MAGIC MIRRORS, DENSE DIAMETERS, Baire Category

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Abstract. An old result of Zamfirescu says that for most convex curves $C$ in the plane most points in $\mathbb{R}^2$ lie on infinitely many normals to $C$, where most is meant in Baire category sense. We strengthen this result by showing that ‘infinitely many’ can be replaced by ‘continuum many’ in the statement. We present further theorems in the same spirit.

1. Introduction

In a 1982 paper [6] Tudor Zamfirescu proved a remarkable result saying that ‘most mirrors are magic’. For the mathematical formulation let $\mathcal{C}$ be the set of all closed convex curves in the plane $\mathbb{R}^2$. Fix some $C \in \mathcal{C}$ and $z \in C$ so that the tangent line, $T(z)$, to $C$ at $z$ is unique, then so is the normal line $N(z)$ to $C$ at $z$. A point $u \in \mathbb{R}^2$ sees an image of another point $v \in \mathbb{R}^2$ via $z$ if $u$ and $v$ and $C$ lie on the same side of $T(z)$ and the line $N(z)$ halves the angle $\angle uzv$. In particular, $u$ sees an image of itself via $z$ if $u \in N(z)$ and $u$ and $C$ are on the same side of $T(z)$.

With the Hausdorff metric $\mathcal{C}$ becomes a complete metric space. It is well-known that the normal $N(z)$ is unique at every point $z \in C$ for most convex curves $C \in \mathcal{C}$ in the Baire category sense, that is, for the elements of a comeagre set of curves in $\mathcal{C}$. Now the ‘most mirrors are magic’ statement is, precisely, that for most convex curves, most points in $\mathbb{R}^2$ (again in Baire category sense) see infinitely many images of themselves. Another theorem from [6], (generalized for higher dimensions in [7]) says that for most convex curves, most points in $\mathbb{R}^2$ see infinitely many images of any given point $v \in \mathbb{R}^2$. Zamfirescu actually proves the existence of countably many images and self-images.

The purpose of this paper is to show that most mirrors are even more magic.

Theorem 1.1. For most convex curves, most points in $\mathbb{R}^2$ see continuum many images of themselves.
Theorem 1.2. For most convex curves \( C \) and for every point \( v \in \mathbb{R}^2 \setminus C \), most points in \( \mathbb{R}^2 \) see continuum many images of \( v \).

The condition \( v \notin C \) in the last theorem is used to avoid some trivial complications in the proof. The statement holds even for \( v \in C \).

Remark. Let \( C^o \) denote the closed convex set whose boundary is \( C \).

The above definition of ' \( u \) sees an image of \( v \) via \( z \in C \)' means that the mirror side of \( C \) is the interior one, that is, the segment \( uz \) intersects the interior of \( C^o \). Theorem 1.1 does not hold when the mirror is on the other side of \( C \) because every point in \( \mathbb{R}^2 \setminus C^o \) lies on exactly one outer normal halfline to \( C \).

The segment \([a, b]\) is an affine diameter of \( C \) if there are distinct and parallel tangent lines to \( a, b \in C \). A result of Zamfirescu in [8] says that for most convex curves \( C \in \mathcal{C} \), most points in \( C^o \) are contained in infinitely many affine diameters of \( C \). This theorem is extended in [2] for typical \( d \)-dimensional convex bodies for every \( d \geq 2 \). In the planar case we again show the existence of continuum many diameters passing through most points in \( C^o \).

Theorem 1.3. For most convex curves \( C \in \mathcal{C} \), most points in \( C^o \) lie in continuum many diameters of \( C \).

It is well-known (see [4] or [9]) that most convex curves are strictly convex and continuously differentiable. Note that for a strictly convex curve \( C \), every point outside \( C^o \) lies on the line of at most one affine diameter as any two affine diameters have a point in common. It is not hard to see, actually, that for a strictly convex \( C \) every point outside \( C^o \) lies on a unique affine diameter.

2. Plan of proof

For \( C \in \mathcal{C} \) let \( \rho(z) \) denote the radius of curvature of \( C \) at \( z \in C \). Let \( \mathcal{D} \) denote the family of all convex curves \( C \in \mathcal{C} \) such that

1. there is a unique tangent line to \( C \) at every \( z \in C \),
2. \( \{ z \in C : \rho(z) = 0 \} \) is dense in \( C \),
3. \( \{ z \in C : \rho(z) = \infty \} \) is dense in \( C \).

It is well-known, see for instance [9], that \( \mathcal{D} \) is comeagre in \( \mathcal{C} \). We are going to show that every \( C \in \mathcal{D} \) has the property required in Theorem 1.1. We will need slightly different conditions for Theorems 1.2 and 1.3. But the basic steps of the proofs are the same. We explain them in this section in the case of Theorem 1.1.

Let \( C \in \mathcal{D} \) and define, for \( z \in C \), the halfline \( N^+(z) \subset N(z) \) that starts at \( z \) and intersects the interior of \( C^o \). Note that every \( u \in \mathbb{R}^2 \) lies on some \( N^+(z) \): namely when \( z \) is any farthest point from \( u \) on \( C \). Set \( L(u) = \{ z \in C : u \in N^+(z) \} \) and define

\[
H = \{ u \in \mathbb{R}^2 : L(u) \text{ is not perfect} \}.
\]
Lemma 2.1. \( H \) is a Borel set.

Write now \( u = (u_1, u_2) \in \mathbb{R}^2 \) and define \( H^{u_2} = \{ u_1 \in \mathbb{R} : (u_1, u_2) \in H \} \). This is just the section of \( H \) on the horizontal line \( \ell(u_2) = \{(x, y) \in \mathbb{R}^2 : y = u_2 \} \). There are two points \( z \in C \) with \( N(z) \) horizontal, so there are at most two exceptional values for \( u_2 \) where \( \ell(u_2) \) coincides with some \( N(z) \).

Lemma 2.2. Apart from those exceptional values, \( H^{u_2} \) is meagre.

These two lemmas imply Theorem 1.1. Indeed, deleting the (one or two) exceptional lines from \( H \) gives a Borel set \( H' \). According to a theorem of Kuratowski (see [3] page 53), if all horizontal sections of the Borel set \( H' \) are meagre, then so is \( H' \), and then \( H \) itself is meagre. So its complement is comeagre, so \( L(u) \) is perfect and non-empty for most \( u \in \mathbb{R}^2 \). The theorem follows now from the fact that a non-empty and perfect set has continuum many points. The proofs of Theorems 1.2 and 1.3 will use the same argument.

For the proof of Lemma 2.2 we need another lemma that appeared first as Lemma 2 in [5]. A function \( g : [0, 1] \rightarrow \mathbb{R}^2 \) is increasing on an interval \( I \subset [0, 1] \) (resp. decreasing on \( I \)) if every \( x, y \in I \) with \( x \leq y \) satisfy \( g(x) \leq g(y) \) (resp. \( g(x) \geq g(y) \)), and \( g \) is monotone in \( I \) if it is either increasing or decreasing there. For the sake of completeness we present the short proof.

Lemma 2.3. Assume \( g : [0, 1] \rightarrow \mathbb{R}^2 \) is continuous and is not monotone in any subinterval of \([0, 1]\). Then the set
\[
B = \{ b \in \mathbb{R} : \{ x : g(x) = b \} \text{ is not perfect} \}
\]
is meagre.

Proof of Lemma 2.3. For each \( b \in B \) the level set \( \{ x : g(x) = b \} \) has an isolated point, and so there is an open interval \( I_b \subset [0, 1] \) with rational endpoints in which \( g(x) = b \) has a unique solution. For a given rational interval \( (p,q) \) define
\[
B(p,q) = \{ b \in B : I_b = (p,q) \}.
\]
If every \( B(p,q) \) is nowhere dense, then we are done since \( B \), as a countable union of nowhere dense sets, is meagre. If some \( B(p,q) \) is not nowhere dense, then there is a non-empty open interval \( I \) in which \( B(p,q) \) is dense. The line \( y = b \), for a dense subset of \( I \), intersects the graph of \( g \) restricted to \( (p,q) \) in a single point. This implies easily that \( g \) is strictly monotone in a subinterval \( (p,q) \), contrary to our assumption. \( \square \)
3. Proof of the lemmas

Fix $C \in \mathcal{D}$ and let $z(\alpha)$ denote the point $z \in C$ where the halfline $N^+(z)$ spans angle $\alpha \in [0, 2\pi)$ with a fixed unit vector in $\mathbb{R}^2$. This is a parametrization of $C$ with $\alpha \in [0, 2\pi]$ and $z(0) = z(2\pi)$. We write $C_{\alpha,\beta}$ for the arc $\{z(\gamma) : \alpha < \gamma < \beta\}$ when $0 \leq \alpha < \beta \leq 2\pi$, and the definition is extended, naturally, to the case when $\alpha < 2\pi < \beta$. We always assume that $\alpha, \beta$ are rational and $\beta - \alpha$ is small, smaller than 0.1, say.

**Proof** of Lemma 2.1. Note first that the set
$$K = \{(u, z) \in \mathbb{R}^2 \times C : u \in N^+(z)\}$$
is closed. Further, $L(u)$ is not perfect if and only if there is a short arc $C_{\alpha,\beta}$ such that $u \in N^+(z)$ for a unique $z \in C_{\alpha,\beta}$. Thus
$$H = \bigcup_{\text{all } C_{\alpha,\beta}} \{u \in \mathbb{R}^2 : u \in N^+(z) \text{ for a unique } z \in C_{\alpha,\beta}\}.$$Let $p : K \to \mathbb{R}^2$ be the projection $p(u, z) = u$. Let $P_{\alpha,\beta}$ be the set of points $u \in \mathbb{R}^2$ such that there are more than one $z \in C_{\alpha,\beta}$ with $u \in N^+(z)$. Then
$$P_{\alpha,\beta} = \bigcup_{\gamma} p(K \cap (\mathbb{R}^2 \times C_{\alpha,\gamma})) \cap p(K \cap (\mathbb{R}^2 \times C_{\gamma,\beta}))$$where the union is taken over all rational $\gamma$ with $\alpha < \gamma < \beta$. Consequently
$$H = \bigcup_{\text{all } C_{\alpha,\beta}} p(K \cap (\mathbb{R}^2 \times C_{\alpha,\beta})) \setminus P_{\alpha,\beta}.$$Since $p(K \cap (\mathbb{R}^2 \times C_{\alpha,\beta}))$ is $F_\sigma$ for every $\alpha < \beta$, it follows that $H$ is indeed Borel. \hfill $\square$

**Proof** of Lemma 2.2. The set $z \in C$ where $N^+(z)$ intersects $\ell(u_2)$ in a single point consists of one or two open subarcs of $C$, as one can check easily. Let $C_1$ be such an arc. It suffices to see that
$$E = H^{u_2} \cap \{u_1 \in \mathbb{R} : (u_1, u_2) = \ell(u_2) \cap N(z) \text{ for some } z \in C_1\}$$is meagre, as $H^{u_2}$ either coincides with this set, or is the union of two such sets.

We may assume that $C_1$ is the graph of a convex function $F : J \to \mathbb{R}$ and $u_2 > F(x)$ on $J$ where $J$ is an open interval. (This position can be reached after a suitable reflection about a horizontal line.) With this notation, $E$ is the set of real numbers $u_1 \in \mathbb{R}$ such that the set of points $x \in J$ for which $(u_1, u_2) \in N^+(x, F(x))$ is not perfect.

Then $F'(x) = f(x)$ is continuous and increasing on $J$. Each $z \in C_1$ is a point $(x, F(x))$ on the graph of $F$. As $\rho(z) = (1 + f(x))^{3/2}/f'(x)$, $f'$ equals zero resp. infinity on a dense set in $J$. The normal $N(z)$ to $z = (x, F(x))$ has equation $(u_2 - F(x))f(x) = x - u_1$, as one checks readily.
With \( g(x) = (u_2 - F(x))f(x) - x \), \( g'(x) = -f(x)^2 + (u_2 - F(x))f'(x) - 1 \) and so on a dense set in \( J \) the value of \( g'(x) \) is positive, and on another dense set in \( J \) it is negative. So \( g \) is not monotone in any subinterval of \( J \). Lemma 2.3 implies now that \( E \) is meagre. 

\[ \square \]

4. Proof of Theorem 1.2

It is known [9] that for most \( C \in \mathcal{D} \) there is a dense set \( E \subset C \) such that at each point \( z \in E \) the lower radii of curvature in both directions \( \rho^+_{i}(z), \rho^-_{i}(z) \) vanish and the upper radii of curvature in both directions \( \rho^+_{s}(z), \rho^-_{s}(z) \) are infinite. We let \( \mathcal{D}_1 \) denote the set of all \( C \in \mathcal{D} \) possessing such a dense set \( E \). We are going to show that for each \( C \in \mathcal{D}_1 \), most points see continuum many images of any given point \( v \in \mathbb{R}^2, v \notin C \).

For \( z \in C \) we define the line \( R(z) \) as the reflected copy (with respect to \( N(z) \) ) of the line through \( v \) and \( z \). Note that \( R(z) \) depends continuously from \( z \). Here we need \( v \notin C \).

If \( u \) sees an image of \( v \) via \( z \), then \( u \in R(z) \). More precisely, \( u \) sees an image of \( v \) via \( z \) iff \( u, v \) and \( C \) are on the same side of \( T(z) \) and \( u \in R(z) \). Let \( R^+(z) \subset R(z) \) be the halfline that starts at \( z \) and is on the same side of \( T(z) \) as \( C \). Also, \( R^+(z) \) is well-defined for all \( z \in C \).

As before, \( \ell(u_2) \) is the horizontal line in \( \mathbb{R}^2 \) whose points have second coordinate equal to \( u_2 \). Define, for fixed \( u_2 \in \mathbb{R} \), \( H^{u_2} = \{u_1 \in \mathbb{R} : (u_1, u_2) \in H\} \). This is the same as the set of first coordinates of all \( u \in H \cap \ell(u_2) \).

In the generic case \( R(z) \) is not horizontal and so \( R(z) \cap \ell(u_2) \) is a single point. But we have to deal with non-generic situations, that is, when \( R(z) \) is horizontal and so coincides with \( \ell(u_2) \) for some \( u_2 \in \mathbb{R} \).

Define \( Z = \{z \in C : R(z) \text{ is horizontal}\} \) and \( U_2 = \{u_2 \in \mathbb{R} : \ell(u_2) = R(z) \text{ for some } z \in Z\} \). Both \( Z \) and \( U_2 \) are closed sets and there is a one-to-one correspondence between them given by \( z \mapsto u_2 \text{ iff } R(z) = \ell(u_2) \).

From now on we assume that \( Z \) is nowhere dense. We will justify this assumption at the end of the proof. Then \( U_2 \) is also nowhere dense. \( C \setminus Z \) is open in \( C \) and so its connected components \( C_1, C_2, \ldots \) are open arcs in \( C \), and there are at most countably many of them.

This time we define \( L(u, C_i) \) as the set of \( z \in C_i \) via which \( u \) sees an image of \( v \). Formally, \( L(u, C_i) = \{z \in C_i : u \in R^+(z)\} \), and define again, for fixed \( u_2 \in \mathbb{R} \),

\[ H^{u_2}_i = \{u_1 \in \mathbb{R} : L((u_1, u_2), C_i) \text{ is not perfect}\}. \]

A very similar proof shows that \( H^{u_2}_i \) is Borel. We omit the details, but mention that the condition \( v \notin C \) is needed to show that the corresponding \( K = \{(u, z) : \ldots\} \) is closed.

**Lemma 4.1.** For \( u_2 \notin U_2 \) the set \( H^{u_2}_i \) is meagre.
Proof. With every $u_1 \in H_{i}^{u_2}$ we associate a (rational) open arc $C_{\alpha,\beta}$ of $C_i$ such that $u = (u_1, u_2) \in R(z)$ for a unique $z \in C_{\alpha,\beta}$, namely for $z_u$. If the set of $u \in H_{i}^{u_2}$ that are associated with $C_{\alpha,\beta}$ is nowhere dense for every rational arc $C_{\alpha,\beta}$, then we are done as $H_{i}^{u_2}$ is the countable union of nowhere dense sets. So suppose that it is not nowhere dense for some $C_{\alpha,\beta}$. Then there is an open interval $I \in \mathbb{R}$ such that $H_{i}^{u_2}$ is dense in $I$.

Choose two distinct points $w^{-}, w^{+}$ from $I \cap H_{i}^{u_2}$. Then $z_{(w^{-}, u_2)}$ and $z_{(w^{+}, u_2)}$ are distinct points and so they are the endpoints of an open subarc $C_{\gamma,\delta}$ of $C_{\alpha,\beta}$. Define the map $h : C_{\gamma,\delta} \to I$ by $h(z) = u_1$ when $(u_1, u_2) = \ell(u_2) \cap R(z)$; $h$ is clearly continuous. It also monotone because its inverse is well-defined on a dense subset $I$.

We show next that this is impossible. Choose $z_0 \in C_{\gamma,\delta} \cap E$ (recall that $E$ is dense in $C$).

We fix a new coordinate system in $\mathbb{R}^2$: the origin coincides with $z_0$, the $x$ axis with $T(z_0)$, the tangent line to $C$ at $z_0$, and the $y$ axis is $N(z_0)$; see the figure. We assume w.l.o.g. that $v_1 < 0$ and $v_2 > 0$ where $v = (v_1, v_2)$. A subarc of $C_{\gamma,\delta}$ is the graph of a non-negative convex function $F : [0, \Delta) \to \mathbb{R}$ such that $F(0) = 0$ and $z = z(x) = (x, F(x))$ and $f(x) = F'(x)$ is an increasing function with $f(0) = 0$. If the lines $R(z(x))$ and $R(z(0))$ intersect, then they intersect in a single point whose $y$ component is denoted by $y(x)$.

Claim 4.2. For every $\varepsilon > 0$ there are $x_1, x_2 \in (0, \varepsilon)$ so that $y(x_1) < 0$ and $0 < y(x_2) < \varepsilon$.

Proof. We use the notation of the figure. The sine theorem for the triangle with vertices $v, 0, z(x)$ implies that $\phi(x) = x \sin \lambda/|v|(1 + o(1))$
where $o(1)$ is understood when $x \to 0$. The slope of the line $R(z(x))$ is 
\[
\tan(\lambda - \phi + 2\psi),
\]
and 
\[
\tan \psi(x) = f(x) = x \cdot \frac{f(x) - 0}{x - 0}.
\]

The liminf and limsup of the last fraction (when $x \to 0$) is the curvature 
\[
\rho_1^+(z_0) = 0 \quad \text{and} \quad \rho_2^+(z_0) = \infty \quad \text{of} \quad C \quad \text{at} \quad z_0 \in E.
\]
Consequently for every integer $n > 0$ there is $x \in (0, 1/n)$ with $\tan \psi(x) < x/n$ and 
also with $\tan \psi(x) > nx$. Then there is $x_1 < 1/n$ such that $\lambda/2 < \lambda - \phi(x_1) + 2\psi(x_1) < \lambda$ which implies, after a simple checking, that $y(x_1) < 0$. Also, there is $x_2 < 1/n$ such that $\lambda - \phi(x_2) + 2\psi(x_2) > \lambda + nx_2/2$. A direct computation show then that $0 < y(x_2) < \varepsilon$ if $n$ is chosen large enough.

We return to the proof of Lemma 4.1. The claim shows that there are $x_1, x_2, x_3 \in (0, \Delta)$ with $x_1 < x_2 < x_3$ such that the lines $R(z(x_1))$ and $R(z(x_3))$ strictly separate the origin and the point $R(z_0) \cap \ell(u_2)$ while $R(z(x_2))$ does not. Writing $z_i = z(x_i), i = 1, 2, 3$ this implies that $z_2$ is between $z_1$ and $z_3$ while $h(z_2)$ is not on the segment $(h(z_1), h(z_3))$.

So $h$ is not monotone. \hfill \square

It is evident that $U_2$, and consequently $U$, is closed and nowhere dense, so $U$ is meagre. The lemma implies, by Kuratowski’s theorem, that $H_i \setminus U$ is meagre. It follows that $H_i$ is meagre and then so is $H = \bigcup H_i$. Thus every point in the complement of $H$ sees an image of $v$ via a perfect set in $C$, except possibly for the points of the meagre set $U$. This perfect set is nonempty, because every point sees an image of $v$ via some $z \in C$ (for instance by Zamfirescu’s result [6, Theorem 1]). So most points see continuum many images of $v$.

Finally we justify the assumption that $Z$ is nowhere dense. This is done by choosing the horizontal direction (which is at our liberty) suitably. So for a given direction $(\cos \theta, \sin \theta)$ write $Z(\theta)$ for the set of $z \in C$ such that $R(z)$ is parallel with this direction. Every $Z(\theta)$ is closed and so there is one (actually, many) among them that contains no non-empty open arc of $C$. Choose the corresponding $\theta$ for the horizontal direction, then $Z = Z(\theta)$ is nowhere dense. \hfill \square

5. Proof of Theorem 1.3

Write $C_1$ for the set of all strictly convex curves $C$ that have a unique tangent line at every $z \in C$. Assume $C \in C_1$ and use the parametrization $z : [0, 2\pi) \to C$ as before. For $z \in C$ with $z = z(\alpha)$ let $z^{\ast} \in C$ be the opposite point, that is $z^{\ast} = z(\alpha + \pi)$. It is evident that $z^{\ast\ast} = z$. Further, $[z, z^{\ast}]$ is always an affine diameter of $C$ and all affine diameters of $C$ are of this form. We need a geometric lemma.
Lemma 5.1. Most convex curves \( C \in C_1 \) have the following property: for every \( \varepsilon > 0 \) every subarc \( C_0 \) of \( C \) contains points \( x, y \) such that

\[
\frac{|x - y|}{|x^* - y^*|} < \varepsilon.
\]

This lemma follows from the results of [1]. To make the paper self contained we give a proof in the next section. From now on we assume that \( C \in C_1 \) has the property in the Lemma.

We use again the same proof scheme: for \( u \in C^o \) define \( L(u) = \{ z \in C : u \in [z, z^*] \} \); this set is nonempty as one can check easily that every point \( u \in C^o \) lies on at least one affine diameter. (This holds for every convex curve, not only for the ones in \( C_1 \).) We set next \( H = \{ u \in C^o : L(u) \) is not perfect \}; and, for fixed \( u_2 \in \mathbb{R}^2 \), \( H^{u_2} = H \cap \ell(u_2) \). The same proof as in Section 3 shows that \( H \) is Borel. We claim that \( H \) is meagre which implies Theorem 1.3.

\( C \) has a horizontal affine diameter and we assume w.l.o.g. that it lies on the line \( \ell(0) \). To see that \( H \) is meagre it suffices to show (by Kuratowski’s theorem) that \( H^{u_2} \) is meagre as a subset of \( \ell(u_2) \) for \( u_2 \neq 0 \). We only consider \( u_2 \in \mathbb{R}, u_2 \neq 0 \) with \( \ell(u_2) \cap C \neq \emptyset \). With each \( u \in H^{u_2} \) we associate an isolated point \( z_u \in C \) and a short rational arc \( C_{\alpha,\beta} \) such that \( z_u \) is the unique \( z \in C_{\alpha,\beta} \) with \( u \in [z, z^*] \). We are done if, for each short rational arc \( C_{\alpha,\beta} \), the set of \( u \in H^{u_2} \) that are associated with \( C_{\alpha,\beta} \) is nowhere dense. So suppose that this fails for some \( C_{\alpha,\beta} \). Then there is an open interval \( I \subset \ell(u_2) \) on which \( H^{u_2} \) is dense. Choose distinct points \( u^- \) and \( u^+ \) from \( I \cap H^{u_2} \) and let \( z^-, z^+ \) be the corresponding isolated points on \( C_{\alpha,\beta} \). We suppose (by symmetry) that \( C_{\alpha,\beta} \) is below the line \( \ell(u_2) \).

From now on we consider the subarc \( C_0 \subset C_{\alpha,\beta} \) whose endpoints are \( z^- \) and \( z^+ \) and its opposite arc \( C_0^- \). We note here that the map \( z \to z^* \) is order preserving on \( C_0 \), meaning that if \( v \in C_0 \) is between \( v_1, v_2 \in C_0 \), then \( v^* \) lies between \( v_1^* \) and \( v_2^* \) on \( C_0^o \).

Define a map \( m : C_0 \to \ell(u_2) \) via \( m(z) = \ell(u_2) \cap [z, z^*] \); \( m \) is continuous. It is one-to-one on a dense subset of \( C_0 \) which implies that \( m \) is order-preserving in the sense that if \( v \in C_0 \) is between \( v_1, v_2 \in C_0 \), then \( m(v) \) lies between \( m(v_1) \) and \( m(v_2) \) on \( \ell(u_2) \). We show that this is impossible.

Using Lemma 5.1 choose two points \( v_1, v_2 \) on \( C_0 \) very close to each other so that \( |v_1 - v_2| \) is much shorter than \( |v_1^* - v_2^*| \). Then the segment \([v_1, v_2]\) is almost parallel with \([v_1^*, v_2^*]\), and the diameters \([v_1, v_1^*]\) and \([v_2, v_2^*]\) intersect in a point very close to \([v_1, v_2]\), so this point is below \( \ell(u_2) \). Now apply Lemma 5.1 on the arc between \( v_1^* \) and \( v_2^* \). We get points \( w_1 \) and \( w_2 \) very close to each other on this arc so that \( |w_1 - w_2| \) is much shorter than \( |w_1^* - w_2^*| \). This time the diameters \([w_1, w_1^*]\) and \([w_2, w_2^*]\) intersect above \( \ell(u_2) \). We assume (by choosing the names \( w_1, w_2 \) properly) that \( v_1^*, w_1, w_2, v_2^* \) come in this order on \( C_0^o \) and so
Given $C \in C_1$ define $A_{k,n}$ as the short arc between $z_k = z(2\pi k/2n)$ and $z_{k+1} = z(2\pi (k+1)/2n)$ where $k = 0, 1, \ldots, 2n - 1$. For positive integers $n, m$ let $F_{n,m}$ be the set of all $C \in C_1$ for which there is $A_{k,n}$ such that for all $x, y \in A_{k,n}$ ($x \neq y$)

$$\frac{|x - y|}{|x^* - y^*|} \geq \frac{1}{m}.$$  

It is easy to see that $F_{n,m}$ is closed in $C_1$, we omit the details. We show next that it is nowhere dense.

Fix a $C \in C_1$ and $\varepsilon > 0$ and let $U(C)$ denote the $\varepsilon$-neighbourhood of $C$. We construct another convex curve $\Gamma \in C_1$ that is contained in $U(C)$ but is not an element of $F_{n,m}$. Fix $k \in \{0, 1, \ldots, n - 1\}$ and consider a fixed arc $A_{k,n}$ and its opposite arc $A^*_{k,n} = A_{k+n,n}$. Let $T_k$ be the tangent line to $C$ at $z((k + \frac{1}{2})\pi/n)$ and $T^*_k$ be the parallel tangent line at $z((k + n + \frac{1}{2})\pi/n)$. Translate $T^*_k$ a little so that the translated copy intersects $C$ in two points $x, y_1$ and the segment $[x, y_1]$ lies in $U(C)$ and is much shorter than $[z_{k+n}, z_{k+1+n}]$. Similarly translate $T_k$ a little so that the translated copy intersects $C$ in $x_2, y_2$ and $[x_2, y_2]$ lies in $U(C)$, and is much shorter than $[z_k, z_{k+1}]$ and, most importantly, it is much shorter than $[x_1, y_1]$, namely, $m|x_2 - y_2| < |x_1 - y_1|$. This is clearly possible.

Now we choose points $w_1$ resp. $w_2$ from the caps cut off from $C^0$ by the segment $[x_1, y_1]$ and $[x_2, y_2]$ so that, for $i = 1, 2$, the triangles $\triangle_i = \text{conv}\{x_i, y_i, w_i\}$ are homothetic. This is possible again. Note that $[x_1, w_1]$ and $[x_2, w_2]$ are parallel, and so are $[y_1, w_1]$ and $[y_2, w_2]$.

The next target is construct a convex curve $\Gamma_k$ from $z_k$ to $z_{k+1}$ going through $x_2$ and $y_2$ that lies in $U(C)$, has a unique tangent at every point, and this tangent coincides with the line through $x_2, w_2$ at $x_2$ and with the line through $y_2, w_2$ at $y_2$. Also, an analogous curve $\Gamma_{k+n}$ is needed from $z_{k+n}$ to $z_{k+1+n}$.

This is quite easy. The unique parabola arc connecting $x_2$ to $y_2$ within $\triangle_2$ that touches the sides $[x_2, w_2]$ at $x_2$ and $[y_2, w_2]$ at $y_2$ is the middle piece of $\Gamma_k$. To connect this arc by a convex curve to $z_k$ (say) within $U(C)$ choose a point $w \in C$ on the arc between $z_k$ and $y_2$ so close to $y_2$ that the triangle $\triangle$ delimited by $T(z)$, the line through $y_2, w_2$, and the segment $[y_2, z]$ lies in $U(C)$. The analogous parabola arc in $\triangle$ gives the next piece of $\Gamma_k$, and then add to this piece the subarc of $C$ between $w$ and $z_k$. The middle piece of $\Gamma_k$ is continued to $z_{k+1}$ the same way.
The convex curve $\Gamma_{k+n}$ connecting $z_{k+n}$ to $z_{k+1+n}$ is constructed the same way. Note that the tangents to $\Gamma_k$ at $x_2$ (resp. $y_2$) are parallel with the tangents to $\Gamma_{k+n}$ at $x_1$ (and $y_1$).

The curves $\Gamma_k$ for $k = 0, \ldots, 2n - 1$ together form a convex curve $\Gamma \in C_1$. It has parallel tangents at $x_1 \in \Gamma_{k+n}$ and $x_2 \in \Gamma_k$, and also at $y_1$ and $y_2$. Thus $[x_1, x_2]$ and $[y_1, y_2]$ are affine diameters of $\Gamma$ and $m|x_1 - y_1| < |x_2 - y_2|$. As this holds for every $k$, $\Gamma \notin F_{n,m}$. Thus $F_{n,m}$ is indeed nowhere dense.

It follows that $C_2 = C_1 \setminus \bigcup_{n,m} F_{n,m}$ is comeagre in $C_1$. We show next that every $C \in C_2$ satisfies the requirement of the lemma. So we are given $\varepsilon > 0$ and a short subarc $C_0$ of $C$. Take a positive integer $m$ with $1/m < \varepsilon$. For a suitably large $n$, $C_0$ contains an arc of the form $A_{k,n}$. As $C \notin F_{n,m}$, there are distinct points $x, y \in A_{k,n}$ with

$$\frac{|x - y|}{|x^* - y^*|} \leq \frac{1}{m} < \varepsilon.$$ 

This finishes the proof. \hfill $\square$

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