1}{2} \sum_{l,q} (l + q)^2 \hbar^2 |A_{l,q}^n|^2 \Delta q
\]

(5.31)

\[
\langle p(n) \rangle = \langle \Psi_n | \hat{p} | \Psi_n \rangle = \sum_{l,q} (l + q)\hbar |A_{l,q}^n|^2 \Delta q
\]

(5.32)

where, as with the double-well, the quasi-momentum has been discretised with step size \( \Delta q \).

5.3 Numerical Results for zero initial momentum

One motivation for studying this rocking system is its relative simplicity when compared with the double well ratchet discussed in chapters 2-4. As a result, the major focus of the work was directed towards obtaining accurate analytical formulae for the observed diffusion and ratchet current. In this way, it was hoped to further the understanding of the underlying processes.
5.3. Numerical Results for zero initial momentum

involved in chaotic Hamiltonian ratchets. However, it must first be demonstrated that the rocking system produces a ratchet effect, and as such, show that the chaotic Hamiltonian ratchet model is indeed generic.

5.3.1 Momentum distributions

It has been demonstrated in the chapters concerning the double-well that the ratchet effect results from an accumulation of asymmetry in the momentum distribution. It is therefore appropriate to demonstrate that this situation is repeated in the rocking system. In Fig. 5.3 one sees the effect on the first moment of the momentum distribution of independently varying the kick strength and the amplitude of the rocking linear term. It is immediately clear that for certain values of each parameter, the first moments are highly asymmetric (implying a large ratchet current) and that increasing $K$ and $A$ induces current reversals.

5.3.2 Average Momentum and Energy

Classical results: vary $K$, $A$

One recalls that the ratchet signature is the saturation of the momentum asymmetry to a finite average momentum, or current, after a characteristic time, the ratchet time $t_r$. The top panel of Fig. 5.4 shows for a selection of values of the kicking strength, $K$, that a non-zero ratchet current does indeed occur in this system. Note that, as with the double well case, there is not a simple $K$-dependence for the ratchet current. In this case, the magnitude of the current decreases as the kicking strength is increased from $K = 2.6$ with a current reversal clearly having occurred by $K = 5$ and a large positive current persisting for $K = 7$. Note that by $K = 9$ a second current reversal has taken
Figure 5.3: Classical momentum distributions for varying $K$ and $A$. One clearly sees a large asymmetry (and therefore ratchet current) for $K = 3.6$ with a current reversal occurring at $K = 5.0$. A similar effect is seen as $A$ is increased, with a current reversal occurring for $A = \pi/2$.

place. One also observes that the ratchet time decreases dramatically as the kick strength is increased.

The lower panel shows that as the amplitude of the rocking linear term is increased from $A = 1.6$ to $A = 5.2$, a current reversal is once again observed. The detailed nature of the dependence of the current on the system parameters will be clarified in the derivation of its analytical form in the next chapter. It should be noted that the dependence of the average current on the chirp parameter, $b$, is in keeping with the behaviour found for the double-well system; therefore no explicit results are shown.
Figure 5.4: Average current for various values of $K$ and $A$. The figure shows clearly that the rocking system produces a ratchet effect, and that the final finite current is dependent on both the kicking strength, $K$, and the amplitude of the rocking linear term, $A$.

**Quantum results for average energy and momentum**

In Fig. 5.5 one observes that dynamical localisation occurs in the rocking ratchet system. As expected, the break-time increases with the kicking strength. The lower panel shows the existence of a finite quantum current, with saturation occurring as a result of asymmetry being frozen in with the onset of localisation.
5.3. Numerical Results for zero initial momentum

Figure 5.5: Average quantum energy and momentum for various values of $K$, showing localisation of the quantum energy and the persistence of a finite average current. In both cases, $b = 0.02$ $A = 2$ and $h = 0.5$.

5.3.3 Timescales

In chapter 3 it was shown numerically (and later proved analytically) that the ratchet time, $t_r \propto 1/(Kb)^2$. For the rocking ratchet, the relationship is found to be:

$$t_r \approx \frac{3}{(bK)^2}$$  \hspace{1cm} (5.33)

Again, this result will be derived analytically in the next chapter.

One can show that Eq. (5.33) holds empirically by considering the two panels of Fig. 5.6. The upper panel of shows the expected $1/K^2$ trend quite clearly, whilst the lower panel shows explicitly the relationship between $t_r$ and $b$. The ratchet time is once again taken to be when the current has
5.4 Rocking Ratchet with Non-zero Initial Momentum

5.4.1 The Chaotic Filter Effect

The previous sections show that for zero initial momentum, the rocking system does indeed admit a ratchet effect. However, the most striking result to emerge from this study occurs for non-zero starting momentum. In common
5.4. Rocking Ratchet with Non-zero Initial Momentum

Figure 5.7: Quantum (solid line) and classical (dotted line) average energy gain for the rocking ratchet. In the top panel, the classical system was allowed to evolve for 60 kicks, whilst the quantum system evolved for 200 kicks. These values were chosen so that the maximum value for both cases was approximately the same. In the bottom panel the results are shown after 100 kicks for both cases.

with the double-well ratchet, the ratchet effect in the rocking system is a direct result of the diffusion rate being momentum dependent. This can be seen quite clearly by plotting numerically the average classical and quantum energy gains as a function of the initial momentum, shown in Fig. 5.7. One notes immediately that the shape of the curve is simpler than the equivalent for the double-well, oscillating as a function of $p$ with approximately sinusoidal form of period $\pi/b$. In the next chapter this behaviour will be derived analytically. The origin of the ratchet effect for zero initial momentum can immediately be attributed to $p_0 = 0$ lying at a point on the curve with max-
Figure 5.8: The figure demonstrates the chaotic filtering effect for two parameter sets. The initial wavepackets (black line) are centred at $\pm \pi/4b$. One can clearly see from the final states (red lines) that the wavepacket moving to the right is only slightly damped, whilst the one moving to the left is greatly dispersed. The system was allowed to evolve for 100 kicks to reach the final wavepackets shown.

The simple nature of the energy gain (diffusion) curve means that it is possible to accurately predict how the wavepacket will spread for any
5.4. Rocking Ratchet with Non-zero Initial Momentum

given initial momentum. A wavepacket with initial momentum centred at a minimum of the diffusion curve will absorb little energy, and be virtually unaffected after a large number of kicks, whilst a wavepacket centred at a diffusion maximum will be strongly dispersed. This effect is shown for two different parameter sets in Fig. 5.8, where the quantum wavepacket moving to the right is virtually unperturbed and the one moving to the left is heavily damped.

In Fig. 5.9 the momentum-selection effect of Fig. 5.8(b) is presented with the appropriate energy gain curves, clearly demonstrating the origin to be the differential diffusion rates at the given starting momenta. As a result of this preferential selection of packets with a given initial momentum, this system is termed the 'chaotic filter'.

5.4.2 Enhancement of Filtering by Dynamical Localisation

One notes that in Fig. 5.7, the minimum of the quantum energy spread is lower than that of the classical energy spread. From the discussion of the quantum results for the double-well system one recalls that the break-time, $t^*$, is proportional to the diffusion coefficient. Ordinarily it is sufficient to use the quasi-linear approximation for calculating the break-time, however in this case it is this dependence on the diffusion coefficient that causes the observed differences between the classical and quantum energy spreading. At a maximum of the diffusion coefficient the break-time will be longer, meaning that the system will take longer to localise than at a minimum, where the break-time will be comparatively short. Thus the system absorbs energy for longer at a diffusion maximum and for a shorter time at a diffusion
minima, when compared with the classical case. Effectively, the break-time is dependent on the initial momentum:

\[ t^*(p_0) \approx \frac{D(p_0)}{\hbar^2} \] (5.34)

Qualitatively, if the ratio of the maxima and minima for the classical energy spread is \( D_{\text{max}}/D_{\text{min}} \), then the similar ratio for the quantum case is: \( (D_{\text{max}}/D_{\text{min}})^2 \).

Figure 5.9: Shows the origin of the filtering effect to be due to the relative diffusion rates in the system. A wavepacket started at an initial momentum corresponding to a maxima of the diffusion coefficient will have substantially increased average kinetic energy, whilst a wavepacket started at a minima will absorb comparatively little energy.
5.4. Rocking Ratchet with Non-zero Initial Momentum

Figure 5.10: The figure shows the classical and quantum energy growth for wavepackets started at $p_0 = \pi/4b$ (red lines) and $p_0 = -\pi/4b$ (black lines). The wavepacket started at $p_0 = -\pi/4b$ takes much longer to localise, clearly demonstrating the momentum dependent nature of the break-time.

In Fig. 5.10 one sees further evidence for the enhancement of the chaotic filtering effect by the action of dynamical localisation. The classical and quantum energy growth is plotted for two wavepackets, one centred around $p_0 = \pi/(4b)$ and the other around $p_0 = -\pi/(4b)$ where $K = 3.2$, $b = 0.01$, $\hbar = 1$ and $A = \pi/2$. Clearly the quantum energy growth for the latter wavepacket, which is subjected to the maximum of the diffusion curve (c.f. Fig. 5.9), takes a much longer time to localise, showing the momentum dependent nature of the break-time.
5.5 Conclusion

In this chapter a second chaotic Hamiltonian ratchet system has been introduced. This ratchet uses a rocking linear term to induce spatial asymmetry and, in common with the double-well ratchet, a chirped sequence of kicks to introduce temporal asymmetry. In the limit of zero initial momentum, the system has been shown to admit a ratchet effect - thus demonstrating that the chaotic Hamiltonian ratchet model is generic.

The ability to create a chaotic filter for moving atoms has also been demonstrated. This phenomenon was shown to be a direct result of the simple sinusoidal nature of $D(p_0)$. One surprising result is that dynamical localisation can actually be used to enhance the filtering effect.

In the next chapter analytical forms for the diffusion coefficient and average current are derived. It will be seen that the simpler form of the rocking ratchet correlations allow a more in-depth analysis to be performed, relative to the double-well system.
CHAPTER 6

The Ratchet Current and Diffusion in the Rocking Ratchet

6.1 Introduction

In the last chapter it was shown that by using a rocking cosinusoidal potential one may create another instance of a chaotic Hamiltonian ratchet. The ratchet effect was again shown to be due to an accumulation of asymmetry in the momentum distributions, resulting in a net classical current. Furthermore, the classical and quantum results obtained in the limit of zero initial momentum demonstrated that the timescales for this new ratchet, the classical ratchet time and quantum break-time, have the same dependence on the system parameters as for the double-well case. In Fig. 6.1, one sees evidence
6.1. Introduction

that (as in the double-well ratchet) the existence of a finite classical current is due to momentum-dependent short time diffusion.

In this chapter, analytical forms for both the diffusion coefficient and ratchet current in the rocking ratchet are obtained using the same method as detailed in chapter 4. These are then compared extensively with numerical results.

![Graph showing differential energy growth](image)

Figure 6.1: Shows the differential energy growth for particles with positive (dashed line) and negative (solid line) momenta. The red line confirms that after the ratchet time, the corrections to the diffusion coefficient vanish, and the energy growth becomes quasi-linear again.
6.2 Derivation of the Classical Diffusion Coefficient

The derivation of the diffusion coefficient for the rocking ratchet follows the same method as that for the double well case covered in chapter 4. In fact, as stated earlier, the derivation (and hence the resulting diffusion coefficient) are much simpler than that of the double well.

One begins by recalling the map:

\[
\begin{align*}
p_j &= p_{j-1} + K \sin x_j + A \\
x_j &= x_{j-1} + p_{j-1}(1 + b) \\
p_{j+1} &= p_j + K \sin x_{j+1} - A \\
x_{j+1} &= x_j + p_j(1 - b)
\end{align*}
\]  
(6.1)

The 'rocking' sign of \( A \) means that one must be careful when dealing with the iteration of the momentum variable. Each iteration will either mean that there are an odd number of \( A \)'s in the summation (in which case one single instance of \( A \) will be present), or an even number (in which case \( A \) will not appear in the momentum summation). In other words there are two cases:

\[
\begin{align*}
p_j &= \sum_{i=1}^{j} K \sin x_i + p_0 = S_j + p_0 \\
p_{j-1} &= \sum_{i=1}^{j-1} K \sin x_i + A + p_0 = S_{j-1} + A + p_0,
\end{align*}
\]  
(6.2)

where it should be noted that \( S \) is defined not as the sum over the derivative of the potential, but as:

\[
S_j = \sum_{i=1}^{j} \sin x_i
\]  
(6.3)
6.2. Derivation of the Classical Diffusion Coefficient

As with the derivation for the double well case, the diffusion coefficient can be calculated as:

\[ \langle \frac{(p - p_0)^2}{2} \rangle = D(t)t, \]  

(6.4)

where one defines the diffusion coefficient in terms of the conditional probability that the system evolves to a state \((x_N, p_N)\) at time \(t = t_N\):

\[ D = \frac{1}{2N} \int Q(x_N, p_N, t_N|x_0, p_0, 0)P(x_0, p_0, 0)(p_N - p_0)^2dx_Ndp_N \]  

(6.5)

Using the same initial conditions as the double well case;

\[ P(x, p, 0) = (2\pi)^{-1}\delta(p - p_0), \]

and substituting the map into Eq. 6.5 one obtains:

\[ Q(x_N, p_N, t_N|x_0, p_0, 0) = \sum_{n_N=0}^{+\infty} \cdots \sum_{n_1=0}^{+\infty} \int_0^{2\pi} \int_0^{2\pi} \delta(p - p_0) \int_0^{2\pi} dx_1 \cdots \int_0^{2\pi} dx_{N-1} \]

\[ \delta(p_N - p_0 - S_{N-1})\delta(x_N - x_{N-1} - (1 - b)(p_0 + S_N + A) + 2\pi n_N) \cdot \]

\[ \delta(x_{N-1} - x_{N-2} - (1 + b)(p_0 + S_{N-1}) + 2\pi n_{N-1}) \cdot \]

\[ \delta(x_2 - x_1 - (1 - b)(p_0 + S_1 + A) + 2\pi n_1)\delta(x_1 - x_0 - (1 + b)p_0 + 2\pi n_1) \]

(6.6)

Once again the Poisson summation formula (Eq. 4.10) is used to turn the \(\delta\)-functions into exponentials; so one can now write the diffusion coefficient as:

\[ D = \frac{1}{2N} \sum_{m_N=-\infty}^{\infty} \cdots \sum_{m_1=-\infty}^{\infty} \prod_{i=0}^{N-1} \int_0^{2\pi} \frac{dx_i}{(2\pi)} S_N^2 \]

\[ e^{im_N(x_N - x_{N-1} - (1 - b)(p_0 + S_N + A))}e^{im_{N-1}(x_{N-1} - x_{N-2} - (1 + b)(p_0 + S_{N-1}))} \cdots \]

\[ e^{im_2(x_2 - x_1 - (1 - b)(p_0 + S_1 + A))}e^{im_1(x_1 - x_0 - (1 + b)p_0)} \]

(6.7)
6.2. Derivation of the Classical Diffusion Coefficient

As expected, by setting $m_j = 0$ for all $j$ one recovers the quasi-linear diffusion rate; $D_{ql} = K^2/4$. Once again, the main corrections to $D_{ql}$ in the rocking ratchet arise from two kick correlations. As with the double well, one must address each correlation in the sequence to build up a picture of how diffusion is affected over an extended kicking period. In this case, with a 2-kick cycle, one need only address two pairs of kicks; using an arbitrary starting point $i$ in the sequence, correlations between kicks $i : i + 2$ and $i + 1 : i + 3$ are investigated. Due to the comparatively simple form of the derivation for the rocking case it is possible to look beyond the 2-kick correlations and obtain contributions from 3-kick correlations, as shown later. As stated earlier in chapter 4, this is possible for the double-well case, but the analysis is complicated and yields no new information concerning the ratchet effect - the contribution from the two kick correlation is sufficient to satisfactorily explain the observed momentum-dependent diffusion rates.

6.2.1 Correlation between kicks $i : i + 2$

The three kick map is defined as follows:

\[
\begin{align*}
p_i &= p_{i-1} + K \sin x_{i-1} + A \\
x_i &= x_{i-1} + p_i(1 - b) \\
p_{i+1} &= p_{i} + K \sin x_{i+1} - A \\
x_{i+1} &= x_i + p_{i+1}(1 + b) \\
p_{i+2} &= p_{i+1} + K \sin x_{i+1} + A \\
x_{i+2} &= x_{i+1} + p_{i+2}(1 - b)
\end{align*}
\]

The generalised form for the two-kick correlation is now (setting $m_{i+2} = \ldots$)
6.2. Derivation of the Classical Diffusion Coefficient

\[ \pm 1 \text{ and } m_{i+1} = -m_{i+2} : \]

\[ C(2, p)^{i+2} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2} - x_i + 1 - (t_{i+2} - t_i))(p_0 + S_{i+1} + A)} . \]

Looking specifically at the case for \( m_{i+2} = -1, m_{i+1} = 1 \) and substituting for the time intervals one finds:

\[ C(2, p)^{i+2}_{m_j=1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{-i(x_{i+2} - x_i + 1 - (1-b)(p_0 + S_{i+1} + A))} . \]

which simplifies to:

\[ C(2, p)^{i+2}_{m_j=1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{-i(x_{i+2} - x_i + 1)(p_0 + S_{i+1} + A)} . \]

\[ e^{i(x_{i+1} - x_i - (1+b)(p_0 + S_i))} \]

The \( S^2 \)-exponent is dealt with as in chapter 5, to give:

\[ C(2, p)^{i+2}_{m_j=-1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{-i(x_{i+2} - x_i + 1)(p_0 + S_{i+1} + A)} . \]

Combining the terms in \( x_2 \) one obtains a second order Bessel function:

\[ C(2, p)^{i+2}_{m_j=-1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{-i(x_{i+2} + x_i)} . \]

\[ J_2(K(1 - b)) e^{i2Kb\sin x_i} e^{i2KbS_{i-1}} e^{i(1-b)A}. \]

One now combines the \( e^{i(x_{i+2} + x_i)} \) term with the sin \( x_i \sin x_{i+2} \) part of the \( S^2 \) term in the usual manner. The \( e^{i2Kb\sin x_i} \) term then yields a single zero-order Bessel function, such that one is left with:

\[ C(2, p)^{i+2}_{m_j=-1} = \frac{1}{2N} e^{i(1-b)A-2p_0b} J_0(2Kb)J_2(K(1 - b)) \int_0^{2\pi} \frac{dx_i - 1}{2\pi} e^{i2KbS_{i-1}} . \]
6.2. Derivation of the Classical Diffusion Coefficient

Once again the exponential term in $S_{i-1}$ effectively yields a series of zero-order Bessel functions, $(J_0(2Kb))^i$ from which one can obtain the time dependence of the system. As with the double well, since $bK \ll 1$ these Bessel functions can be approximated to unity for $i - 1$ not too large.

Combining this with the similar expression for $m_{i+2} = 1, m_{i+1} = -1$ and performing the summation over all terms, $N$, one obtains the final form:

$$C(2, p)^{13} = \frac{K^2}{2} J_2(K(1 - b))J_0(2bK)\cos((1 - b)A - 2p_0b) \quad (6.15)$$

Note that the factor $1/2$ appears since this particular correlation type only addresses half the kicks in the sequence.

6.2.2 Correlation between kicks $i + 1 : i + 3$

The three kick map encompassing kicks $i + 1$ and $i + 3$ is:

$$p_{i+1} = p_i + K\sin x_i - A$$
$$x_{i+1} = x_i + p_{i+1}(1 + b)$$
$$p_{i+2} = p_{i+1} + K\sin x_{i+1} + A$$
$$x_{i+2} = x_{i+1} + p_{i+2}(1 - b)$$
$$p_{i+3} = p_{i+2} + K\sin x_{i+3} - A$$
$$x_{i+3} = x_{i+2} + p_{i+3}(1 + b) \quad (6.16)$$

By setting $m_{i+3} \pm 1$ and $m_{i+2} = -m_{i+3}$ in Eq. (6.7) one obtains the generalised form for the two-kick correlation to be:

$$C(2, p)^{i+1;i+3} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_{N}^{2} e^{\pm i(x_{i+3} - x_{i+2} - (t_{i+3} - t_{i+2})(p_0 + S_{i+2}))} .$$

$$e^{\mp i(x_{i+2} - x_{i+1} - (t_{i+2} - t_{i+1})(p_0 + S_{i+1} + A))} \quad (6.17)$$
6.2. Derivation of the Classical Diffusion Coefficient

For \( m_{i+3} = -1, m_{i+2} = 1 \):

\[
C(2, p)^{i+1:i+3} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{-i(x_{i+3} - x_{i+2} - (1+b)(p_0 + S_{i+2}))} \cdot e^{i(x_{i+2} - x_{i+1} - (1-b)(p_0 + S_{i+1} + A))} \quad (6.18)
\]

which can be simplified to:

\[
C(2, p)^{i+1:i+3} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{-i(x_{i+3} - 2x_{i+2} + x_{i+1})} e^{i2p_0b} \cdot e^{-i(1+b)K \sin x_{i+1} e^{i2KbS_1} e^{i(1-b)A}} \quad (6.19)
\]

having taken into account the \( S \)-dependent exponent as before.

Following through the rest of the analysis in the usual way, the total term for the correlation between kicks \( i + 1 : i + 3 \) is:

\[
C(2, p)^{i+1:i+3} = \frac{K^2}{2} J_2(K(1 + b))J_0(2bK) \cos(2p_0b - (1 - b)A) \quad (6.20)
\]

Therefore the total correction term resulting from the two-kick correlations is:

\[
C(2, p) = K^2 J_0(2bK) \cos(2p_0b - (1 - b)A)[J_2(K(1 + b)) + J_2(K(1 - b))] \quad (6.21)
\]

6.2.3 Further Correction Terms

In the case of the rocking ratchet where the analysis is somewhat simpler, it is possible to investigate correlations beyond the two-kick case. It should be noted however that most of these additional terms have a negligible effect on the diffusion coefficient, and it is in general still acceptable to just consider the corrections due to two kick correlations.

The first two extra terms to consider arise from three-kick correlations. Consider the case where in the exponential of Eq. (6.7) one has \( m_{i+2} = \pm 1, \)
6.2. Derivation of the Classical Diffusion Coefficient

$m_{i+1} = 0$ and now $m_i = 1$. One therefore has:

$$C(3, p)^{i+2} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 \exp \left( i (x_{i+2} - x_{i+1} - (1-b)(x_0 + S_{i+1} + A)) \right) \cdot$$

$$e^{i(x_i - x_{i-1} - (1-b)(x_0 + S_{i-1} + A))}$$

(6.22)

So for $m_{i+2} = 1$, $m_{i+1} = 0$ and now $m_i = -1$:

$$C(3, p)^{i+2}_{m_{i+2}=1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 \exp \left( i (x_{i+2} - x_{i+1}) \right) \cdot$$

$$e^{-i(1-b)S_{i+1} + i(1-b)S_{i-1}}$$

$$= \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 \exp \left( i (x_{i+2} - x_{i+1}) \right) \cdot$$

$$e^{i(1-b)K \sin x_{i+1} - i(1-b)K \sin x_i}$$

(6.23)

which picks out the $V'(x_{i+2})V'(x_{i-1})$ term in $S^2$ and when combined with the term for $m_{i+2} = -1$ gives a contribution:

$$C(3, p) = K^2 J_1((1 - b)K)^2$$

(6.24)

By performing the same analysis, but moving the starting point along one kick, a similar term is arrived at with $(1 - b) \rightarrow (1 + b)$ in the Bessel function argument. Thus the whole term is:

$$C(3, p)^{(1)} = \frac{K^2}{2} [J_1((1 - b)K)^2 + J_1((1 + b)K)^2]$$

(6.25)

which resembles the $J_1^2$ term obtained by Rechester and White for the correction to the standard map [63]. Note that this term is symmetrical in momentum and therefore does not contribute to the asymmetric energy growth.

A similar term is obtained for $m_{i+2} = \pm 1$, $m_{i+1} = -2m_{i+2}$ and $m_i = m_{i+2}$.
where the general form is:

\[ C(3, p)^{i+2}_{m_i+i=1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2}-x_{i+1}-(1-b)(p_0+S_{i+1}+A))} e^{-i(x_{i+1}-x_i-(1+b)(p_0+S_i))} e^{i(x_i-x_{i-1}-(1-b)(p_0+S_{i-1}+A))} \]

(6.26)

and the specific case of \( m_{i+2} = 1, m_{i+1} = -2 \) and \( m_i = 1 \) gives:

\[ C(3, p)^{i+2}_{m_i+i=1} = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2}-3x_{i+1}+3x_i-x_{i-1})} e^{i(4b(p_0-2(1-b)A))} e^{-i(1-b)S_{i+1}+2i(1+b)S_i-i(1-b)S_{i-1}} \]

\[ = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2}-3x_{i+1}+3x_i-x_{i-1})} e^{i(4b(p_0-2(1-b)A))} e^{-i(1-b)K \sin x_{i+1}+i(1-b)S_i+2i(1+b)S_i-i(1-b)S_{i-1}} \]

\[ = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2}-3x_{i+1}+3x_i-x_{i-1})} e^{i(4b(p_0-2(1-b)A))} e^{-i(1-b)K \sin x_{i+1}+(1+3b)S_i-i(1-b)S_{i-1}} \]

\[ = \frac{1}{2N} \int_0^{2\pi} \frac{dx_i}{2\pi} S_N^2 e^{i(x_{i+2}-3x_{i+1}+3x_i-x_{i-1})} e^{i(4b(p_0-2(1-b)A))} e^{-i(1-b)K \sin x_{i+1}+(1+3b)K \sin x_i-4bS_{i-1}} \]

(6.27)

which eventually gives a term resembling the \( J_2^3 \) correction to the Standard Map:

\[ C(3, p)^{(2)} = K^2[J_3((1-b)K)]J_3((1+3b)K) + J_3((1+b)K)]J_3((1-3b)K) \]

\[ \cdot \cos(4p_0b - 2(1-b)A) \]

(6.28)

note that the index (2) is used to differentiate from Eq. (6.25).

It is possible to obtain corrections due to 4-kick correlations, but these will involve products of three Bessel functions resulting in negligible corrections to the overall diffusion rate.
6.2.4 Total Corrected Diffusion Coefficient

Combining all these terms together, one finds that the total corrected diffusion coefficient for the rocking ratchet is:

\[
D = \frac{K^2}{2} \left[ 1 - J_1((1 + b)K)J_1((1 - b)K) \\
+ J_0(2bK) \cos(2p_0b - (1 - b)A)[J_2(K(1 + b)) + J_2(K(1 - b))] \\
+ [J_3((1 - b)K)J_3((1 + 3b)K) + J_3((1 + b)K)J_3((1 - 3b)K)] \\
\cdot \cos(4p_0b - 2(1 - b)A) \right] \tag{6.29}
\]

Note that the first two terms are symmetric in momentum and therefore do not contribute to the ratchet effect. One must also recall that this form for the diffusion coefficient is only valid in the region where \( bK \ll 1 \).

6.3 Derivation of Average Current

In chapter 4 it was shown that an analytical form for the average current in a ratchet system can be obtained using a similar method to that used for the diffusion coefficient. The derivation begins from the following definition of the average current:

\[
\langle p \rangle = \frac{1}{N} \int Q(x_N, p_N, t_N | x_0, p_0, 0) P(x_0, p_0, 0) (p_N - p_0) dx_N dp_N \tag{6.30}
\]

One again makes use of the recursion relation Eq. (6.6) and the Poisson summation formula to obtain:

\[
\langle p \rangle = \frac{1}{N} \sum_{m_N = -\infty}^{\infty} \cdots \sum_{m_1 = -\infty}^{\infty} \prod_{i=0}^{N} \int_0^{2\pi} \frac{dx_i}{(2\pi)} S_N \\
e^{im_N(x_N-x_{N-1}-(1-b)(p_0+S_{N-1}+A))}e^{im_{N-1}(x_{N-1}-x_{N-2}-(1+b)(p_0+S_{N-1}))} \cdots \\
e^{im_2(x_2-x_1-(1-b)(p_0+S_1+A))}e^{im_1(x_1-x_0-(1+b)p_0)} \tag{6.31}
\]
6.3. Derivation of Average Current

In the case of the current, when \( m_j = 0 \) in the above equation there is no contribution analogous to the quasi-linear term present in the diffusion coefficient. This is a result of the fact that one effectively has an integral over \( \sin x \) as opposed to \( \sin^2 x \) for the diffusion. The first order correlation terms are also zero for all \( j \). Therefore, as before, one investigates higher order terms, beginning with the two-kick correlations where \( m_j = \pm 1 \) and \( m_{j-1} = -m_j \).

6.3.1 Average current between kicks \( i \) and \( i + 2 \)

The three kick map is as in Eq. (6.8). For the first case, with \( m_{i+2} = +1 \) one finds:

\[
\langle p \rangle_{i+2} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} K \sin x_{i+2} e^{i(x_{i+2}-2x_{i+1}+x_i)} e^{-i(1-b)A} e^{-2ip_0b} \cdot e^{-(1-b)K \sin x_{i+1}} e^{2ibK \sin x_i} e^{2ibS_{i-1}}
\]

(6.32)

The integral over the \( x_{i+2} \) terms gives:

\[
K \int_0^{2\pi} \frac{dx_{i+2}}{2\pi} \sin x_{i+2} e^{ix_{i+2}} = K \int_0^{2\pi} \frac{dx_{i+2}}{2\pi} e^{ix_{i+2}} - e^{-ix_{i+2}} e^{ix_{i+2}} = \frac{iK}{2}
\]

(6.33)

For the \( x_{i+1} \) terms:

\[
K \int_0^{2\pi} \frac{dx_{i+1}}{2\pi} e^{-2ix_{i+1}} e^{-i(1-b)K \sin x_{i+1}}
\]

\[
= K \int_0^{2\pi} \frac{dx_{i+1}}{2\pi} e^{-2ix_{i+1}} \sum_n J_n((1 - b)K) e^{-inx_{i+1}} = J_2((1 - b)K)
\]

(6.34)
6.3. Derivation of Average Current

Similarly, the $x_i$ terms give:

$$K \int_0^{2\pi} \frac{dx}{2\pi} x_i e^{2ibK \sin x_i} = K \int_0^{2\pi} \frac{dx}{2\pi} e^{ix} \sum_n J_n(2bK) e^{inx} = J_{-1}(2bK) = -J_1(2bK)$$

(6.35)

Now consider the remaining terms for $x_1 \ldots x_{i-1}$. These give a product of $i - 1$ terms of the type:

$$\int_0^{2\pi} \frac{dx}{2\pi} e^{2ibK \sin x} = \int_0^{2\pi} \frac{dx}{2\pi} \sum_n J_n(2bK) e^{inx} = J_0(2bK)$$

(6.36)

However, this correlation between kicks $i : i + 2$ acts on only half the kicks, therefore the contribution from the $S_{i-1}$ terms couples terms in $J_0^3$. Thus if one now performs the sum over $i$ for the whole term, the only term that depends on $i$ is that involving $J_0$:

Firstly one has

$$(J_0(2bK))^{i-1} = \frac{(J_0(2bK))^i}{(J_0(2bK))}$$

(6.37)

Then, one performs the sum and notes that it links only terms in $J_0^3$. So:

$$\sum_{j=2}^{\infty} [(J_0(2bK))]^{i-1} = \frac{1}{(J_0(2bK))} \sum_{j=2}^{\infty} [(J_0(2bK))^2]^j$$

$$= (J_0(2bK))^3 \sum_{j=0}^{\infty} [(J_0(2bK))^2]^j$$

$$= \frac{(J_0(2bK))^3}{1 - (J_0(2bK))^2}$$

(6.38)

Finally, after combination with the result for $m_{i+2} = -1$ one obtains:

$$\langle p \rangle^{i+2} = -\frac{K}{2} \sin((1-b)A - 2p_0 b)J_2((1-b)K)J_1(2bK) \frac{(J_0(2bK))^2}{1 - (J_0(2bK))^2}$$

(6.39)
6.3. Derivation of Average Current

6.3.2 Average current between kicks $i + 1$ and $i + 3$

The current between the second pair of kicks is essentially the same, except that one now has a product $(J_0(2bK))^4$ terms:

$$\sum_{j=2}^{\infty}[(J_0(2bK))^2]^j = (J_0(2bK))^4 \sum_{j=0}^{\infty}[(J_0(2bK))^2]^j = \frac{(J_0(2bK))^4}{1 - (J_0(2bK))^2}$$

(6.40)

also, $1 - b \rightarrow 1 + b$ in the argument to the second order Bessel function.

Therefore, the average current for the $i + 1 : i + 3$ correlation is:

$$\langle p \rangle_{i+1;i+3} = -\frac{K}{2} \sin((1 - b)A - 2p_0b)J_2((1 + b)K)J_1(2bK) \frac{(J_0(2bK))^4}{1 - (J_0(2bK))^2}$$

(6.41)

6.3.3 3-Kick Correlations

As with the diffusion coefficient there are contributions to the ratchet current from higher order correlations. Again in general these are small, but it will be shown below that there are instances where the effect of these additional terms can be noticed.

The most important term arises from the case when: $m_{i+2} = 1, m_{i+1} = -2$ and now $m_i = 1$:

$$\langle p \rangle_{i-1;i+2} = \frac{1}{N} \int_0^{2\pi} \frac{dx_i}{2\pi} K \sin x_{i+2}e^{i(x_{i+2} - 3x_{i+1} + 3x_i - x_{i-1})}e^{i(4p_0 - 2(1-b)A)}$$

$$e^{-i(1-b)K \sin x_{i+1} + i(1 + 3b)K \sin x_i - 4bS_{i-1}}$$

(6.42)

Using the same procedure as followed above, one obtains the term:

$$\langle p \rangle_{i-1;i+2} = -K \frac{J_1(4bK) \sin(2(1 - b)A - 4p_0b)}{1 - (J_0(4bK))^2} \cdot [J_3((1 - b)K)J_3((1 + 3b)K) + J_3((1 + b)K)J_3((1 - 3b)K)]$$

(6.43)
where any \( J_0 \) terms in the numerator have been approximated to unity under the assumption that \( K b \ll 1 \). The power in the denominator comes from the fact that although this correction effectively covers a three-kick cycle, there are in fact only two distinct types of kicks. As such, the power \( 3/2 \) address the fact that certain kicks will be sampled twice.

### 6.3.4 Total Average Current

Combining the the terms, one finds that the total average current is:

\[
\langle p \rangle = -K \frac{\sin((1 - b)A - 2p_0 b)J_1(2bK)}{1 - (J_0(2bK))^2} \\
\cdot [J_2((1 - b)K) + J_2((1 + b)K)] \\
- K \frac{\sin(2(1 - b)A - 4p_0 b)J_1(4bK)}{1 - (J_0(4bK))^3} \\
\cdot [J_3((1 - b)K)J_3((1 + 3b)K) + J_3((1 + b)K)J_3((1 - 3b)K)]
\]

(6.44)

### 6.4 The Ratchet Time

The derivation of the ratchet time for the rocking ratchet once again makes use of the \( J_0 \) product that appears in both the diffusion and current formulae. In this case one starts from:

\[
\sum_{i=0}^{T-2} \left[ (J_0(2bK))^2 \right]^i = \frac{1 - (J_0(2bK))^2} {1 - (J_0(2bK))^2}^{T-1}
\]

(6.45)

If the ratchet time is again estimated to be the time at which the series is 95% of it’s value at \( T = \infty \), then one finds that:

\[
t_r - 1 \approx t_r = \frac{\ln(5/100)}{2 \ln((J_0(2bK)))}
\]

(6.46)
6.5. Comparison with Numerical Results

Now using the small argument form of $J_0$ one obtains:

$$\ln(J_0(2bK)) = \ln \left(1 - \frac{(2Kb)^2}{4}\right) \approx -(Kb)^2$$  \hspace{1cm} (6.47)

Therefore the 'one-kick' ratchet time is:

$$t_r^{(1-kick)} = \frac{\ln(20)}{2(Kb)^2}$$  \hspace{1cm} (6.48)

There are however, two contributions to the ratchet time. Thus $t_r = 2t_r^{(1-kick)}$:

$$t_r = \frac{\ln(20)}{(Kb)^2} \approx \frac{3}{(Kb)^2}$$  \hspace{1cm} (6.49)

Comparison with the results in Fig. 5.6 shows this result to be in excellent agreement with the numerics.

6.5 Comparison with Numerical Results

6.5.1 Diffusion

The simple shape of the energy gain curve for the rocking ratchet has already been introduced via Fig 5.7 in the previous chapter. The comparison between the theoretical form for the energy gain (Eq. (6.29)) and the classical numerical results from Fig 5.7 is shown in Fig 6.2. In general there is good agreement between the two curves, however there is a discrepancy between the analytical form and the numerics for the dip in the $K = 3.2$ plot.

For the double-well it was demonstrated that the terms in the diffusion coefficient which induce transport are those which have an explicit dependence on both the momentum, and the period-one chirp parameter, $b$. In the
6.5. Comparison with Numerical Results

In both cases, $b = 0.01$ and $A = \pi/2$.

case of the rocking ratchet these terms are:

$$D_{\text{asym}} = K^2 \left[ J_0(2bK) \cos(2p_0b - (1 - b)A)[J_2(K(1 + b)) + J_2(K(1 - b))] + [J_3((1 - b)K)J_3((1 + 3b)K) + J_3((1 + b)K)J_3((1 - 3b)K)] \cdot \cos(4p_0b - 2(1 - b)A) \right]$$  \hfill (6.50)

In Figure 6.3 these terms are plotted as a function of the kick strength, $K$, along with numerical results from which the non-transporting terms (effectively $D_0 = K^2/2 - K^2J_1(K)^2$) have been removed. One can clearly see that there is excellent agreement between the numerics and analytics. It should be noted that as with the double-well, the transport in the system is dominated by the second order correlation ($C(2,p)$) term - in this case the $J_2$ term. As $K$ increases it is clear that the agreement begins to deteriorate.
6.5. Comparison with Numerical Results

One attributes this to the breakdown of the $Kb \ll 1$ assumption meaning that the previously neglected $J_0(nKb)$ terms are beginning to have an effect.

6.5.2 Current

It now just remains to show that the average current as given by Eq. (6.44) also compares well with the calculated results. The panels in Figs. 6.4 and 6.5 show that this is indeed the case. The top panel of Fig. 6.4 shows in full the oscillating nature of the current as the kick strength is increased, which was suggested previously by the results shown in Fig 5.4. One notes that the agreement appears to deteriorate at low $K$, where the increasing regularity of
6.5. Comparison with Numerical Results

Figure 6.4: Average momentum as a function of $K$, $b$ and initial momentum. In each case the numerical results show good agreement with the form of the current given by Eq (6.44. Unless otherwise stated, parameters are $K = 4$, $A = \pi/2$, with $b = 0.03$ for the top panel, and $b = 0.04$ for the bottom.

the phase space begins to affect the accumulation of momentum asymmetry.

The middle panel shows that the analytical form for the current accurately predicts the expected inverse dependence on the chirp parameter, $b$. In the bottom panel, the current is compared against numerical results for varying initial momentum. In this case, the average current appears to follow a saw-tooth shape. This is a direct result of the analytical current effectively containing the first two harmonics of a Fourier series in $\sin 2p_0$.

Finally the dependence of the current on the strength of the rocking linear term, $A$, is illustrated in Fig. 6.5. As with the dependence on the initial momentum, the average current as a function of $A$ is effectively a
6.5. Comparison with Numerical Results

![Graph comparing numerical results with analytical formula]

Figure 6.5: Average momentum as a function of \( \Delta \). This figure shows the 'double-well' nature of the average current dependence on \( \Delta \). In the top panel \( K = 4 \) and in the bottom panel \( K = 5 \) with \( b = 0.02 \) in each case.

'double-well', two-term Fourier series: \( \langle p \rangle = C_1 \sin \Delta + C_2 \sin 2\Delta \). The relative strength of the two 'harmonics' is set by the other parameters in Eq (6.44), and clearly for \( K = 5 \) the curve is highly asymmetric.

These results for \( p_0 \) and \( \Delta \) highlight one of the difficulties that arise from using equations that contain a number of Bessel function. The second term in the current formula is for a large majority of the time overshadowed by the first term, since \( J_2(K) \) is in general larger than \( J_3^2(K) \). However as Fig 6.6 shows, when \( K = 5 \), the first term in Eq. (6.44) weakens considerably since \( J_2(5) \approx 0 \), whilst \( J_3(5) \) is close to a maximum, hence the shape of the curve in Fig. 6.5. One must therefore be careful not to overlook terms which, for the most part, may give no appreciable contribution to either the current or
6.6 Conclusion

The derivations in this chapter and the subsequent comparison with numerical results show that, as with the double well, the ratchet mechanism is a result of momentum and chirp dependent corrections to the quasi-linear diffusion. The comparatively simple nature of the rocking system also allows three-kick correlation terms to be investigated. One notes that whilst these higher order terms provide negligible contribution to the diffusion over the major part of the parameter range, at certain points they do become significant.
Experimental Simulation of Ratchet Systems

7.1 Introduction

So far, the ability to create a Chaotic Hamiltonian Ratchet has been demonstrated purely theoretically - in this chapter the actual experimental realisation is discussed.

For both the double-well and rocking cases, the experiment is performed using ultra-cold cesium atoms subjected to an optical lattice that is pulsed on and off to simulate the periodic kicking. Naturally, the period of the pulsing can be varied to introduce the necessary temporal asymmetry. This chapter begins by briefly covering the techniques used to laser-cool atoms to micro-Kelvin temperatures, and how one can use a Magneto-Optical Trap (MOT) to confine the atoms. The creation of a one-dimensional optical lattice from two counter-propagating beams is also introduced.
7.2. Laser Cooling and Trapping

The conversion of the dimensionless Hamiltonian used thus far to real variables is given, before progressing to a discussion of the potential experimental arrangements for each of the two ratchet systems. It will be shown that in each case reproducing the appropriate potential has inherent difficulties which, whilst possible to overcome, make the experimental realisation a challenging and subtle task. The chapter finishes by covering initial results for the rocking ratchet that have been obtained by the Laser Cooling Group at UCL.

7.2 Laser Cooling and Trapping

The ability to perform laser cooling of atoms to temperatures of the order of a few micro-Kelvin was first suggested in 1975 by Hänsch and Schawlow [94] and independently by Wineland and Dehmelt [95]. The method they proposed, known as Doppler cooling, gave rise to a theoretical lower bound for the atomic temperature of about $140 \mu$K. However it soon became clear that in practice, temperatures well below this limit could be easily achieved [96, 97, 98]. In 1989, J. Dalibard et al. [99] explained this unexpected development using a model known as Sisyphus cooling. This new theory not only predicted temperatures that concurred with experimental results, but also suggested the possibility of trapping the atoms in potential wells. With the addition of a spatially varying magnetic field, one can sustain a cloud of trapped atoms for many hours. To create an optical lattice from this amorphous cloud, one simply switches off the trapping beams and applies a second set of beams to the cloud. These lattice beams have varying arrangements depending on how many dimensions one wishes the lattice to have. The interested reader is referred to references [100] and [101].
7.2. Laser Cooling and Trapping

7.2.1 Doppler Cooling

Before explaining the modern theory governing laser cooling, it is worth covering the original Doppler cooling theory. Although the theory easily generalises to three dimensions, the following discussion is for the one-dimensional case. In this arrangement, two counter-propagating laser beams, detuned slightly from atomic resonance, form a standing wave. A simple two-level atom moving along the standing wave with speed $v$ will in effect see a Doppler shift in the frequencies of the two laser beams. The beam travelling in the opposite direction to the atom will appear to be frequency shifted closer to resonance, whilst the one travelling in the same direction as the atom will get farther away from resonance. This frequency difference gives rise to a radiation pressure imbalance - the atom will naturally absorb more photons from the beam closer to resonance. When a photon is absorbed, the atom recoils in the direction of the beam that supplied the photon. The subsequent re-emission of the absorbed photon is a spontaneous process with the photon being emitted in a random direction. The net result is a force opposing the motion of the atom. Thus the atom is slowed down and effectively cooled.

The Doppler cooling theory, as mentioned above, predicts a lower limit for the temperature to which atoms can be cooled. This limit arises from the re-emission of the absorbed photons. Although this process is random in nature and the force on the atom due to re-emission averages to zero, momentum exchange fluctuations are introduced which (as with Brownian motion) affect the atomic momentum distribution. The result is a gradual heating of the atoms over time. The balance between the cooling force on the atoms, and the heating gives rise to the Doppler limit:

$$T_D = \frac{\hbar \Gamma}{2k_B} \tag{7.1}$$
where $k_B$ is Boltzmann's constant, and $\Gamma$ is the linewidth for the atomic transition. The experiment uses Caesium which has $\Gamma = 5.22\text{MHz}$ giving a Doppler Cooling temperature limit of $T_D \approx 120\mu K$.

If this technique is extended to three dimensions, the resulting Doppler shifts cause the motion of the atoms to be effectively confined to a small region of space. When restricted in this manner, the atoms behave like a viscous fluid and as a result are often referred to as an optical molasses. It was using this arrangement of six laser beams, and a time-of-flight measurement technique that P. Lett et. al. at the National Institute of Standards and Technology (NIST) observed sodium molasses temperatures of only $40\mu K$ [96, 102]. This surprising result led to obvious questions concerning the validity of the Doppler cooling theory. These queries have now been answered by the highly intuitive Sisyphus Cooling mechanism introduced below.

### 7.2.2 Trapping the atoms: the magnetic field

In order to trap the atoms for an appreciable length of time, it is necessary to apply an inhomogeneous magnetic field. This has the effect of incurring Zeeman shifts on the atomic sublevels. To illustrate this, consider a multilevel atom with a non-degenerate ground state $|J_g = 0\rangle$ and a triply degenerate excited state $|J_e = 1\rangle$ (see Fig. 7.1).

The magnetic field induces spatially dependent energy shifts on the $|J_e = 1, m_j = \pm 1\rangle$ levels as shown. Thus for an atom not at the origin and subjected to a slightly detuned beam, one of the sublevels appears closer to atomic resonance, and one appears further away than before (the $|J_e = 1, m_j = 0\rangle$ level is unaffected). Therefore, atoms far away from the origin will experience a greater radiation pressure than those close to it. The resulting photon
absorption-emission cycle means that the atoms are effectively trapped in a small volume of space. Note that Doppler cooling produces a velocity-dependent confining force, whilst the addition of a magnetic field gradient produces spatially-dependent confinement.

It should be noted that in the usual arrangement one uses six lasers beams (one for each of the positive and negative cartesian axes) slightly detuned from atomic resonance, and a 3-dimensional magnetic field gradient to confine the atoms. This is then known as a Magneto-Optical-Trap (MOT).
The problem of anomalously low temperatures was solved in 1989 by J. Dalibard and C. Cohen-Tannoudji using a cooling mechanism called Sisyphus cooling [99, 103]. The theory relies on optical pumping rates and polarisation gradients, and as such can also be called polarisation-gradient cooling. The first important point to note is that atoms such as Caesium are not simple two level systems, they have multiple Zeeman sublevels within the ground and excited states. For the two level Doppler case above the lifetime for the atomic transition is $\tau = 1/\Gamma$, which determines the minimum achievable temperature. In a multilevel atom one can 'pump' the atom from one ground state sublevel to another (via an excited state) with characteristic time $\tau_p = 1/\Gamma'$. Here, $\Gamma'$ is the mean rate at which absorption-spontaneous emission cycles occur between sublevels. Thus by increasing the pumping time, one can decrease the rate (i.e. the width) of the transition and decrease the
Doppler cooling limit. For an alkali atom such as caesium, $\tau_p > \tau$.

\[
\begin{array}{cccc}
|e, -3/2> & |e, -1/2> & |e, +1/2> & |e, +3/2> \\
1 & 2/3 & 1/3 & 2/3 & 1 \\
\sigma^- & \pi & \sigma^+ & \sigma^- & \pi & \sigma^+ \\
|g, -1/2> & |g, +1/2> \\
\end{array}
\]

Figure 7.3: Square of Clebsch-Gordon coefficients and polarisations necessary for a $|J_g = 1/2 \rightarrow |J_e = 3/2 >$ transition.

One now considers an atom in a one dimensional molasses, formed using two counterpropagating beams (tuned slightly below atomic resonance) with orthogonal linear polarisations (known as lin_Llin). The result of this arrangement is to form a light field with spatially varying polarisation - i.e. a polarisation gradient (as shown in Fig. 7.2). The polarisation varies from circular to linear every eighth of a wavelength, and from left/right circular, vertical/horizontal linear in the manner shown. The atomic transition used is from ground state $|J_g = 1/2 >$ to excited state $|J_e = 3/2 >$, as shown in Fig. 7.3.

An atom in the $|g, 1/2 >$ ground state sublevel situated at a $\sigma^-$ site will be excited to the $|e, -1/2 >$ state from where it will (on average) decay into the $|g, -1/2 >$ ground state. It is worth noting that if the atom is initially in the $|g, -1/2 >$ state, then absorbing a $\sigma^-$ photon will take it to the $|e, -3/2 >$ from where it can only decay back to the $|g, -1/2 >$ sublevel. Thus all atoms at a particular polarisation site are pumped into the same sublevel with
characteristic time $\tau_p$.

![Figure 7.4: Schematic representation of Sisyphus cooling in a one dimensional lattice.](image)

When the atom is subjected to the polarisation gradient, the two ground state sublevels undergo light shifts in sinusoidal fashion - shown in Fig. 7.4. If one now considers an atom, initially in the $|g, \frac{1}{2}\rangle$ state, moving from left to right in the figure, it will climb the potential hill and move from a region of $\sigma^+$ light to $\sigma^-$. Eventually the probability of absorbing a $\sigma^-$ photon will be such that the atom is excited (or pumped) to the $|e, -\frac{1}{2}\rangle$ level before decaying to the $|g, -\frac{1}{2}\rangle$ state from where the cycle begins again. Thus in analogy with the fate of Sisyphus, doomed forever to roll a stone up a
The example highlighted above results in the creation of a one dimensional linear lattice, so-called because of the initial polarisations of the lattice beams. However, optical lattices can be created in one, two or three dimensions and the spatial symmetry of the potential surface can be easily controlled by appropriate arrangement of laser beams and their polarisation. Furthermore, by detuning the lattice beams far (2000-3000 linewidths) from atomic resonance such that spontaneous scattering (and therefore dissipation and decoherence) are negligible, one can prepare a large proportion of the atoms in the ground state. As a result, optical lattices provide a versatile framework within which to perform many different experiments where the absence of dissipation is essential, including demonstrating chaos phenomena such as the ratchet systems detailed in this thesis.

7.3 Experimental Realisation of the Double-Well Ratchet

7.3.1 The Real Hamiltonian

Throughout this thesis, the theoretical analysis has been performed using a dimensionless Hamiltonian, enabling the system to be described by a small number of parameters which could be independently varied. However, to
investigate the system experimentally one must convert these dimensionless variables into real quantities and obtain a 'real' Hamiltonian for the system.

It can be shown [104] (see Appendix A) that a simple two-level atom in a standing wave of light created by two counterpropagating laser beams can be described by the centre-of-mass Hamiltonian:

\[ H = \frac{p^2}{2M} + V_0 \sin(2k_L x) \]  

(7.2)

where \( M \) is the mass of the atom (in this case caesium) and \( k_L \) is the wave number of the light.

The depth of the potential well is given by:

\[ V_0 = \frac{\hbar \Omega^2}{8\Delta_L} \]  

(7.3)

where the Rabi frequency, \( \Omega \) is given by:

\[ \Omega = \frac{2\mu E_0}{\hbar} \]  

(7.4)

with \( \mu \) the atomic dipole moment, \( \Delta_L \) the detuning from atomic resonance and \( E_0 \) the amplitude of a single travelling wave component of the standing wave. For a pulsed system, the Hamiltonian becomes:

\[ H = \frac{p^2}{2M} + V_0 \sin(2k_L x) \sum_{n=-\infty}^{\infty} F(t - nT) \]  

(7.5)

where \( F(t) \) is a pulse centred around \( t = 0 \) and with width \( t_p \). One notes that in practice a \( \delta \)-kick pulse is impossible to achieve.

This Hamiltonian can now be rescaled to dimensionless units using the
following transformations:

\[
\begin{align*}
2k_Lx & \rightarrow x' \\
\frac{2k_LT}{M}p & \rightarrow p' \\
t & \rightarrow \tau \\
8V_0 \frac{\omega_r T^2}{\hbar} & \rightarrow k \\
4k_L^2 T^2 / M \frac{H}{H'} & \rightarrow H'
\end{align*}
\]

(7.6)

where \( \omega_r = \hbar k_L^2 / 2M \) is the atomic recoil frequency. One therefore obtains a Hamiltonian for the system similar to that used for the Kicked rotor in chapter 2:

\[
H' = \frac{p'^2}{2} + k \sin(x') \sum_{n=-\infty}^{\infty} f(\tau - n)
\]

(7.7)

One further point to note is that by considering the commutator \([x', p'] = i\hbar'\) one finds the scaled Planck constant to be: \( \hbar' = 8\omega_r T \).

In principle, to create the double-well potential needed for the ratchet one simply combines two standing waves of the kind described above, albeit with one having twice the spatial period. In practice it is not quite that simple, as will be discussed below.

### 7.3.2 The Effect of the Finite Pulse Width

The pulse \( f(\tau) \) now has unit amplitude and a scaled duration: \( \alpha = t_p / T \).

In the limit that the pulse becomes a delta function, and for a function \( \beta = \int_{-\infty}^{\infty} f(\tau) d\tau \) one finds that \( \beta k \rightarrow K \), the classical kick strength. Due to time constraints, the effect of a finite pulse width on the theoretical ratchet system was not incorporated into the theoretical simulations - however a few general points are worth noting. In [65], B. G. Klappauf et. al. showed that
by rewriting the sequence of kicks as a discrete Fourier series and analyzing the primary resonances, an effective kick strength can be obtained which (for a square pulse) takes the form:

$$K' = \alpha k \frac{\sin(\alpha p'/2)}{\alpha p'/2}$$  \hspace{1cm} (7.8)

This effective kick strength $K'$ clearly depends strongly on momentum; as $p'$ increases, $K'$ falls rapidly to zero, in accordance with the shape of the sinc function. If one examines the classical phase space, one observes a solid torus (or momentum boundary) at the value of $\pm p'$ for which $K' = 0$. One clearly needs to ensure that the initial momentum distribution is narrow enough to allow the accumulation of classical asymmetry (and quantum localisation) to occur before the momentum boundary is reached. If one considers an atom that traverses one period of the potential in the time $t_p$ (such that $vt_p = \frac{1}{2}\lambda$), then self-correlation can occur and the average force on the atom, or average momentum transferred to it, drops to zero and diffusion stops. It is then possible to obtain an estimate of the momentum boundary location to be:

$$\frac{p}{2\hbar k} = \frac{M\lambda^2}{8\pi \hbar t_p}$$  \hspace{1cm} (7.9)

Thus to be certain that the momentum boundary occurs at high $p$, one wishes to make the pulse width $t_p$ as small as possible.

A further consequence of the sinc-function dependence of $K'$ is the resultant modulation of the diffusion coefficient as a function of momentum, even in the quasi-linear case. Any wavepacket starting at non-zero momentum could display asymmetry simply as a result of the sinc-function modulation. In order to avoid this possibility one must ensure that the initial momentum distribution is sufficiently narrow and centred about $p = 0$. 

7.3. Experimental Realisation of the Double-Well Ratchet
7.3. Experimental Realisation of the Double-Well Ratchet

7.3.3 Experimental Arrangement

The experimental arrangement needed to create a double well ratchet surface is at once elegant and difficult to implement. One begins by using a standard three-dimensional Magneto-Optical Trap to cool and confine a cloud of caesium atoms in a vacuum cell, forming an optical molasses. The average atomic temperature in the molasses is well below the Doppler limit indicating that as well as three dimensional Doppler cooling, some sub-Doppler cooling also occurs. Although no lattice is formed, there are polarisation gradients in the molasses and hence sisyphus cooling is present, along with another sub-Doppler cooling mechanism known as orientation cooling.

![Schematic arrangement for beams used in double-well ratchet. Note all beams are far detuned and the polarisations are out of the page.](image)

Figure 7.5: Schematic arrangement for beams used in double-well ratchet. Note all beams are far detuned and the polarisations are out of the page.

In order to obtain the appropriate double-well lattice one uses a 'bow-tie' arrangement of beams (demonstrated in Fig. 7.5), which are switched on after the cooling phase. Note that the beam polarisations are all parallel, and out of the plane. This means that (in contrast to Sisyphus cooling) all
the Zeeman states have the same light shift and the atoms are all effectively on the same sinusoidal potential surface - thus each atom feels the same kick. Furthermore, the beams are detuned a long way from atomic resonance (~ 2000 - 3000 linewidths). In this regime, spontaneous scattering and therefore decoherence and dissipation, are minimal - an atom can reasonably accurately be thought of as a ball rolling on a hard surface.

The optical potential for a single beam is given by the form (see [105] and Appendix A):

\[ U(x) = -\frac{2}{3}U_1|\varepsilon_L(x)|^2 \hat{I} + \frac{i}{3}[\varepsilon_L^*(x) \otimes \varepsilon_L(x)] \cdot \hat{\sigma} \]  

(7.10)

where \( \varepsilon_L(x) \) is the local polarisation and \( U_1 \) is the single beam light shift. As the polarisations are all parallel, when calculating the total light shift the second term in the above equation is zero. However as the arrangement stands the four beams (designated \( E_1 - E_4 \) in the figure) are coherent, meaning that the total light shift is effectively given by:

\[ \hat{U} \propto |E_1 + E_2 + E_3 + E_4|^2 \]  

(7.11)

This results in unwanted cross-terms between \( E_1, E_2 \) and \( E_3, E_4 \). To get round this problem, a frequency shift is applied to the beams, \( E_1 \) and \( E_3 \). An Acousto-Optic-Modulator (AOM) is used to shift the frequency of \( E_1 \) up by 80MHz and the frequency of \( E_3 \) down by 80MHz, and then the interaction of the four beams is averaged over a time \( t = \frac{1}{160} \text{MHz} \). The light field is now given by:

\[ \hat{U} \propto |E_1 + E_2|^2 + |E_3 + E_4|^2 \alpha \sin(2k_Lx) + \sin(k_Lx + \phi) \]  

(7.12)

Note that beams \( E_1 \) and \( E_2 \) create a lattice with half the spatial frequency (i.e. double the period) of the one created by beams \( E_3 \) and \( E_4 \). One also
notes that a phase $\phi$ has been introduced in the above formula - this shall be discussed below.

Clearly, one now has the appropriate double-well potential with which to perform the chaotic ratchet experiments. It has already been shown in Eq. (7.6) that the kick strength for each term can be altered by changing the depth of the potential well, $V_0$, or the (average) time between kicks, $T$. The kick itself is applied by switching the potential on for a time, $t_p$ - set to be as short as possible, to approximate the $\delta$-pulse. The effective $\hbar$ can also be controlled by varying $T$. Thus, in principle, one has all the tools with which to perform a successful experiment.

7.3.4 Problems Encountered

There are, unfortunately, a number of difficulties which must be overcome in order to successfully demonstrate a double-well chaotic Hamiltonian ratchet. Firstly, there are standard experimental limitations, such as how short one can make the kicks is dependent on the function generator used to switch the beams on and off. The kick strength attainable is dependent on the depth of the potential well, and hence is limited by the maximum intensity of the lasers used.

However, the most serious stumbling block concerns the phase difference between the two parts of the potential. This phase fluctuates rapidly and whilst it can be locked to a fixed value (using an interferometer arrangement), at the time of writing it has proved impossible to measure what this value is due to the lack of a reference point. A secondary issue that compounds this problem arises when considering the angle between beams $E_1$ and $E_3$. If this angle is not exactly $60^\circ$, then a phase difference once again occurs.
between the two terms in the potential. This particular phase difference is spatially dependent, meaning that the lattice may (in the worst case) change symmetry from one side of the atomic cloud to the other. To maintain a potential asymmetric in the same sense across the cloud, one needs to ensure that this phase difference is $< \frac{x}{2}$. However, the angular tolerance needed to do this is very small - further adding to the difficulty of the experiment.

Whilst it is hoped that these problems may be overcome in the future, at the present time it has not been possible to obtain any conclusive results for the double-well ratchet.

7.4 Experimental Realisation of the Rocking Ratchet

7.4.1 Real Hamiltonian

The conversion of the Hamiltonian from the dimensionless case to real variables is essentially the same as for the double well, with the exception of the linear rocking term. In the experimental realisation described below, this term is introduced by using an accelerating lattice. The experiment is then performed in the rest frame of the lattice, rather than the rest frame of the lab.

The accelerating frame gives rise to a momentum change in the following manner:

\[
\begin{align*}
x' &= x - \frac{1}{2}at^2 \\
p' &= mv - mat \\
\Rightarrow p &= mv' + mat
\end{align*}
\] (7.13)
So the kinetic energy term in the Hamiltonian becomes:

\[
\frac{p^2}{2m} = \frac{p'^2}{2m} + ma(v't + \frac{1}{2}at^2)
\]

\[
= \frac{p'^2}{2m} + ma(vt - \frac{1}{2}at^2)
\]

\[
= \frac{p'^2}{2m} + ma(vt + \frac{1}{2}at^2)
\]

\[
= \frac{p'^2}{2m} + max'
\]  

(7.14)

where the second term is equivalent to the rocking linear term in the dimensionless Hamiltonian, and the alternating sign is introduced by changing the sense of the acceleration, \(a\). The full Hamiltonian is then:

\[
H' = \frac{p'^2}{2m} + k \cos(x') \sum_{n=-\infty}^{\infty} f(\tau - n) + max'
\]  

(7.15)

### 7.4.2 Experimental Arrangement

The experimental set-up for the rocking ratchet experiment is considerably simpler than that of the double-well case. Once again, a cloud of cold cesium atoms is prepared using a three dimensional MOT. The cosinusoidal potential polarisations are parallel to avoid creating Zeeman state dependent light shifts (see Fig. 7.6).

In order to create the rocking linear term one exploits a common property of the lattice. For a fixed frequency difference (\(\Delta \nu\)) between the two lattice beams, the standing wave (in the rest frame of the lab) has a constant velocity. If one introduces a frequency difference which varies with time (\(\Delta \nu(t)\)), one creates an accelerating lattice. Experimentally it is possible to control both the magnitude and sign of the frequency difference as a function of time. To create a rocking inertial term the frequency ramp resembles a
7.4. Experimental Realisation of the Rocking Ratchet

![Schematic arrangement for beams used in rocking ratchet](image)

Figure 7.6: Schematic arrangement for beams used in rocking ratchet. Note both beams are far detuned and the polarisations are out of the page. Also shown is the form of the frequency ramp.

zig-zag, where the acceleration, $a$, is the gradient - see Fig. 7.6. In real units, the acceleration takes the form:

$$a = \frac{\lambda}{2} \frac{d(\Delta \nu(t))}{dt}$$  \hspace{1cm} (7.16)

The case where $\nu_1 = \nu_2$ in Fig 7.6 samples the diffusion coefficient (in the rest frame of the lattice) around $p_0 = 0$, but it is not possible to sample over a full $2\pi$ oscillation due to limitations of the frequency generator - the maximum value attainable is $A = 0.6\pi$. Thus, to complete the picture, a fixed frequency difference (such that $\nu_1 \neq \nu_2$) is introduced between the two beams before the ramp is applied to one of them. This has the effect of moving the sampling range to about a value $p \neq 0$. By working through the Hamiltonian one finds that the amplitude of the linear term is given by:

$$A = 2\pi t_p(\Delta \nu_{\text{max}})$$  \hspace{1cm} (7.17)
where $t_p$ is the pulse duration and $\Delta \nu_{max}$ is the maximum value of the frequency ramp. This value is obtained by integrating $A$ over the kick period:

$$A' = \frac{1}{T} \int_{-T/2}^{T/2} Adt$$

$$= \frac{1}{T} \int_{-tp/2}^{tp/2} Adt$$

$$= \frac{Atp}{T} \quad (7.18)$$

The limits on the integral alter since the term is zero except during the pulse.

7.5 Results

Before discussing the results from the rocking ratchet experiment, it should be noted that the Laser Cooling group at UCL began their investigations by considering the chirped kicked rotor - effectively the rocking ratchet case with $A = 0$. In [106] they use a moving optical potential to exploit the momentum dependent nature of the diffusion coefficient and create asymmetry in the resulting momentum distribution. This work is expanded in [107] to include the effects of the finite-width of the pulse used in the experiment.

The rocking ratchet experiment, as detailed above, has now been performed successfully and representative results are shown in Figs. 7.7 and 7.8. The momentum distributions shown in Fig. 7.7 are typical of those obtained in the experiment by fluorescence imaging using a CCD camera. One can clearly see the asymmetric nature of the final distributions, and the fact that direction of the resulting current depends on the sign of $A$.

In Fig. 7.8 one sees the variation of the average momentum with $(2p\theta + A)/\pi$. Note that in accordance with Eq. (6.44), the form is sinusoidal. Also shown is the numerical curve obtained from the quantum simulation. One
notes that the agreement with the period is excellent, but there is a significant discrepancy with the magnitude of the experimental result. Some difference between theory and experiment is to be expected, particularly when one considers that the effect of the finite pulse width is not taken into account in the theoretical simulation. Furthermore, the numerical code considers the evolution of one (approximately) Gaussian wavepacket that is narrow in
7.5. Results

Figure 7.8: Average momentum as a function of $(2pb + A)/\pi$ for the rocking ratchet. Parameters are $K = 2.6$, $b = 0.06$ and the system has evolved for 300 kicks. Also shown is the corresponding quantum average momentum obtained from the numerical simulation.

both position and momentum, whereas in the experiment the distribution is effectively narrow in momentum but stretched out across a number of periods of the lattice (effectively $x$ from $0 \rightarrow 2\pi$ in phase space). Thus, a more accurate simulation would model the evolution of wavepackets at each point in $x$, or alternatively consider the superposition of many eigenstates. Such a simulation is under construction, and it is hoped that this will demonstrate good correspondence with experiment.
7.6 Conclusion

In this chapter, the possible experimental realisation of both the double-well and rocking ratchets has been addressed. The motivation for using optical lattices as an experimental framework is clear; one can prepare a cloud of atoms in a well-defined quantum state, and then subject them to a pulsed potential in a noise free environment. The experimental arrangements used for each ratchet are given, and the problems encountered with the double-well case that have (at the time of writing) inhibited the demonstration of this system are discussed.

The rocking ratchet experiment has, however, been successfully performed. The results obtained thus far are extremely encouraging, showing a momentum and $A$-dependent current that is in good agreement with theoretical predictions. It is expected that a new simulation which incorporates experimental factors, such as the finite pulse width, will show even better agreement with observed data. It is hoped that further results from the rocking ratchet experiment will verify the theory for a range of parameters, and that a successful demonstration of the double-well ratchet will be possible in the near future.
8.1 Summary

The work contained in this thesis describes the next step in the evolution of the study of ratchet systems - the realisation of a chaotic Hamiltonian ratchet capable of producing a ratchet effect even in the limit of global chaos. Prior to this work, it was believed that a mixed phase space was necessary for a ratchet effect to be observed. However, the ratchet models presented here do not rely on the presence of any special features such islands, KAM, or can-tori.

The existence of directed transport in both the classical and quantum regimes has been demonstrated for two systems, showing the generic nature of the underlying processes. Both models are based on the \( \delta \)-kicked rotor, with chirped kicks providing the necessary temporal asymmetry, whilst spatial asymmetry is introduced initially by the use of a double-well potential, and secondly through the action of a rocking symmetric potential. The key result,
for both systems, is the presence of a finite classical ratchet current, reached after a finite saturation time, known as the ratchet time.

For the double well ratchet, detailed in chapters 2-4, a thorough study of the dependence of the ratchet effect on each of the system parameters is presented. One notes that the dynamics is complex, with many current reversals and a subtle dependence of the magnitude of the current on each parameter. Dynamical localisation is evident in the quantum case, and it is shown that the onset of localisation can be used to ‘freeze in’ the ratchet current - vital for any experiment. One finds that the clearest experimental signature will be seen when the quantum break-time and the classical ratchet time coincide.

In chapter 4, the ratchet effect is shown to be due to different short time classical diffusion rates for particles with negative momenta from those with positive momenta. Furthermore, these corrections to the diffusion rate are shown to depend on the chirp parameter, $b$. A momentum and $b$-dependent diffusion coefficient is then derived by considering previously neglected correlations between successive kicks in the evolution sequence. A similar analysis is used to obtain an analytical form for the average current and the ratchet time. These results exhibit good agreement with numerical simulations.

The current in the rocking ratchet, considered in chapters 5 & 6, displays the same general trends as the double-well case for zero initial momentum. Furthermore, the existence of the finite ratchet current is once again shown to be due to differing diffusion rates, demonstrating the generic nature of the model. The rocking ratchet was also studied in the regime of non-zero starting momentum. In this regime, the system was shown to demonstrate a filtering effect - where atoms travelling in one direction are strongly damped and those travelling in the opposite manner are unaffected. Analytical forms
for the diffusion coefficient, average current and ratchet time for the rocking ratchet are derived in chapter 6 and are once again shown to have an excellent fit to numerical results.

Finally, the possible experimental realisation of both ratchets is discussed, with preliminary results for the rocking ratchet shown to give good qualitative agreement with the theory.

In summary, during the course of this research a new generic ratchet model has been successfully simulated numerically, modelled analytically and demonstrated experimentally.

8.2 Ongoing Research, Future Work and Possible Applications

The completion of this work on chaotic Hamiltonian ratchets, and the successful explanation of the origin of the ratchet current allows one to extend the work and investigate related issues. In particular, work is ongoing regarding the onset of dynamical localisation in the rocking ratchet. By examining the eigenstates of the Floquet operator, the dependence of the localisation length and break-time on the initial momentum and the chirp parameter can be found.

Also, the role of correlations in other systems is being investigated, specifically in the two-kick case where $b$ is large. In this ‘double-kick’ case, the dependence of the energy growth on correlations is much more subtle and, at the time of writing, is not clearly understood. This work was motivated by recent experiments performed by the Laser Cooling Group at UCL who observed diffusive energy growth which could not be accounted for by using
the rocking ratchet diffusion coefficient (i.e. in the limit of small $b$).

Future extensions of the work might include adding noise to the system, to see whether transport is enhanced or destroyed and to test the robustness of dynamical localisation in these ratchet systems. The effect of a finite kick width could also be incorporated, to aid comparison with experiment.

When considering possible applications of this work it must be noted that one can only consider systems where quantum effects are present. The ratchet relies on dynamical localisation to lock-in the classical current - a purely classical ratchet of this type would not be practical, since the energy grows without bound and any asymmetry is lost in the expansion of the momentum distribution. The most likely practical application arises from the filtering effect that arises from the rocking ratchet scheme. It is hypothesised that this ability to preferentially select atoms of certain velocities could be used to guide atoms in so-called atoms chips. Atom chips are essentially to coherent matter wave optics, what integrated circuits are to electronics. Instead of electrons moving inside a semiconductor device, atoms are transported via potentials created above a surface [108, 109, 110, 111]. The last reference details the creation of a Bose-Einstein condensate above an atom chip, which is then used as an atom conveyor belt. In order to combine this work with the chaotic filter proposed here, one must determine the robustness of the ratchet effect and dynamical localisation in a BEC. Very recently, Raizen (and collaborators) [112] have begun such an investigation for the simple quantum kicked rotor in which they investigated dynamics of a periodically kicked BEC ring. The next logical step would be to open out the ring and study a linear kicked rotor, before moving to a true ratchet potential.

Whilst this field is still in its infancy, it is envisaged that atom chips will provide a new framework in which to study and implement concepts such as
8.2. Ongoing Research, Future Work and Possible Applications

interferometry, atom lithography and quantum information processing. With an ever increasing need to be able to control the flow of atoms, it is hoped that the concepts contained in this thesis will aid the advance of this new and exciting research.
All our science, measured against reality, is primitive and childlike -
and yet it is the most precious thing we have.

*Albert Einstein*
APPENDIX A

Derivation of Light Field Hamiltonian and Optical Potential

This appendix serves to clarify the origin of Eqs (7.2), (7.10), the two-level atom Hamiltonian and the optical potential for the lattice used in the experiment.

A.1 Two-Level Atom Hamiltonian

Presented here is an outline of the derivation of the atom-light Hamiltonian for a two-level atom in a standing wave of laser light, the interested reader is referred to [104] for more details.

The standing wave is formed by two counter-propagating waves:

\[ E(x, t) = \hat{z}E_0 \sin(k_x x)(e^{-i\omega_L t} + e^{i\omega_L t}) \]  

(A.1)
where $E_0$ is the single-beam amplitude, $k_L$ is the wavenumber of the laser light and $\omega_L$ is the laser frequency.

The free-evolution atomic Hamiltonian is then written in terms of the excited state $|e\rangle$ (the ground state being $|g\rangle$) and the atomic resonance frequency, $\omega_0$, as:

$$H_A = \frac{p^2}{2m} + \hbar \omega_0 |e\rangle \langle e|$$  \hspace{2cm} (A.2)

In the dipole approximation, the atom-field interaction Hamiltonian is then:

$$H_{AF} = -d \cdot E$$  \hspace{2cm} (A.3)

where the dipole operator can be written as:

$$d = (a + a^\dagger)(e|d|g)$$  \hspace{2cm} (A.4)

with $a = |g\rangle \langle e|$ being the atomic lowering operator. After making the rotating wave approximation (where terms rotating at twice the optical frequencies are replaced by their zero average values, denoted by the tilde-notation i.e. $\tilde{a}$), the interaction Hamiltonian can then be written as:

$$H_{AF} = \frac{\hbar \Omega}{2} (\tilde{a} + \tilde{a}^\dagger) \cos k_L x$$  \hspace{2cm} (A.5)

where the maximum Rabi frequency is defined as:

$$\Omega = \frac{2\langle e|d_s|g\rangle E_0}{\hbar}$$  \hspace{2cm} (A.6)

Note that the free atomic Hamiltonian in the rotating wave approximation is now:

$$H_A = \frac{p^2}{2m} + \hbar \Delta_L |\tilde{e}\rangle \langle \tilde{e}|$$  \hspace{2cm} (A.7)

where $\Delta_L = \omega_L - \omega_0$ is the detuning from resonance, and $|\tilde{e}\rangle = e^{i\omega_L t}|e\rangle$ is the excited state in the rotating frame.
Since detuning is large, spontaneous emission is negligible and the appropriate Schrödinger equation is:

\[(\hat{H}_A + \hat{H}_{AF})|\psi\rangle = i\hbar \partial_t |\psi\rangle\]  

(A.8)

One now expands the state vector \(|\psi\rangle\) into a product of internal and external states, such that \(|\psi\rangle = |\psi_e(t)\rangle|\bar{e}\rangle + |\psi_g(t)\rangle|g\rangle\) where the \(|\psi(t)\rangle\) are centre-of-mass states. The Schrödinger equation can now be separated into the coefficients of the ground \(|g\rangle\) and excited \(|e\rangle\) states:

\[
i\hbar \partial_t \psi_e = \frac{p^2}{2m} \psi_e + \left(\frac{\hbar\Omega}{2} \cos k_L x\right) \psi_g - \hbar \Delta_L \psi_e
\]

\[
i\hbar \partial_t \psi_g = \frac{p^2}{2m} \psi_g + \left(\frac{\hbar\Omega}{2} \cos k_L x\right) \psi_e
\]

(A.9)

These equations can be greatly simplified by using the adiabatic approximation. Under this approximation one can assume that internal atomic motion occurs on timescales much shorter than the slow centre-of-mass motion that is of interest here. One therefore assumes that internal motion is damped instantaneously, \(\partial_t \psi_e = 0\), since it is this variable that carries the internal time dependence - \(|\psi_g\rangle\) is defined to be at zero energy. Noting again that \(\hbar \Delta_L \gg p^2/2m\), the Schrödinger equation simplifies to:

\[
i\hbar \partial_t \psi_g = \left(\frac{p^2}{2m}\right) \psi_g + V_0 \cos(2k_L x) \psi_g
\]

(A.10)

and the centre-of-mass Hamiltonian is therefore:

\[
H = \frac{p^2}{2m} + V_0 \cos(2k_L x)
\]

(A.11)

which, apart from an arbitrary phase shift, is Eq. (7.2). Note that, as in the main text, \(V_0\) is defined as:

\[
V_0 = \frac{\hbar \Omega^2}{8 \delta L}
\]

(A.12)
A.2 Optical Potential

Following Jessen and Deutsch’s Physical Review A paper entitled “Quantum-state control in optical lattices” [105], one considers laser light described generally by the form:

\[ E_L(x, t) = \text{Re}[E_L(x)e^{-i\omega_L t}] \]  \hfill (A.13)

In addition one assumes that magnetic fields in the trapping region are static and that the laser detuning from atomic resonance is large, such that atoms in the ground state are subject to the potential:

\[ \hat{U}(x) = -E_L^*(x) \cdot \hat{\alpha} \cdot E_L(x) - \hat{\mu} \cdot \mathbf{B} \]  \hfill (A.14)

where \( \hat{\mu} = \hbar \gamma \hat{F} \) is the magnetic dipole operator, \( \gamma \) is the gyromagnetic ratio and \( \hat{F} \) is the total angular momentum operator. The atomic polarizability tensor, in the far off-resonant regime, is defined as:

\[ \hat{\alpha} = -\sum_e \frac{\hat{d}_{ge}\hat{d}_{eg}}{\hbar\Delta_{ge}} \]  \hfill (A.15)

with \( \hat{d}_{eg} \) being the electric dipole operator between the excited and ground states and \( \Delta_{ge} \) the detuning from resonance.

For an alkali-metal atom, such as the Caesium atom used in the experiments detailed in Chapter 7, which has a number of hyperfine excited states, the polarizability tensor becomes:

\[ \hat{\alpha} = -\sum_{F,F'} \frac{P_F \hat{d}_{F'F} \hat{d}_{F'}}{\hbar\Delta_{F,F'}} \]  \hfill (A.16)

where the \( P_F \) and \( P_{F'} \) are projection operators onto the ground and excited hyperfine states. Now consider the simpler case of the \(|J = 1/2\rangle \rightarrow |J' = 3/2\rangle\). In the case where the detuning is much larger than the hyperfine
splittings (as in the experiments) the polarizability tensor for the Caesium atom reduces to:

\[ \hat{\alpha} = P_F \hat{\alpha}(J \rightarrow J') P_F \]  \hspace{1cm} (A.17)

where \( \hat{\alpha}(J \rightarrow J') \) is the polarizability tensor of the \(|J\rangle \rightarrow |J'\rangle \) transition. The key point to note here is that despite the complicated level structure of the alkali metal atoms, in the limit of large detuning the optical potential has similar properties to that of the \(|J = 1/2 \rangle \rightarrow |J' = 3/2 \rangle \).

Jessen and Deutsch show in appendix B of their paper that this rank-2 polarizability tensor can be written as a sum of rank 0, 1 and 2 tensors for which (because it acts on a 2-dimensional Hilbert space) the rank 2 component vanishes. They demonstrate that it can therefore be written in terms of the identity (\( \hat{I} \)) and Pauli spin (\( \hat{\sigma}_i \)) operators:

\[ \hat{\alpha}_{ij}(J \rightarrow J') = \hat{\alpha}(\frac{2}{3} \hat{\delta}_{ij} \hat{I} - \frac{i}{3} \epsilon_{ijk} \hat{\sigma}_k) \]  \hspace{1cm} (A.18)

where the characteristic polarizability tensor, \( \hat{\alpha} \), of the \(|J \rangle \rightarrow |J'\rangle \) transition is shown in appendix A of [?] to be:

\[ \hat{\alpha} \equiv \frac{|\langle J' \rangle | d | J \rangle |^2}{\hbar \Delta_{F,F'}} \]  \hspace{1cm} (A.19)

By expressing the optical lattice field as \( E_L(x) = \text{Re}[E \bar{E}_L(x)e^{-i\omega t}] \), where \( \bar{E}_L(x) \) is the local polarization and assuming that each beam has amplitude \( E_1 \) one can therefore combine Eqs. (A.14),(A.18),(A.19) to write the optical potential as:

\[ \hat{U}(x) = -\frac{2}{3} U_1 |\bar{E}_L(x)|^2 \hat{I} + \frac{i}{3} U_1 [\bar{E}_L(x) \otimes \bar{E}_L(x)] \cdot \hat{\sigma} \]  \hspace{1cm} (A.20)

where now \( U_1 = \hat{\alpha} E_1^2 / 4 \). The first, Zeeman-state (i.e. hyperfine state) independent term is essentially a shift proportional to the local intensity of the laser field and the square of the laser polarisation. The second term, which is
Zeeman-state dependent, is usually described as an effective static magnetic field of the form $\mathbf{B}_{\text{eff}}(\mathbf{x}) \cdot \hat{\sigma}$. One notes that for the experiments detailed in this thesis one wishes to ensure that all the Zeeman states, and therefore all the atoms, feel the same potential. Since the effective magnetic field is a function of the ellipticity of the light field (through the term $\mathcal{c}_L^*(\mathbf{x}) \otimes \mathcal{c}_L(\mathbf{x})$), one can ensure this term vanishes by making the polarisations of all laser beams parallel to each other.
APPENDIX B

Papers

Three papers are presented in this appendix:

Proposal for a Chaotic Hamiltonian Ratchet Using Cold Atoms in Optical Lattices
T. S. Monteiro, P. A. Dando, N. A. C. Hutchings and M. R. Isherwood

Chaotic Filtering of Moving Atoms in Pulsed Optical Lattices by Control of Dynamical Localisation
T. Jonckheere, M. R. Isherwood and T. S. Monteiro

Chaotic Hamiltonian Ratchets for Pulsed Periodic Double-Well Potentials: Classical Correlations and the Ratchet Current
To Appear in Physical Review E


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