Asymptotic analysis of the transmission eigenvalue problem for a Dirichlet obstacle coated by a thin layer of non-absorbing media

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We consider the transmission eigenvalue problem for an impenetrable obstacle with Dirichlet boundary condition surrounded by a thin layer of non-absorbing inhomogeneous material. We derive a rigorous asymptotic expansion for the first transmission eigenvalue with respect to the thickness of the thin layer. Our convergence analysis is based on a Max-Min principle and an iterative approach which involves estimates on the corresponding eigenfunctions. We provide explicit expressions for the terms in the asymptotic expansion up to order three.

Keywords: Transmission eigenvalues, thin layers, asymptotic methods, inverse scattering

1 Introduction

Transmission eigenvalues appear in the study of scattering by inhomogeneous media and are closely related to non-scattering frequencies Cakoni & Haddar (2012), Blasten et al. (preprint). Such eigenvalues provide information about material properties of the scattering media Cakoni et al. (2010b) and can be determined from scattering data Cakoni et al. (2010a), Kirsch & Lechleiter (to appear). Hence they can play an important role in a variety of inverse problems in target identification and non-destructive testing Giorgi & Haddar (2012). The transmission eigenvalue problem is a non-selfadjoint and non-linear problem that is not covered by the standard theory of eigenvalue problems for elliptic operators. In the past few years transmission eigenvalues have become an important area of research in inverse scattering theory. Since the first proof of existence of transmission eigenvalues in Cakoni et al. (2010b) and Päivärinta & Sylvester (2008), the interest in the transmission eigenvalue problem has increased, resulting in a number of important advancements. For an update survey on the topic we refer the reader to Cakoni & Haddar (2012).

In this paper we consider the transmission eigenvalue problem corresponding to the scattering by an impenetrable obstacle with Dirichlet boundary condition coated by a thin layer of non-absorbing inhomogeneous material. The existence and discreteness of transmission eigenvalue problem is investigated in Cakoni et al. (2010a) (see also Lakshtanov & Vainberg (preprint)). In the two-dimensional case this problem models the scattering of TE-polarized electromagnetic waves (written in terms of the electric field) by an infinitely long cylindrical prefect conductor coated by a thin layer of non-magnetic dielectric material. In the three dimensional case it models the scattering of acoustic waves by a sound-soft object surrounded by acoustically non-absorbing material. It is well known (see e.g. Bendali & Lem-
rabet (1996)) that the first order approximation to the scattering problem for a coated perfect conductor is an exterior boundary value problem with impedance type boundary condition where the impedance function depends inverse proportionally to the thickness of the layer, here denoted by $\delta$. The corresponding "non-scattering" frequencies for this approximate model become the eigenvalues of a non-coercive Robin eigenvalue problem, which is studied by the authors of this paper in Cakoni et al. (preprint).

The main concern of this study is to develop a rigorous asymptotic expansion for transmission eigenvalues as $\delta \to 0$. Our asymptotic analysis is based on an iterative and constructive approach. We restrict ourselves here to the first transmission eigenvalue. As expected this transmission eigenvalue is close to the first Dirichlet eigenvalue up to order $\delta$, result that is proven directly in this paper by using the Max-Min principle. Then, the main idea of our approach is, roughly speaking, having proven convergence of order $k$ for the asymptotic expansion of the transmission eigenvalue, we next prove estimates of order $k$ for the corresponding eigenfunctions by using standard approximation results for the eigenfunctions of the negative Laplacian with Dirichlet boundary conditions. Then, we deduce convergence at order $k + 1$ for the eigenvalues by using the Max-Min principle. Although our analysis can in principle be carried through for any order, for sake of simplicity we provide here explicit expressions only for the terms up to order three in the asymptotic expansion of the first transmission eigenvalue. The explicit construction of the asymptotic expansion is simplified by the fact that the first eigenvalue of the Dirichlet problem is simple. The extension of our analysis to higher order transmission eigenvalues is challenging, first because explicit construction of the asymptotic is complicated and second because one looses the characterization of the transmission eigenvalues in terms of a Max-Min principle.

From practical point of view, the second order expansion provides in fact a formula for the thickness of the layer in terms of the first (measurable) transmission eigenvalue. Unfortunately, the refractive index of the layer does not appear in the first three terms of the asymptotic expansion. Of course, the refractive index will show in higher order terms but then the obtained reconstruction formula would be highly unstable with respect to noise in the transmission eigenvalue. A better model to capture both the thickness and the refractive index in the first order term in the context of electromagnetic scattering is to write the problem in terms of the magnetic field, which would lead to Neumann boundary condition on the boundary of the inclusion. Unfortunately the transmission eigenvalue problem for inhomogeneous media containing an inclusion with Neumann boundary condition is still open. Moreover, no Max-Min principle is available in this case which is the corner stone of our approach.

The structure of the paper is as follows. In the next section we formulate the problem and recall some relevant results on the transmission eigenvalue problem for an inhomogeneous media containing an inclusion with Dirichlet boundary condition. In Section 3 we derive the formal asymptotic expansion for transmission eigenvalues and provide explicit formulas for the terms up to order three. Section 4 is dedicated to the rigorous convergence proof of the asymptotic expansion derived in the previous section for the first transmission eigenvalue. For our analysis we need various technical results that to our knowledge are not available in the literature, in particular elliptic a priori estimates and trace lemma with explicit dependance on $\delta$. To keep the reader focused in the main goal of the paper, we prove all the auxiliary results needed for our analysis in Appendix.

2 Formulation of the problem

We consider an impenetrable object coated with a thin layer of non-absorbing penetrable material with refractive index $n$ which occupies the region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ where $\Omega$ is bounded and simply connected with smooth enough (to become precise later) boundary $\Gamma$. We denote by

$$\Omega_\delta = \{ x \in \Omega \text{ such that } d(x, \Gamma) > \delta \}$$
and by
\[ \Gamma_\delta = \{ x \in \Omega \text{ such that } d(x, \Gamma) = \delta \} \]
its boundary. The simply connected domain \( \Omega_\delta \) (see Figure 1) represents here the impenetrable object and \( \Omega \setminus \overline{\Omega_\delta} \) represents the thin layer. The scattering of an incident wave \( u^i \), which here for simplicity is assumed to be an entire solution of the Helmholtz equation (one could also consider the incident field to be a point source located outside \( \Omega \)), by such a structure gives rise to a scattered field \( u^s = u - u^i \), with \( u \) being the total field, that satisfies
\[
\begin{cases}
\Delta u + k^2 n u = 0 & \text{in } \Omega \setminus \overline{\Omega_\delta}, \\
\Delta u + k^2 u = 0 & \text{in } \Omega_{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}, \\
\frac{\partial u}{\partial \nu} = 0, & [u] = 0 \text{ on } \Gamma, \\
u \text{ is unitary normal to } \Gamma \text{ directed inward to } \Omega, \\
u^+ - v^- & \text{denotes the jump of } v \text{ across } \Gamma \text{ where } v^+ \text{ is the exterior trace of } v \text{ and } v^- \text{ is the interior trace of } v \text{ on } \Gamma. \\
The corresponding transmission eigenvalue problem is to find the values of \( k^2_\delta \) such that there exists a non trivial solution \((w_\delta, v_\delta)\) to the following homogeneous coupled problem
\[
\begin{cases}
\Delta w_\delta + k^2_\delta n w_\delta = 0 & \text{in } \Omega \setminus \overline{\Omega_\delta}, \\
\Delta v_\delta + k^2_\delta v_\delta = 0 & \text{in } \Omega, \\
\frac{\partial v_\delta}{\partial \nu} = \frac{\partial w_\delta}{\partial \nu}, & v_\delta = w_\delta \text{ on } \Gamma, \\
w_\delta = 0 & \text{on } \Gamma_\delta.
\end{cases}
\]

**Definition 2.1** The values \( k^2_\delta > 0 \) for which (2.2) has a non trivial solution \((w_\delta, v_\delta)\) to the following homogeneous coupled problem
\[
\begin{cases}
\Delta w_\delta + k^2_\delta n w_\delta = 0 & \text{in } \Omega \setminus \overline{\Omega_\delta}, \\
\Delta v_\delta + k^2_\delta v_\delta = 0 & \text{in } \Omega, \\
\frac{\partial v_\delta}{\partial \nu} = \frac{\partial w_\delta}{\partial \nu}, & v_\delta = w_\delta \text{ on } \Gamma, \\
w_\delta = 0 & \text{on } \Gamma_\delta.
\end{cases}
\]

The scattering layered object

![Fig. 1. The scattering layered object](image-url)
It is shown in Cakoni et al. (2010a) that the real transmission eigenvalues (the wavenumber $k$ is related to the interrogating frequency) can be determined from measured far field (or near field) scattering data. Note that the transmission eigenvalue problem is non-selfadjoint and complex eigenvalues may occur but, from practical point of view as discussed in Introduction and the fact that only real transmission eigenvalues are proven to exist, here we are interested only on real transmission eigenvalues (see e.g. Cakoni & Haddar (2012)). Our main goal in this paper is to derive rigorous asymptotic expansions for transmission eigenvalues in terms of the thickness of the layer $\delta$ as $\delta \to 0$.

The transmission eigenvalue problem for an inhomogeneity containing an impenetrable inclusion with Dirichlet boundary condition is investigated in Cakoni et al. (2012) and Lakshtanov & Vainberg (preprint) (our problem (2.2) is exactly of that form) where the discreteness and existence of real transmission eigenvalues is shown under appropriate assumptions on the refractive index $n$. For the sake of reader’s convenience and later use we summarize the main results from Cakoni et al. (2012).

The first step in the analysis of (2.2) consists in reformulating it as an eigenvalue problem for a fourth order equation. To this end, introducing

$$u_\delta = \begin{cases} w_\delta - v_\delta \text{ in } \Omega \setminus \overline{\Omega_\delta} \\ -v_\delta \text{ in } \Omega_\delta \end{cases} \tag{2.3}$$

we obtain that this $u_\delta$ satisfies

$$(\Delta + k_\delta^2 \frac{1}{1-n}(\Delta + k_\delta^2 n))u_\delta = 0 \text{ in } \Omega \setminus \overline{\Omega_\delta}.$$\tag{2.4}

Equation (2.4) together with the fact that $u_\delta$ must be in $H^1_0(\Omega)$ and satisfy the Helmholtz equation in $\Omega_\delta$ suggest that to arrive at a variational formulation equivalent to the eigenvalue problem (2.2) we need to introduce the space

$$W_\delta := \left\{ u \in H^1_0(\Omega) \cap H^1_0(\Omega \setminus \overline{\Omega_\delta}) \text{ such that } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \right\}$$

equipped with the norm

$$\|u\|_{W_\delta}^2 := \|u\|_{H^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega \setminus \overline{\Omega_\delta})}^2.$$ \tag{2.4}

Then it is shown in Cakoni et al. (2012) that $k_\delta^2 > 0$ is a transmission eigenvalue (according to Definition 2.1) with associated eigenfunctions $(w_\delta, v_\delta)$ if and only if $u_\delta$ defined by (2.3) solves

$$A_{k_\delta} u_\delta - k_\delta^4 B u_\delta = 0 \tag{2.5}$$

where the bounded linear self-adjoint operators $A_k : W_\delta \to W_\delta$ and $B : W_\delta \to W_\delta$ are given by

$$(A_k u, v)_{W_\delta} := \int_{\Omega \setminus \overline{\Omega_\delta}} \frac{1}{1-n} (\Delta u + k^2 u)(\Delta v + k^2 v) \, dx + k^4 \int_{\Omega} u v \, dx + k^2 \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

$$(B u, v)_{W_\delta} := 2 \int_{\Omega} u v \, dx.$$ \tag{2.6}

In the following we denote $n_* = \inf_{\Omega \setminus \overline{\Omega_\delta}} n(x)$ and $n^* = \sup_{\Omega \setminus \overline{\Omega_\delta}} n(x)$. The operators $A_k$ and $B$ satisfy the following properties.
exists a constant $C > 0$ such that for all $\delta > 0$

$$|(A_ku, u)_{W_\delta}| \geq C\|u\|^2_{W_\delta}.$$ 

**Proof.** The proof can be found in (Cakoni et al., 2012, Theorem 2.1). The fact that the coercivity constant $C$ is independent of $\delta$ is clear in this proof.

We remark that if $k\delta$ and $u_\delta$ do not satisfy (2.5), then $(w_\delta, v_\delta)$ are obtained from $u_\delta$ by

$$w_\delta = \frac{1}{k_\delta^2(1-n)}(\Delta u_\delta + k_\delta^2 u_\delta) \text{ in } \Omega \setminus \Gamma_\delta,$$

$$v_\delta = \begin{cases} 
n -u_\delta \text{ in } \Omega_\delta, \\
 \frac{1}{k_\delta^2(1-n)}(\Delta u_\delta + k_\delta^2 u_\delta) \text{ in } \Omega \setminus \Gamma_\delta. 
\end{cases}$$

(2.6) (2.7)

The following result proven in Cakoni et al. (2012) is the starting point of our discussion.

**Theorem 2.3** Assume that $0 < n_* < n(x) < n^* < 1$. There exist an infinite discrete set of transmission eigenvalues and $+\infty$ is the only accumulation point. At this point we choose to normalize the $w_\delta$ and $v_\delta$ so that

$$\|u_\delta\|_{L^2(\Omega)} = 1.$$ 

The following regularity result for the eigenfunctions $(w_\delta, v_\delta)$ holds true.

**Lemma 2.1** Assume that $\Gamma$ is a $C^{k+2}$-boundary and $n \in C^{k+2}(\overline{\Omega} \setminus \Omega_\delta)$ with $k \geq 2$. Then $w_\delta \in H^k(\Omega \setminus \Gamma_\delta)$ and $v_\delta \in H^k(\Omega)$.

**Proof.** First since $\Delta v_\delta = -k_\delta^2 v_\delta$, using interior elliptic regularity for the Laplacian, we have that $v_\delta \in C^\infty(\omega)$ for all open set $\omega \subset \subset \Omega$. Hence its trace and its normal derivative trace on $\Gamma_\delta$ are in $H^{k+2-1/2}(\Gamma_\delta)$ and $H^{k+2-3/2}(\Gamma_\delta)$ respectively. Using the same argument but this time for the Laplace operator with homogeneous Dirichlet boundary condition on $\Gamma_\delta$ we can conclude that the trace of the normal derivative of $w_\delta$ on $\Gamma_\delta$ is also in $H^{k+2-3/2}(\Gamma_\delta)$. Hence we can easily obtain that on $\Gamma_\delta$ we have

$$\begin{cases} 
\Delta u_\delta \in H^{k+2-1/2}(\Gamma_\delta), \\
\frac{\partial}{\partial n} u_\delta \in H^{k+2-3/2}(\Gamma_\delta) 
\end{cases}$$

where $\nu_\delta$ is the unit normal to $\Gamma_\delta$ directed inward $\Omega_\delta$. Since $u_\delta$ satisfies (2.4) in $\Omega \setminus \Gamma_\delta$ with homogeneous boundary conditions on $\Gamma$, regularity results for the bilaplacian implies that $u_\delta$ is in $H^{k+2}(\Omega \setminus \Gamma_\delta)$ (see e.g. Agmon (1965)). We finally obtain the result by using (2.6) and (2.7).

From now on we assume that the refractive index satisfies

$$0 < n_* < n(x) < n^* < 1.$$ 

This assumption ensure existence of the interior transmission eigenvalues (Theorem 2.3) but is more restrictive than the one proposed in Lakshtanov & Vainberg (preprint) that allows $n$ to be greater than 1 provided the thickness of the layer is sufficiently small. Nevertheless, when $n > 1$ the operator $A_k\delta$ is sign indefinite and we loose the Max-Min principle which is the main ingredient of our approach.
3 Formal Asymptotic Expansion

3.1 Preliminary material

For the sake of simplicity, here we perform the asymptotic expansion in the two dimensional case. The extension to three dimensional case is purely a technical issue and it is possible to obtain similar asymptotic expansions by using the same approach. Having limited ourselves to the two dimensional case and assuming that the boundary is $C^{k+2}$-smooth for $k \geq 2$, we can parametrize $\Gamma$ as

$$\Gamma = \{x_F(s), \ s \in [0; s_0]\}$$

where the periodic function $x_F : [0; s_0] \rightarrow \mathbb{R}^2$ is in $C^{k+2}([0; s_0])$ for some $s_0 > 0$. Moreover, we can choose this parameterization such that the tangent vector $\tau(s) := \frac{dx_F}{ds}(s)$ to the surface $\Gamma$ at the arbitrary point $x_F(s)$ is a unit vector. Then denoting by $\nu(s)$ the inward unit normal vector to $\Gamma$ at the point $x_F(s)$ and we can define the curvature $\kappa(s)$ by

$$\frac{d\tau}{ds}(s) = -\kappa(s)\nu(s).$$

Based on this parameterization of the curve $\Gamma$, we obtain the following parameterization of the surface $\Gamma_\delta$

$$\Gamma_\delta = \{x_F(s) + \delta(s)\nu(s), \ s \in [0; s_0]\}$$

(3.1)

where $\delta \in C^\infty([0; s_0])$ is a periodic function of sufficiently small values. Let us define by

$$\eta_0 := \inf_{s \in [0, s_0]} \frac{1}{|\kappa(s)|},$$

and $\Omega_0 := \{x \in \mathbb{R}^2, \ \text{dist}(x, \Gamma) \leq \eta_0\}$. Then the map

$$\varphi : [0, s_0] \times [-\eta_0, \eta_0] \rightarrow \Omega_0$$

$$(s, \eta) \mapsto x_F(s) + \eta \nu(s)$$

is a $C^{k+2}$-diffeomorphism, in other words, for every point $x \in \Omega_0$ there exists a unique $(s, \eta) \in [0, s_0] \times [-\eta_0, \eta_0]$ such that

$$x = x_F(s) + \eta \nu(s).$$

Next, for any function $u$ defined on $\Omega_0$ we can define $\tilde{u}$ in $[0, s_0] \times [-\eta_0, \eta_0]$ by

$$\tilde{u}(s, \eta) := u \circ \varphi(s, \eta)$$

(3.2)

and the gradient of $u$ in the local coordinates $(s, \eta)$ writes as

$$\nabla u = \frac{1}{1 + \eta \kappa} \frac{\partial}{\partial s} \tilde{u} \tau + \frac{\partial}{\partial \eta} \nu.$$  

Furthermore, using integration by parts we have that the divergence of a vector field $\vec{u} = u_\tau \tau + u_n \nu$ writes as

$$\text{div} \vec{u} = \frac{1}{1 + \eta \kappa} \frac{\partial}{\partial s} \tilde{u}_\tau + \frac{1}{1 + \eta \kappa} \frac{\partial}{\partial \eta} (1 + \eta \kappa) \tilde{u}_n.$$  

We finally denote by $J_{s, \eta} := |\det(\nabla \varphi(s, \eta))| = 1 + \eta \kappa(s)$ the Jacobian of the change of variables.
Formal derivation of the asymptotic expansion

Let us now turn our attention to the transmission eigenvalue problem (2.2). To be able to carry on our computations, we assume that the function \( \delta \) used in (3.1) to define the interior boundary is of the form

\[
\delta(s) = \delta_0 g(s)
\]

for some constant \( \delta_0 > 0 \) and some strictly positive \( C^\infty \)-function \( g \) independent of \( \delta_0 \) such that \( |\delta_0 g(s)| < \eta_0 \). To simplify the notations and since there is no ambiguity, we make no distinction between \( g \) as a function of local and global variables. Then, we postulate the following ansatz for the interior transmission eigenvalues and the associated eigenfunctions:

\[
k^2 \delta = \sum_{j=0}^{\infty} \delta_j \lambda_j
\]

\[
w_\delta(x) = \tilde{w}_\delta(s, \xi) = \sum_{j=0}^{\infty} \delta_j \tilde{w}_j(s, \xi)
\]

\[
v_\delta(x) = \sum_{j=0}^{\infty} \delta_j v_j(x)
\]

for \( \xi = \eta/\delta_0 \). We remark that the functions \( \tilde{w}_j \) are defined on \( G := \{ (s, \xi) \in [0, s_0] \times [0, \max(g)] , \xi \leq g(s) \} \) which is independent of \( \delta_0 \) and we define \( w_k(x) = \tilde{w}_k(s, \eta/\delta_0) \). Using (2.2) and the expressions for the gradient and divergence operators in the local coordinates, we obtain that \( (\tilde{w}_\delta, v_\delta) \) satisfies

\[
\frac{1}{1 + \xi \delta_0 \kappa} \frac{\partial}{\partial s} \frac{1}{1 + \xi \delta_0 \kappa} \frac{\partial}{\partial s} \tilde{w}_\delta + \frac{1}{1 + \xi \delta_0 \kappa} \frac{\partial}{\partial \xi} \frac{1}{1 + \xi \delta_0 \kappa} \frac{\partial}{\partial \xi} \tilde{w}_\delta + k_n^2 n \tilde{w}_\delta = 0 \text{ in } G
\]

(3.4)

together with the boundary conditions

\[
\begin{cases}
\tilde{w}_\delta(s, g(s)) = 0 & s \in [0, s_0], \\
\tilde{w}_\delta(s, 0) = \tilde{v}_\delta(s, 0) & s \in [0, s_0], \\
\frac{1}{\delta_0} \frac{\partial \tilde{w}_\delta}{\partial \xi} |_{\xi=0} = \frac{\partial \tilde{v}_\delta}{\partial \eta} |_{\eta=0} & s \in [0, s_0]
\end{cases}
\]

where \( \tilde{v}_\delta \) is defined by (3.2). Let us multiply (3.4) by \( \delta_0^2 (1 + \xi \delta_0 \kappa)^3 \) to obtain

\[
\sum_{k=0}^{5} \delta_0^k A_k \tilde{w}_\delta = 0
\]

where the \( (A_k)_{k=0,\ldots,5} \) are differential operators of order 2 at maximum with the following expression for the few first terms

\[
A_0 = \frac{\partial^2}{\partial \xi^2},
\]

\[
A_1 = 3\xi \kappa \frac{\partial^2}{\partial \xi^2} + \kappa \frac{\partial}{\partial \xi},
\]

\[
A_2 = 3\xi^2 \kappa^2 \frac{\partial^2}{\partial \xi^2} + 2\xi \kappa \frac{\partial^2}{\partial \xi} + \frac{\partial^2}{\partial s^2} + \lambda_0 n,
\]

\[
A_3 = \cdots
\]
Hence, by equating the terms of same order in $\delta$, the function $\hat{w}_k$ for $k \in \mathbb{N}$, solves

$$A_0 \hat{w}_k = -\sum_{l=1}^{5} A_l \hat{w}_{k-l} \quad \text{in } G,$$

$$\hat{w}_k(s, g(s)) = 0 \quad s \in [0, s_0],$$

$$\hat{w}_k(s, 0) = \tilde{v}_k(s, 0) \quad s \in [0, s_0],$$

$$\frac{\partial \hat{w}_k}{\partial \xi}(s, 0) = \frac{\partial \tilde{v}_{k-1}}{\partial \eta}(s, 0) \quad s \in [0, s_0],$$

with the convention that $\hat{w}_k = v_k = 0$ for negative $k$. The functions $v_k$ also satisfy

$$\Delta v_k + \lambda_0 v_k = -\sum_{l=1}^{k} \lambda_l v_{k-l}.$$  (3.9)

Now we can easily obtain the formal expansion at any order by solving (3.5)–(3.9) recursively.

**Order 0.**

From (3.5) we have

$$\frac{\partial^2 \hat{w}_0}{\partial \xi^2} = 0 \quad \text{in } G$$

and using the boundary conditions (3.6) and (3.8) we obtain $\hat{w}_0 = 0$ on $G$. Equation (3.7) together with (3.9) give that $(\lambda_0, v_0)$ solves

$$\begin{cases}
\Delta v_0 + \lambda_0 v_0 = 0 \quad \text{in } \Omega, \\
v_0 = 0 \quad \text{on } \Gamma.
\end{cases}$$

(3.10)

and hence we define $(\lambda_0, v_0)$ as being an eigenpair of the $-\Delta$ in $\Omega$ with Dirichlet boundary condition and $\|v_0\|_{L^2(\Gamma)} = 1$. We remark that $v_0$ is not uniquely determined (since it can be any Dirichlet eigenfunction), but this will be made precise later in the convergence analysis. Nevertheless, we assume that $\lambda_0$ is simple, which is the case for example for the first Dirichlet eigenvalue of $-\Delta$ in a Lipschitz and connected domain. The latter assumption is necessary to simplify the formal analysis to come.

**Order 1.**

Having determined $\hat{w}_0$ and $v_0$ we iterate the process and obtain that $\hat{w}_1$ is the solution to

$$\frac{\partial^2 \hat{w}_1}{\partial \xi^2} = 0 \quad \text{in } G$$

with boundary conditions (3.6) and (3.8). That gives

$$\hat{w}_1(s, \xi) = \frac{\partial v_0}{\partial \eta}(s, 0) \xi - g(s) \frac{\partial v_0}{\partial \eta}(s, 0) \quad \text{in } G$$

The function $v_1$ solves

$$\begin{cases}
\Delta v_1 + \lambda_0 v_1 = -\lambda_1 v_0 \quad \text{in } \Omega, \\
v_1 = -g \frac{\partial v_0}{\partial \nu} \quad \text{on } \Gamma.
\end{cases}$$

(3.11)
Since $\lambda_0$ is a simple eigenvalue for the operator $-\Delta$ with a Dirichlet boundary condition, to ensure uniqueness of $v_1$ we have to constraint $v_1$ to be orthogonal to $v_0$ in $L^2(\Omega)$. This compatibility condition gives a unique definition for $\lambda_1$. By multiplying the first equation of (3.11) by $v_0$ and by integrating by part we obtain

$$\lambda_1 = \int_{\Gamma} g \left| \frac{\partial v_0}{\partial \nu} \right|^2 ds.$$  \hfill (3.12)

Here we see the simplification due to the assumption that $\lambda_0$ is simple. If this does not hold, then the definition of $\lambda_1$ does not seem to be obvious.

ORDER 2.
To obtain the next term in the asymptotic expansion we iterate the process once more, which yields to the following equation for $\hat{w}_2$

$$\frac{\partial^2 \hat{w}_2}{\partial \xi^2} + \kappa \frac{\partial \hat{w}_2}{\partial \xi} = 0 \quad \text{in } \mathcal{G}.$$  \hfill (3.13)

The boundary conditions (3.6) and (3.8) on $\Gamma$ and $\Gamma_{\delta}$ respectively, imply that $\hat{w}_2$ is given by

$$\hat{w}_2(s, \xi) = -\frac{\kappa(s)}{2} \frac{\partial v_0}{\partial \eta}(s, 0) \xi^2 + \frac{\partial \hat{v}_1}{\partial \eta}(s, 0) \xi + \frac{\kappa(s)}{2} \frac{\partial v_0}{\partial \eta}(s, 0) g(s)^2 - \frac{\partial \hat{v}_1}{\partial \eta}(s, 0) g(s)$$  \hfill (3.14)

in $\mathcal{G}$ where $\kappa$ is the curvature defined in Section 3.1. From this we deduce that $v_2$ solves

$$\begin{cases}
\Delta v_2 + \lambda_0 v_2 = -\lambda_1 v_1 - \lambda_2 v_0 \quad \text{in } \Omega, \\
v_2 = \frac{\kappa}{2} \frac{\partial v_0}{\partial \nu} g^2 - \frac{\partial \hat{v}_1}{\partial \nu} g \quad \text{on } \Gamma.
\end{cases}$$  \hfill (3.15)

Once more, we have to constraint $v_2$ to be orthogonal to $v_0$ and this uniquely defines $\lambda_2$ as being

$$\lambda_2 = -\int_{\Gamma} \left( \frac{\kappa}{2} \frac{\partial v_0}{\partial \nu} g^2 - \frac{\partial \hat{v}_1}{\partial \nu} g \right) \frac{\partial v_0}{\partial \nu} ds.$$  \hfill (3.16)

ORDER K.
Now it becomes clear how to recursively obtain each of the terms in the asymptotic expansion. In particular, for $k > 1$ we assume that the functions $\hat{w}_l$ and $v_l$ as well as the real numbers $\lambda_l$ are well defined for $l < k$. Assume moreover that for all $0 < l < k$,

$$\int_{\Omega} v_l v_0 dx = 0$$

and that $\|v_0\|_{L^2(\Omega)} = 1$. The first step consists in computing $\hat{w}_k$ by solving (3.5) together with the boundary conditions (3.6) and (3.8). This uniquely determines $w_k$ which leads to an explicit formula for $\hat{w}_k$. Then, by (3.7) and (3.9), $v_k$ is uniquely defined as being the solution to

$$\begin{cases}
\Delta v_k + \lambda_0 v_k = -\sum_{l=1}^{k} \lambda_l v_{k-l} \quad \text{in } \Omega, \\
v_k = w_k \quad \text{on } \Gamma, \\
\int_{\Omega} v_k v_0 ds = 0
\end{cases}$$

and that $\|v_0\|_{L^2(\Omega)} = 1$. The first step consists in computing $\hat{w}_k$ by solving (3.5) together with the boundary conditions (3.6) and (3.8). This uniquely determines $w_k$ which leads to an explicit formula for $\hat{w}_k$. Then, by (3.7) and (3.9), $v_k$ is uniquely defined as being the solution to

$$\begin{cases}
\Delta v_k + \lambda_0 v_k = -\sum_{l=1}^{k} \lambda_l v_{k-l} \quad \text{in } \Omega, \\
v_k = w_k \quad \text{on } \Gamma, \\
\int_{\Omega} v_k v_0 ds = 0
\end{cases}$$

and that $\|v_0\|_{L^2(\Omega)} = 1$. The first step consists in computing $\hat{w}_k$ by solving (3.5) together with the boundary conditions (3.6) and (3.8). This uniquely determines $w_k$ which leads to an explicit formula for $\hat{w}_k$. Then, by (3.7) and (3.9), $v_k$ is uniquely defined as being the solution to

$$\begin{cases}
\Delta v_k + \lambda_0 v_k = -\sum_{l=1}^{k} \lambda_l v_{k-l} \quad \text{in } \Omega, \\
v_k = w_k \quad \text{on } \Gamma, \\
\int_{\Omega} v_k v_0 ds = 0
\end{cases}$$
where the last equation uniquely defines \( \lambda_k \) as being
\[
\lambda_k = -\int_{\Gamma} w_k \frac{\partial v_0}{\partial \nu} ds.
\]

Of course the asymptotic expansion obtained above is only formal at this point. The next section is dedicated to its convergence analysis.

4 Convergence analysis

Our main goal in this section is to rigorously justify the asymptotic expansion formally obtained in the previous section. To this end, for sake of simplicity of presentation and to avoid secondary technical difficulties we assume that the thickness of the thin layer is constant (i.e. \( g \equiv 1 \)), that \( n \) is in \( C^\infty(\overline{\Omega} \setminus \Omega_\delta) \) and that \( \Gamma \) is of class \( C^\infty \) as well. Moreover, we only perform the convergence analysis for the first transmission eigenvalue that we denote \( \lambda_1^\delta := (k_1^\delta)^2 \). More specifically, in the following we justify the expansion
\[
\lambda_1^\delta = \lambda_0 + \delta \lambda_1 + \delta^2 \lambda_2 + O(\delta^3)
\]
where \( \lambda_0 \) is the first Dirichlet eigenvalue for the \( -\Delta \) operator in \( \Omega \), \( \lambda_1 \) and \( \lambda_2 \) are given in the previous section and \( O(x) \) stands for a generic function in \( C^\infty(\mathbb{R}^+) \) such that
\[
|O(x)| \leq C|x|
\]
for some constant \( C > 0 \) independent of \( x \in \mathbb{R}^+ \). The main ingredient to arrive at such a result is to establish explicit a priori estimates with respect to \( \delta \) for the solutions of the interior transmission problem
\[
\begin{cases}
\Delta w_\delta = f_1 & \text{in } \Omega \setminus \overline{\Omega_\delta}, \\
\Delta v_\delta = f_2 & \text{in } \Omega, \\
\frac{\partial v_\delta}{\partial \nu} - \frac{\partial w_\delta}{\partial \nu} = f_3, & v_\delta - w_\delta = f_4 \text{ on } \Gamma, \\
w_\delta = 0 & \text{on } \Gamma_\delta.
\end{cases}
\]

These stability estimates are stated in the two following propositions. The proof of these propositions requires a few technical lemmas which we state and prove in Appendix in order to maintain the main focus of this paper.

**Proposition 4.1** Let \( v_\delta \in H^2(\Omega) \) and \( w_\delta \in H^2(\Omega \setminus \overline{\Omega_\delta}) \) be such that for some \( s \geq 0 \)
\[
\begin{align*}
&\|\Delta w_\delta\|_{L^2(\Omega \setminus \overline{\Omega_\delta})} \leq O(\delta^s) \quad (4.2) \\
&w_\delta = 0 \text{ on } \Gamma_\delta \\
&\|\Delta v_\delta\|_{L^2(\Omega)} \leq O(\delta^s) \quad (4.3) \\
&\left\| \frac{\partial v_\delta}{\partial \nu} - \frac{\partial w_\delta}{\partial \nu} \right\|_{H^{1/2}(\Gamma)} \leq O(\delta^s) \quad (4.4) \\
&\|v_\delta - w_\delta\|_{H^{3/2}(\Gamma)} \leq O(\delta^s). \quad (4.5)
\end{align*}
\]

Then, for sufficiently small \( \delta > 0 \),
\[
\|w_\delta\|_{H^2(\Omega \setminus \overline{\Omega_\delta})} \leq O(\delta^s) \text{ and } \|w_\delta\|_{L^2(\Omega \setminus \overline{\Omega_\delta})} \leq O(\delta^{s+1}).
\]
Proof. First we prove by contradiction that \( \|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta^s) \). Assume to the contrary that the latter is not true, then we can state that (up to an extracted subsequence)

\[ \gamma_\delta := \delta^s \|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)}^{-1} \xrightarrow{\delta \to 0} 0. \]

Since \( w_\delta \) is in \( H^2(\Omega \setminus \Gamma_\delta) \) and since \( \|v_\delta - w_\delta\|_{H^{3/2}(\Gamma)} \leq O(\delta^s) \), from classic elliptic regularity for the Laplacian with Dirichlet boundary condition there exists a constant \( C \) independent of \( \delta \) such that

\[ \|v_\delta\|_{H^2(\Omega)} \leq C \left( \|\Delta v_\delta\|_{L^2(\Omega)} + \|w_\delta\|_{H^{3/2}(\Gamma)} + O(\delta^s) \right) \]

and by using (4.3), Lemma A.4 and the fact that \( \gamma_\delta \) is bounded when \( \delta \to 0 \) we deduce that

\[ \frac{\|v_\delta\|_{H^2(\Omega)}}{\|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)}} \leq \frac{1}{\|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)}} \left( C \|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)} + O(\delta^s) \right) \leq C \]

for \( C > 0 \) independent of \( \delta \). Hence by (4.4) and the trace theorem we have that

\[ \frac{1}{\|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)}} \left\| \frac{\partial w_\delta}{\partial \nu} \right\|_{H^{1/2}(\Gamma)} \leq C. \]

Now by applying Lemma A.2 and Lemma A.3 to the function \( w_\delta/\|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)} \), we have that there exists a constant \( C > 0 \) independent of \( \delta \) such that

\[ 1 \leq C \left( \frac{\|\Delta w_\delta\|_{L^2(\Omega \setminus \Gamma_\delta)}}{\|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)}} + \varepsilon(\delta) \right) \]

where \( \varepsilon(\delta) \to 0 \) when \( \delta \) goes to 0. Hence by using (4.2) since \( \gamma_\delta = \delta^s \|w_\delta\|_{H^2(\Omega \setminus \Gamma_\delta)}^{-1} \to C \) when \( \delta \) goes to 0 we obtain

\[ 1 \leq \varepsilon(\delta) \xrightarrow{\delta \to 0} 0. \]

Thus the first estimate of the lemma holds, and then an application of \( (A.18) \) together with \( (A.19) \) imply the second estimate. \( \square \)

**Proposition 4.2** Let \( w_\delta \in H^2(\Omega \setminus \Gamma_\delta) \) be such that for some \( s \geq 0 \) we have

\[ \|\Delta w_\delta\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta^s) \]

and

\[ w_\delta = 0 \quad \text{on} \quad \Gamma_\delta, \quad \left\| \frac{\partial w_\delta}{\partial \nu} \right\|_{H^{-1/2}(\Gamma)} \leq O(\delta^{s+1/2}). \]

Then, for sufficiently small \( \delta \),

\[ \|w_\delta\|_{H^1(\Omega \setminus \Gamma_\delta)} \leq O(\delta^{s+1/2}) \quad \text{and} \quad \|w_\delta\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta^{s+1}). \]

**Proof.** First by \( (A.18) \) and Lemma A.4 there exists \( C > 0 \) such that for sufficiently small \( \delta > 0 \) we have

\[ \|w_\delta\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq C \delta^{1/2} \|w_\delta\|_{H^1(\Omega \setminus \Gamma_\delta)}. \quad (4.6) \]
An application of Green’s identity in \( \Omega \setminus \overline{\Omega}_\delta \) yields
\[
\int_{\Omega \setminus \overline{\Omega}_\delta} |\nabla w_\delta|^2 \, dx = \int_{\Omega \setminus \overline{\Omega}_\delta} -\Delta w_\delta \, dx - \int_{\Gamma} \frac{\partial w_\delta}{\partial \nu} \, ds,
\]
and using the assumptions of the lemma and (4.6) we finally obtain
\[
\int_{\Omega \setminus \overline{\Omega}_\delta} |\nabla w_\delta|^2 \, dx \leq O(\delta^{s+1/2}) \|w_\delta\|_{H^1(\Omega \setminus \overline{\Omega}_\delta)}
\]
which proves the first estimate. We obtain the second estimate by simply using (4.6). □

Now with help of the above propositions we are able to prove the convergence of our asymptotic expansion.

4.1 The convergence of the zero order approximation

We start with the convergence of the zero order term in the expansion, which can be easily obtained from the expression satisfied by the first transmission eigenvalue.

**Theorem 4.3** Let \( \lambda_1^\delta \) be the first real interior transmission eigenvalue, then for sufficiently small \( \delta > 0 \),
\[
\lambda_1^\delta = \lambda_0 + O(\delta).
\]

**Proof.** We first observe that for \( \lambda < \lambda_0 \), where \( \lambda_0 \) is the first Dirichlet eigenvalue for \(-\Delta\) in \( \Omega \), the operator \( A_\lambda - \lambda^2 B \) defined in Section 2 (where we set \( \lambda := k^2 \)) is injective. Indeed for all \( u \in W_\delta \),
\[
(A_\lambda - \lambda^2 B)u = \int_{\Omega \setminus \overline{\Omega}_\delta} \frac{1}{1-n} |\Delta u + \lambda u| \, dx - \lambda^2 \int_{\Omega} |u|^2 \, dx + \lambda \int_{\Omega} |\nabla u|^2 \, dx \geq \lambda \|\nabla u\|_{L^2(\Omega)}^2 \left( 1 - \frac{\lambda}{\lambda_0} \right)
\]
where we have used the Poincaré inequality. Hence we necessarily have
\[
\lambda_0 \leq \lambda_1^\delta. \tag{4.7}
\]

On the other hand it is possible to characterize \( \lambda_1^\delta \) via the Max-Min principle (see Cakoni et al. (2012) for details) as
\[
2(\lambda_1^\delta)^2 = \inf_{u \in W_\delta} (A_{\lambda_1^\delta} u, u)_{W_\delta} \quad \|u\|_{L^2(\Omega)} = 1
\]
\[
= \inf_{u \in W_\delta} \int_{\Omega \setminus \overline{\Omega}_\delta} \frac{1}{1-n} |\Delta u + \lambda_1^\delta u|^2 \, dx + (\lambda_1^\delta)^2 + \lambda_1^\delta \int_{\Omega} |\nabla u|^2 \, dx.
\]

Now if we take \( u = u_{1, D}^\delta \) where \( u_{1, D}^\delta \) the first Dirichlet eigenfunction for \(-\Delta\) in \( \Omega_\delta \) associated with the eigenvalue \( \lambda_1^{1, D} \) extended by 0 outside \( \Omega_\delta \) such that \( \|u_{1, D}^\delta\|_{L^2(\Omega_\delta)} = 1 \) (note that due to the zero boundary condition on \( \Gamma_\delta \) the extension by zero of \( u_{1, D}^\delta \) is in \( W_\delta \)) we obtain
\[
(\lambda_1^\delta)^2 \leq \lambda_1^{1, D} \lambda_1^{1, D}
\]
or equivalently
\[ \lambda_{\delta,D}^1 \leq \lambda_{D}^1 \]  \hspace{1cm} (4.8)

since \( \lambda_{\delta,D}^1 \) is bounded below by \( \lambda_0 \). To conclude, we remark that the first Dirichlet eigenvalue for \( -\Delta \) is Fréchet differentiable with respect to the shape (see Henrot (2006)). A consequence of this result is that there exists a constant \( C > 0 \) such that for all \( \delta \) sufficiently small
\[ |\lambda_{\delta,D}^1 - \lambda_0| \leq C\delta. \]

This fact together with the lower and upper bounds (4.7) and (4.8) respectively, ends the proof. \( \square \)

**Remark 4.1** Theorem 4.3 is the cornerstone of our analysis since it allows to uniquely define the Dirichlet eigenvalue which is the first term in the asymptotic expansion of the transmission eigenvalue. The other terms in (4.1) are then uniquely defined.

### 4.2 The convergence of the first order approximation

In order to proceed with next order approximation, we must first prove some estimates for the first order approximation of the corresponding eigenfunction. To this end let us define
\[ e_{w}^1 := w_{\delta}^1 - \delta w_1 \quad \text{in} \quad \Omega \setminus \Omega_\delta \]  \hspace{1cm} (4.9)

extended by 0 in \( \Omega_\delta \) and
\[ e_{v}^1 := v_{\delta}^1 - v_0 \quad \text{in} \quad \Omega \]  \hspace{1cm} (4.10)

where \( v_0 \) is a solution to (3.10) such that \( \|v_0\|_{L^2(\Omega)} = 1 \), \( w_1(x) := \hat{w}_1(s, \xi) \) with \( x = \varphi(s, \delta \xi) \), and \( (w_{\delta}^1, v_{\delta}^1) \) are the eigenfunctions corresponding to the first transmission eigenvalue \( \lambda_{\delta}^1 \). We also extend \( w_{\delta}^1 \) by 0 inside \( \Omega_\delta \). We remark that since \( w_{\delta}^1 \) and \( w_1 \) vanish on \( \Gamma_\delta \), the functions \( e_{w}^1 \) and \( w_{\delta}^1 \) are continuous across the interface \( \Gamma_\delta \). Let us begin with a lemma that provides \( \delta \)-explicit a priori estimates for \( w_{\delta}^1 \) which will enable us to derive estimates for the first order approximation of \( w_{\delta}^1 \) and \( v_{\delta}^1 \).

**Lemma 4.1** There exists a constant \( C > 0 \) such that for sufficiently small \( \delta > 0 \) we have
\[ \|w_{\delta}^1\|_{H^2(\Omega \setminus \Gamma_\delta)} \leq C \quad \text{and} \quad \|w_{\delta}^1\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta). \]

**Proof.** We show that Proposition 4.1 applies to \( (v_{\delta}^1, w_{\delta}^1) \) for \( s = 0 \). To this end, since
\[ 2(\lambda_\delta^1)^2 = (A_{\lambda_\delta^1}^1 u_{\delta}^1, u_{\delta}^1)_{W_1^1} \]

and since \( A_{\lambda_\delta^1}^1 \) is coercive with a coercivity constant independent of \( \delta \) (see Proposition 2.2), there exists a constant \( C \) independent of \( \delta \) such that
\[ \|u_{\delta}^1\|_{H^1(\Omega)} + \|\Delta u_{\delta}^1\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq C \]  \hspace{1cm} (4.11)

where \( u_{\delta}^1 \) is defined by (2.3). A straightforward consequence of (4.11) is that there exists another constant \( C \) still independent of \( \delta \) such that
\[ \|w_{\delta}^1\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|w_{\delta}^1\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq C. \]

Since \( -\Delta w_{\delta}^1 = \lambda_{\delta}^1 n w_{\delta}^1 \) and \( -\Delta v_{\delta}^1 = \lambda_{\delta}^1 v_{\delta}^1 \), by using Proposition 4.1 with \( s = 0 \) we have that there exists \( C > 0 \) such that
\[ \|w_{\delta}^1\|_{H^2(\Omega \setminus \Gamma_\delta)} \leq C \quad \text{and} \quad \|w_{\delta}^1\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta), \]

which ends the proof. \( \square \)
LEMMA 4.2 The following error estimates hold true for sufficiently small \( \delta > 0 \)

\[
\| e^1_v \|_{H^1(\Omega)} \leq O(\delta),
\]

(4.12)

and

\[
\| e^1_w \|_{H^1(\Omega)} \leq O(\delta), \quad \| e^1_w \|_{L^2(\Omega)} \leq O(\delta^{3/2}), \quad \| w^1_3 \|_{L^2(\Omega)} \leq O(\delta^{3/2}).
\]

(4.13)

Proof. The idea of the proof is to establish that \( v^1_3 \) is a quasi Dirichlet eigenfunction for \(-\Delta\) in the domain \( \Omega \) and then apply Lemma B.1. From the first estimate of Lemma 4.1 and the inequality (A.19) the trace of \( w^1_3 \) on \( \Gamma \) satisfies

\[
\| w^1_3 \|_{H^{1/2}(\Gamma)} \leq O(\delta^{1/2}).
\]

Let us define \( \theta^1_w \) a lifting of \( w^1_3 \) in \( H^1(\Omega) \) such that \( \theta^1_w |_\Gamma = w^1_3 |_\Gamma \) and that \( \| \theta^1_w \|_{H^1(\Omega)} \leq O(\delta^{1/2}) \). Let us set \( \tau^1_3 := v^1_3 - \theta^1_w \) and show that \( \tau^1_3 \) is close to \( v_0 \). Indeed, for all \( \psi \in H^1_0(\Omega) \) we have that

\[
\left| \int_\Omega \nabla \tau^1_3 \cdot \nabla \psi - \lambda_0 \tau^1_3 \psi \, dx \right| = \left| (\lambda^1_3 - \lambda_0) \int_\Omega v^1_3 \psi \, dx - \int_\Omega (\nabla \theta^1_w \cdot \nabla \psi - \lambda_0 \theta^1_w \psi) \, dx \right|
\]

\[
\leq O(\delta^{1/2}) \| \psi \|_{H^1(\Omega)}.
\]

Moreover since \( \| u^1_3 \|_{L^2(\Omega)} = 1 \), Lemma 4.1 implies that

\[
\| \tau^1_3 \|_{L^2(\Omega)} - 1 \leq O(\delta^{1/2})
\]

(4.14)

and hence by virtue of Lemma B.1 there exists \( C > 0 \) and \( v_0 \) solution to (3.10) with \( \| v_0 \|_{L^2(\Omega)} = 1 \) such that for all \( \delta \) sufficiently small we have

\[
\| \tau^1_3 - v_0 \|_{H^1(\Omega)} \leq O(\delta^{1/2}).
\]

(4.15)

Note that this last inequality uniquely determines the function \( v_0 \). From the definition of \( \tau^1_3 \) and the above bound on the lifting, (4.15) takes the form

\[
\| e^1_v \|_{H^1(\Omega)} \leq O(\delta^{1/2}).
\]

(4.16)

But since \( w_1 \) solves (3.5) we have

\[
\Delta w_1 = \frac{1}{(1 + \delta \xi \kappa)^3} \left( \sum_{k=0}^{5} \delta^{k-2} A_k w_1 - \lambda^1_3 n \hat{w}_1 \right)
\]

whence

\[
\| \Delta w_1 \|_{L^2(\Omega, \partial\Omega)} \leq C(\delta^{-1/2} \| A_1 \hat{w}_1 \|_{L^2(\setminus \Omega)} + \delta^{1/2} \| \hat{w}_1 \|_{L^2(\setminus \Omega)}) \leq C\delta^{-1/2}.
\]

Therefore

\[
\| \Delta e^1_w \|_{L^2(\Omega, \partial\Omega)} \leq O(\delta^{1/2}),
\]

(4.17)

and in addition we also have by (4.16)

\[
\| \Delta e^1_v \|_{L^2(\Omega)} \leq O(\delta^{1/2}), \quad \left\| \frac{\partial e^1_v}{\partial \nu} - \frac{\partial e^1_w}{\partial \nu} \right\|_{H^{1/2}(\Gamma)} = 0
\]

(4.18)
and
\[ \| e_1^w - e_1^v \|_{H^{3/2}(\Gamma)} = \| \delta w_1 \|_{H^{3/2}(\Gamma)} \leq \mathcal{O}(\delta). \]

An application of Proposition 4.1 now implies
\[ \| e_1^w \|_{H^2(\Omega \setminus \Gamma_0)} \leq \mathcal{O}(\delta^{1/2}) \]
and then thanks to (A.19) applied \( e_1^w \), we can improve the bound on \( w_1^\delta \) as follows
\[ \| w_1^\delta \|_{H^{1/2}(\Gamma)} \leq \mathcal{O}(\delta^1) \]

Since \( \| w_1^\delta \|_{H^{1/2}(\Gamma)} \leq C \) for all \( \delta > 0 \). Now repeating the previous steps of the proof allows us to improve the bound on \( e_1^v \) as follows
\[ \| e_1^v \|_{H^1(\Omega)} \leq \mathcal{O}(\delta). \]

Finally, this last inequality together with (4.17) and the fact that
\[ \frac{\partial e_1^w}{\partial \nu} = \frac{\partial e_1^w}{\partial \nu} \text{ on } \Gamma \]
yield the desired bounds for \( e_1^w \) thanks to Proposition 4.2.

As a consequence of the error estimates derived in Lemma 4.2 we can now obtain the desired first order convergence result which is stated in the theorem below.

**Theorem 4.4** The following asymptotic expansion for the first transmission eigenvalue \( \lambda_1^\delta \) holds true for sufficiently small \( \delta > 0 \),
\[ \lambda_1^\delta = \lambda_0 + \delta \lambda_1 + \mathcal{O}(\delta^2), \]
where \( \lambda_0 \) is the first Dirichlet eigenvalue for \(-\Delta\) in \( \Omega \) and \( \lambda_1 \) is defined by (3.12).

**Proof.** Let \( u_1^\delta \in W_\delta \) be defined by (2.3) with \( v_1^\delta \) and \( w_1^\delta \) normalized such that \( \| u_1^\delta \|_{L^2(\Omega)} = 1 \). Then from Cakoni et al. (2012) we have that
\[ (\lambda_1^\delta)^2 = \int_{\Omega \setminus \Gamma_0} \frac{1}{1 - n} |\Delta u_1^\delta + \lambda_1^\delta u_1^\delta|^2 dx + \lambda_1^\delta \int_{\Omega} |\nabla u_1^\delta|^2 dx. \]
and using the definition (2.3) of \( u_1^\delta \) and the equations for \( v_1^\delta \) this becomes
\[ \lambda_1^\delta = \int_{\Omega \setminus \Gamma_0} \lambda_1^\delta (1 - n) |w_1^\delta|^2 dx + \int_{\Omega} |\nabla u_1^\delta|^2 dx. \quad (4.19) \]
From (4.13) the first term in (4.19) is of order \( \delta^3 \), hence we need to develop only the second term. To this end we can write
\[ \int_{\Omega} |\nabla u_1^\delta|^2 dx = \int_{\Omega} |\nabla (e_1^w - e_1^v - v_0 + \delta w_1)|^2 dx \]
\[ = \int_{\Omega} |\nabla e_1^w|^2 dx + \int_{\Omega} |\nabla e_1^v|^2 dx + \int_{\Omega} |\nabla v_0|^2 dx + \delta^2 \int_{\Omega \setminus \Gamma_0} |\nabla w_1|^2 dx \]
\[ - 2 \int_{\Omega} \nabla v_0 \nabla (w_1^\delta - e_1^v) dx + 2 \delta \int_{\Omega} \nabla (e_1^w - e_1^v) \nabla w_1 dx - 2 \int_{\Omega} \nabla e_1^v \nabla e_1^w dx. \quad (4.20) \]
Recall that from Lemma 4.2 we have that
\[ \int_{\Omega} |\nabla e_1^v|^2 \, dx \leq O(\delta^2), \quad \int_{\Omega} |\nabla e_1^w|^2 \, dx \leq O(\delta^2) \text{ and } \int_{\Omega} \nabla e_1^v \nabla e_1^w \, dx \leq O(\delta^2). \]
Furthermore, from the definition of \( v_0 \) we have that
\[ \int_{\Omega} |\nabla v_0|^2 \, dx = \lambda_0, \]
and from the definition of \( w_1 \) we have that
\[ \delta^2 \int_{\Omega \setminus \Gamma_0} |\nabla w_1|^2 \, dx = \delta^2 \int_0^{\delta_0} \left( \left| \frac{\partial v_0}{\partial \nu} \right|^2 \int_0^\delta \frac{1}{\delta^2} J_{\eta, \eta} \, d\eta \right) \, ds + O(\delta^2) = \delta \lambda_1 + O(\delta^2). \]
In addition we also know that \( \| u_3^1 \|_{L^2(\Omega)} = 1 \) that is
\[ 1 = \int_{\Omega} |u_3^1|^2 \, dx = \int_{\Omega} |u_3^1 + v_0|^2 - 2(u_3^1 + v_0)v_0 + |v_0|^2 \, dx. \]
The estimates of Lemma 4.2 together with the definitions (4.9) and (4.10) give that
\[ \| u_3^1 + v_0 \|_{L^2(\Omega)} \leq O(\delta) \]
and hence since \( \| v_0 \|_{L^2(\Omega)} = 1 \) we have
\[ \left| \int_{\Omega} v_0(u_3^1 + v_0) \, dx \right| \leq O(\delta^2). \] (4.21)
Recalling that \( v_0 \) is a Dirichlet eigenfunction for \(-\Delta\) with corresponding eigenvalue \( \lambda_0 \), from (4.21) we now obtain
\[ \left| \int_{\Omega} \nabla v_0 \nabla (w_3^1 - e_1^v) \, dx \right| = \left| \int_{\Omega} \nabla v_0 \nabla (u_3 + v_0) \, dx \right| = \lambda_0 \left| \int_{\Omega} v_0(u_3 + v_0) \, dx \right| \leq O(\delta^2). \]
Finally, noting that
\[ \int_{\Omega \setminus \Gamma_s} \nabla (e_1^w - e_1^v) \nabla w_1 \, dx = \int_{\Omega \setminus \Gamma_s} -\Delta(e_1^w - e_1^v)w_1 \, dx - \int_{\Gamma} \frac{\partial}{\partial \nu} (e_1^w - e_1^v)w_1 \, ds \]
and using \( \frac{\partial e_1^w}{\partial \nu} = \frac{\partial e_1^v}{\partial \nu} \) we obtain
\[ \left| \int_{\Omega \setminus \Gamma_s} \nabla (e_1^w - e_1^v) \nabla w_1 \, dx \right| = \left| \int_{\Omega \setminus \Gamma_s} -\Delta(e_1^w - e_1^v)w_1 \, dx \right|. \] (4.22)
But, (4.17) and (4.18) imply
\[ \| \Delta(e_1^w - e_1^v) \|_{L^2(\Omega \setminus \Gamma_s)} \leq C \delta^{1/2}. \]
This last estimate together with equality (4.22) and \( \|w_1\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq C\delta^{1/2} \) gives
\[
\int_{\Omega \setminus \Gamma_\delta} \nabla(e_1^w - e_0) \nabla w_1 \, dx \leq C\delta.
\]
We have estimated all the terms in (4.20) and thus the expression (4.19) for \( \lambda^1_\delta \) finally yields the estimate
\[
\lambda^1_\delta = \lambda_0 + \delta \lambda_1 + O(\delta^2).
\]

\( \square \)

4.3 The convergence of the second order approximation

The goal of this section is twofold. We complete the rigorous justification of the asymptotic expansion (4.1), and present a constructive procedure how to iteratively obtain the converges of any order in the asymptotic expansion of transmission eigenvalues. To this end, before proceeding with the convergence of the eigenvalues we need to improve the rate of convergence of the corresponding eigenfunctions. This is possible by adding a correction term to the eigenfunctions \( v^1_\delta \) and \( w^1_\delta \). Let us consider the following error functions
\[
e_2^w := w_1^\delta - C^1_\delta(\delta w_1 + \delta^2 w_2) \quad \text{in } \Omega \setminus \Gamma_\delta
\]
extended by 0 in \( \Omega_\delta \) and
\[
e_2^v := v_1^\delta - C^1_\delta(v_0 + \delta v_1) \quad \text{in } \Omega
\]
where \( w_2(x) := \hat{w}_2(s, \xi) \) and \( v_1 \) are defined in Section 3.2 and \( C^1_\delta := \|u^1_\delta + \delta v_1\|_{L^2(\Omega)} \). As before, the error \( e_2^w \) is continuous across the interface \( \Gamma_\delta \) since it vanishes on \( \Gamma_\delta \). We remark that since \( \|u^1_\delta\|_{L^2(\Omega)} = 1 \) we have that
\[
C^1_\delta = 1 + O(\delta).
\]
We now proceed as in Lemma 4.2 to to give a first estimate on \( e_2^w \), which on its turn provides a \( H^1 \) bound for \( e_2^v \) and then iterate the procedure to obtain the optimal bounds for \( e_2^w \) and \( e_2^v \).

**Lemma 4.3** The following a priori estimates hold for \( \delta > 0 \) sufficiently small:
\[
\|e_2^w\|_{H^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta) \quad \text{and} \quad \|e_2^v\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta^3).
\]

**Proof.** First of all, from the definition of \( w_1 \) and \( w_2 \) we have for \( \delta > 0 \) sufficiently small
\[
\|w_1\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta^{1/2}), \quad \|w_2\|_{L^2(\Omega \setminus \Gamma_\delta)} \leq O(\delta^{3/2}).
\]
Moreover, from Section 3.2
\[
\Delta(w_1 + \delta w_2) = \frac{1}{(1 + \delta \kappa)^3} \left( \sum_{k=0}^5 \delta^{k-2} A_k(\hat{w}_1 + \delta \hat{w}_2) - \lambda^1_\delta n(\hat{w}_1 + \delta \hat{w}_2) \right)
\]
and since \( A_0 \hat{w}_1 = 0 \) and \( A_0 \hat{w}_2 + A_1 \hat{w}_1 = 0 \) we now have
\[
\Delta(w_1 + \delta w_2) = \frac{1}{(1 + \delta \kappa)^3} \left( A_1 \hat{w}_2 - \lambda^1_\delta n(\hat{w}_1 + \delta \hat{w}_2) + \sum_{k=2}^5 \delta^{k-2} A_k(\hat{w}_1 + \delta \hat{w}_2) \right)
\]
which yields in view of (4.13) and (4.24)
\[
\|\Delta e_2^w\|_{L^2(\Omega)} \lesssim O(\delta^{3/2}).
\] (4.25)

Next,
\[
\|e_2^w - e_2^v\|_{H^{3/2}(\Gamma)} = \|C_1^1\delta^2 w_2\|_{H^{3/2}(\Gamma)} \lesssim O(\delta^2),
\]
\[
\frac{\partial e_2^w}{\partial n} = \frac{\partial e_2^v}{\partial n}
\]
on \Gamma,
and
\[
\|\Delta e_2^v\|_{L^2(\Omega)} \lesssim O(\delta)
\]
whence by applying Lemma 4.1 with \(s = 1\) we obtain the result. \(\Box\)

Similarly to the previous section we are now able to obtain convergence rates for the eigenfunctions.

**Lemma 4.4** The following error estimates hold for \(\delta > 0\) sufficiently small
\[
\|e_2^w\|_{H^1(\Omega)} \lesssim C\delta^{3/2},
\]
and
\[
\|e_2^w\|_{H^1(\Omega)} \lesssim O(\delta^{3/2}), \quad \|e_2^w\|_{L^2(\Omega)} \lesssim O(\delta^{3/2}).
\]

**Proof.** We proceed exactly as in the proof of Proposition 4.2. To this end, let us define \(v_2 := (C_1^2)^{-1}(v_3^1 - \delta v_1)\) and then since \(C_1^2 = 1 + O(\delta)\), we have that
\[
v_2|_\Gamma = (C_1^2)^{-1}(v_3^1|_\Gamma - \delta v_1|_\Gamma - \delta^2 v_2|_\Gamma) + (C_1^2)^{-1}\delta^2 v_2|_\Gamma \\
= (C_1^2)^{-1}(v_3^1|_\Gamma - C_1^2(\delta v_1|_\Gamma - \delta^2 v_2|_\Gamma)) + O(\delta^2) \\
= (C_1^2)^{-1}v_2|_\Gamma + O(\delta^2).
\]
Using (A.19) together with Lemma 4.3 we see that
\[
\|v_2\|_{H^{1/2}(\Gamma)} \lesssim O(\delta^{1/2})\|e_2^w\|_{H^2(\Omega \setminus \Gamma)} + O(\delta^2) \lesssim O(\delta^{3/2}).
\]

Let us denote by \(\theta_{v_2}\) a lifting of \(v_2\) in \(\Omega\) such that \(\theta_{v_2}|_\Gamma = v_2|_\Gamma\) and \(\|\theta_{v_2}\|_{H^1(\Omega)} \lesssim O(\delta^{3/2})\) then consider \(v_2 := v_2 - \theta_{v_2}\). Obviously \(v_2 \in H_0^1(\Omega)\) and for all \(\psi \in H_0^1(\Omega)\) we have
\[
\int_{\Omega} \nabla v_2 \cdot \nabla \psi - \lambda_0 v_2 \psi \, dx = \int_{\Omega} \nabla v_2 \cdot \nabla \psi - \lambda_0 v_2 \psi \, dx \\
- \int_{\Omega} \nabla \theta_{v_2} \cdot \nabla \psi + \lambda_0 \theta_{v_2} \psi \, dx.
\] (4.26)

We can estimate the second term easily by using the bound on the lifting and the Cauchy-Schwarz inequality
\[
\left| \int_{\Omega} \nabla \theta_{v_2} \cdot \nabla \psi + \lambda_0 \theta_{v_2} \psi \, dx \right| \leq O(\delta^{3/2})\|\psi\|_{H^1(\Omega)}.
\]
Next we consider the first term in (4.26) containing $v_2$. For all $\psi \in H^1_0(\Omega)$ from Theorem 4.4 and Lemma 4.2 we can write

$$
\int_{\Omega} \nabla (v_1^1 - \delta v_1) \cdot \nabla \psi \, dx = \lambda_0^1 \int_{\Omega} v_1^1 \psi \, dx - \delta \lambda_1 \int_{\Omega} v_1 \psi \, dx - \delta \lambda_0 \int_{\Omega} v_0 \psi \, dx
$$

$$
= (\lambda_0 + \delta \lambda_1) \int_{\Omega} v_1^1 \psi \, dx - \delta \lambda_0 \int_{\Omega} v_1 \psi \, dx - \delta \lambda_1 \int_{\Omega} v_0 \psi \, dx + O(\delta^2) \| \psi \|_{H^1(\Omega)}
$$

$$
= \lambda_0 \int_{\Omega} (v_1^1 - \delta v_1) \psi \, dx + O(\delta^2) \| \psi \|_{H^1(\Omega)}.
$$

Next by using (4.23) we obtain that there exists another constant $C$ still independent of $\delta$ such that

$$
\left| \int_{\Omega} (\nabla v_2 \cdot \nabla \psi - \lambda_0 v_2 \psi) \, dx \right| \leq C \delta^2 \| \psi \|_{H^1(\Omega)}
$$

for all $\psi \in H^1_0(\Omega)$. We can now apply Lemma B.1 to $v_2$ since by (4.13) and the definition of $C_1^1$, we have

$$
\| v_2 \|_{L^2(\Omega)} = (C_1^1)^{-1} \| u_1^1 + \delta v_1 \|_{L^2(\Omega)} + O(\delta^{3/2}) = 1 + O(\delta^{3/2})
$$

to obtain

$$
\| \delta v_2 - v_0 \|_{H^1(\Omega)} \leq O(\delta^{3/2}).
$$

From the bound on the lifting $\theta v_2$ and (4.23) the latter becomes

$$
\| v_1^1 - C_1^1 v_0 - C_1^1 \delta v_1 \|_{H^1(\Omega)} \leq O(\delta^{3/2}).
$$

Now it is clear that we can improve the bound on $e_2^w$ by applying Lemma 4.1 with $s = 3/2$, since

$$
\| \Delta e_2^w \|_{L^2(\Omega)} \leq O(\delta^{3/2}). \quad (4.27)
$$

Therefore we arrive at the following improved estimate

$$
\| e_2^w \|_{H^2(\Omega \setminus \Gamma)} \leq O(\delta^{3/2}), \quad \| e_2^w \|_{L^2(\Omega \setminus \Gamma)} \leq O(\delta^{5/2}).
$$

**Remark 4.2** As in Lemma 4.2 we can improve the bound on $e_2^w$ if we choose

$$
C_1^1 = \| u_1^1 + \delta v_1 - \delta w_1 \|_{L^2(\Omega)}.
$$

Indeed, in this case

$$
\| \delta v_2 \|_{L^2(\Omega)} = 1 + O(\delta^2)
$$

and since $\| e_2^w \|_{H^2(\Omega \setminus \Gamma)} \leq O(\delta^{3/2})$ we deduce that $\| e_2^w \|_{H^{1/2}(\Gamma)} \leq O(\delta^2)$ which implies that

$$
\| e_2^w \|_{H^1(\Omega)} \leq O(\delta^2).
$$

The estimates obtained in Lemma 4.4 lead to the following result.
Theorem 4.5 The following expansion for the first transmission eigenvalue holds true for $\delta > 0$ sufficiently small

$$
\lambda_0^1 = \lambda_0 + \delta \lambda_1 + \delta^2 \lambda_2 + \mathcal{O}(\delta^3),
$$

where $\lambda_0$ is the first Dirichlet eigenvalue for $-\Delta$ in $\Omega$, and $\lambda_1$ and $\lambda_2$ are defined by (3.12) and (3.14) respectively.

**Proof.** Similarly to the proof of Theorem 4.4 we expand the definition of $\lambda_0^1$ by using the approximate eigenfunctions

$$
w_{\text{app}}^2 := C_2^1 (\delta w_1 + \delta^2 w_2), \quad \text{and} \quad v_{\text{app}}^2 := C_2^1 (v_0 + \delta v_1)
$$

and we extend $w_{\text{app}}^2$ by 0 inside $\Omega_\delta$. From the characterization (4.19) of $\lambda_0^1$ and the bound (4.13) we have

$$
\lambda_0^1 = \int_{\Omega} |\nabla u_0^1|^2 \, dx + \mathcal{O}(\delta^3).
$$

This writes

$$
\int_{\Omega} |\nabla u_0^1|^2 \, dx = \int_{\Omega} |\nabla (e_w^2 - e_v^2 - v_{\text{app}}^2 + w_{\text{app}}^2)|^2 \, dx
$$

$$
= \int_{\Omega} |\nabla e_w^2|^2 \, dx + \int_{\Omega} |\nabla e_v^2|^2 \, dx + \int_{\Omega} |\nabla v_{\text{app}}^2|^2 \, dx + \int_{\Gamma_\delta} |\nabla w_{\text{app}}^2|^2 \, dx
$$

$$
- 2 \int_{\Omega} \nabla v_{\text{app}}^2 \nabla (w_0^2 - e_v^2) \, dx + 2 \int_{\Omega \setminus \Gamma_\delta} \nabla (e_w^2 - e_v^2) \nabla w_{\text{app}}^2 \, dx - 2 \int_{\Omega} \nabla e_v^2 \nabla e_w^2 \, dx.
$$

There are several terms to evaluate which we consider one by one in the following.

**Step 1: Computation of the $\mathcal{O}(\delta^3)$ terms.** From Lemma 4.4 we can see easily

$$
\int_{\Omega} |\nabla e_w^2|^2 \, dx = \mathcal{O}(\delta^3), \quad \int_{\Omega} |\nabla e_v^2|^2 \, dx = \mathcal{O}(\delta^3) \quad \text{and} \quad \int_{\Omega} \nabla e_v^2 \nabla e_w^2 \, dx = \mathcal{O}(\delta^3).
$$

Furthermore,

$$
\int_{\Omega \setminus \Gamma_\delta} \nabla (e_w^2 - e_v^2) \nabla w_{\text{app}}^2 \, dx = - \int_{\Omega \setminus \Gamma_\delta} \Delta (e_w^2 - e_v^2) w_{\text{app}}^2 \, dx - \int_{\Gamma_\delta} \frac{\partial}{\partial\nu} (e_w^2 - e_v^2) w_{\text{app}}^2 \, d\Gamma
$$

and recalling that $\frac{\partial e_w^2}{\partial\nu} = \frac{\partial e_v^2}{\partial\nu}$ we obtain

$$
\int_{\Omega \setminus \Gamma_\delta} \nabla (e_w^2 - e_v^2) \nabla w_{\text{app}}^2 \, dx = - \int_{\Omega \setminus \Gamma_\delta} \Delta (e_w^2 - e_v^2) w_{\text{app}}^2 \, dx.
$$

But (4.25) and (4.27) give

$$
\| \Delta (e_w^2 - e_v^2) \|_{L^2(\Omega \setminus \Gamma_\delta)} \leq \mathcal{O}(\delta^{3/2})
$$

which complemented with (4.24) gives

$$
\left| \int_{\Omega \setminus \Gamma_\delta} \nabla (e_w^2 - e_v^2) \nabla w_{\text{app}}^2 \, dx \right| \leq \mathcal{O}(\delta^2).
$$
Step 2: Computation of $\int_{\Omega} |\nabla v_{\text{app}}^2| \, dx$. From its definition we have after integration by part:

$$\int_{\Omega} |\nabla v_{\text{app}}^2| \, dx = (C_\delta^1)^2 \left( \int_{\Omega} |\nabla v_0|^2 \, dx + 2\delta \int_{\Omega} \nabla v_0 \cdot \nabla v_1 \, dx + \delta^2 \int_{\Omega} |\nabla v_1|^2 \, dx \right)$$

$$= (C_\delta^1)^2 \left( \lambda_0 + 2\delta \lambda_0 \int_{\Omega} v_0 v_1 \, dx + 2\delta \lambda_1 + \delta^2 \int_{\Omega} (\lambda_0 |v_1|^2 + \lambda_1 v_0 v_1) \, dx + \delta^2 \int_{\Gamma} \frac{\partial v_1}{\partial \nu} \frac{\partial v_0}{\partial \nu} \, ds \right)$$

$$= \lambda_0 \|v_{\text{app}}^2\|_{L^2(\Omega)}^2 + 2(C_\delta^1)^2 \delta \lambda_1 + \delta^2 \int_{\Omega} \nabla v_1 \cdot \nabla v_1 \, dx.$$  

Step 3: Computation of $\int_{\Omega \setminus \Omega_\delta} |\nabla w_{\text{app}}^2| \, dx$. To this end we first write

$$\int_{\Omega \setminus \Omega_\delta} |\nabla w_{\text{app}}^2| \, dx = (C_\delta^1)^2 \delta^2 \int_{\Omega \setminus \Omega_\delta} |\nabla w_1|^2 \, dx + 2(C_\delta^1)^2 \delta^3 \int_{\Omega \setminus \Omega_\delta} \nabla w_1 \cdot \nabla w_2 \, dx + (C_\delta^1)^2 \delta^4 \int_{\Omega \setminus \Omega_\delta} |\nabla w_2|^2 \, dx. \quad (4.29)$$

From the definition of $w_1$ and by using local coordinates we have

$$\delta^2 \int_{\Omega \setminus \Omega_\delta} |\nabla w_1|^2 \, dx = \delta^3 \int_0^1 \frac{1}{\delta^2} \left| \frac{\partial \hat{w}_1}{\partial \xi}(s, \xi) \right|^2 (1 + \delta \kappa \xi) \, d\xi \, ds + O(\delta^3)$$

$$= \delta \lambda_1 + \delta^2 \frac{\kappa}{2} \lambda_1 + O(\delta^3).$$

Similarly, using the definition of $w_2$ we have

$$\delta^3 \int_{\Omega \setminus \Omega_\delta} \nabla w_1 \cdot \nabla w_2 \, dx = \delta^4 \int_0^1 \int_0^1 \frac{1}{\delta^2} \frac{\partial \hat{w}_1(s, \xi)}{\partial \xi} \frac{\partial \hat{w}_2(s, \xi)}{\partial \xi} \, d\xi \, ds \, d\xi \, ds + O(\delta^3)$$

$$= \delta^2 \int_0^1 \left( -\kappa \frac{\partial v_0}{\partial \nu} + \frac{\partial v_1}{\partial \nu} \right) \frac{\partial v_0}{\partial \nu} \, d\xi \, ds + O(\delta^3)$$

$$= -\delta^2 \frac{\kappa}{2} \lambda_1 + \delta^2 \int_{\Gamma} \frac{\partial v_1}{\partial \nu} \frac{\partial v_0}{\partial \nu} \, d\xi \, ds + O(\delta^3).$$

For the last term of (4.29) we simply have

$$\delta^4 \int_{\Omega \setminus \Omega_\delta} |\nabla w_2|^2 \, dx \leq O(\delta^3).$$

Next we need to estimate the constant $(C_\delta^1)^2$. Indeed

$$(C_\delta^1)^2 = \int_{\Omega} |u_0^1 + \delta v_1|^2 \, dx = \int_{\Omega} |u_0^1|^2 \, dx + 2\delta \int_{\Omega} u_0^1 v_1 \, dx + \delta^2 \int_{\Omega} |v_1|^2 \, dx$$

$$= 1 + \delta^2 \int_{\Omega} |v_1|^2 \, dx + 2\delta \int_{\Omega} (u_0^1 + v_0) v_1 \, dx = 1 + O(\delta^2) \quad (4.30)$$
Hence since \( \|u_3^1 + v_0\|_{L^2(\Omega)} \leq \mathcal{O}(\delta) \). Plugging everything into (4.29) we finally obtain
\[
\int_{\Omega} \frac{1}{\delta \lambda_1} |\nabla w_{\text{app}}|^2 \, dx = \delta \lambda_1 + \delta^2 \left( 2 \int \frac{\partial u_1}{\partial \nu} \frac{\partial v_0}{\partial \nu} \, ds - \frac{\kappa}{2} \lambda_1 \right) + \mathcal{O}(\delta^3).
\]

**Step 4: Computation of** \( \int_\Omega \nabla v_{\text{app}}^2 \cdot \nabla (w_3 - e_v^2) \, dx \). To this end, we make use of the equation satisfied by \( v_{\text{app}}^2 \) together with the normalization of \( u_3^1 \) to simplify it. First we can write
\[
\int_\Omega \nabla v_{\text{app}}^2 \cdot \nabla (w_3^1 - e_v^2) \, dx = - \int_\Omega \Delta v_{\text{app}}^2 (w_3^1 - e_v^2) \, dx - \int_\Gamma \frac{\partial v_{\text{app}}^2}{\partial \nu} v_{\text{app}}^2 \, ds
\]
\[
= C_3^1 \int_\Omega (\lambda_0 v_0 + \delta \lambda_0 v_1 + \delta \lambda_1 v_0)(w_3^1 - e_v^2) \, dx
\]
\[
- (C_3^1)^2 \int_\Gamma \left( \frac{\partial v_0}{\partial \nu} + \delta \frac{\partial v_1}{\partial \nu} \right) \delta v_1 \, ds.
\]
From (4.13) and Lemma 4.4 we have that
\[
C_3^1 \delta^2 \int_\Omega \lambda_1 v_1 (w_3^1 - e_v^2) \, dx = \mathcal{O}(\delta^3)
\]
whence
\[
\int_\Omega \nabla v_{\text{app}}^2 \cdot \nabla (w_3^1 - e_v^2) \, dx = (\lambda_0 + \delta \lambda_1) \int_\Omega v_{\text{app}}^2 (w_3^1 - e_v^2) \, dx + (C_3^1)^2 \delta \lambda_1 + \delta^2 \int_\Gamma \frac{\partial v_1}{\partial \nu} \frac{\partial v_0}{\partial \nu} \, ds + \mathcal{O}(\delta^3).
\]

We now use the normalization of \( u_3^1 \) to obtain
\[
\|u_3^1\|_{L^2(\Omega)}^2 = 1 = \int_\Omega |u_3^1 + v_{\text{app}}^2 - v_{\text{app}}^2|^2 \, dx
\]
\[
= \int_\Omega |u_3^1 + v_{\text{app}}|^2 \, dx - 2 \int_\Omega v_{\text{app}}^2 (u_3^1 + v_{\text{app}}) \, dx + \int_\Omega |v_{\text{app}}^2|^2 \, dx
\]
\[
= \mathcal{O}(\delta^3) - 2 \int_\Omega v_{\text{app}}^2 (u_3^1 + v_{\text{app}}) \, dx + \int_\Omega |v_{\text{app}}^2|^2 \, dx.
\]

Hence since \( u_3^1 + v_{\text{app}}^2 = w_3^1 - e_v^2 \), we have that
\[
-2 \int_\Omega \nabla v_{\text{app}}^2 \cdot \nabla (w_3^1 - e_v^2) \, dx = (\lambda_0 + \delta \lambda_1) \left( 1 - \|v_{\text{app}}^2\|_{L^2(\Omega)}^2 \right) - 2(C_3^1)^2 \delta \lambda_1 - 2\delta^2 \int_\Gamma \frac{\partial v_1}{\partial \nu} \frac{\partial v_0}{\partial \nu} \, ds + \mathcal{O}(\delta^3).
\]

The expansion (4.30) and the definition of \( v_{\text{app}}^2 \) yield
\[
\left| 1 - \|v_{\text{app}}^2\|_{L^2(\Omega)}^2 \right| \leq \mathcal{O}(\delta^2)
\]
and consequently
\[
-2 \int_\Omega \nabla v_{\text{app}}^2 \cdot \nabla (w_3^1 - e_v^2) \, dx = \lambda_0 \left( 1 - \|v_{\text{app}}^2\|_{L^2(\Omega)}^2 \right) - 2(C_3^1)^2 \delta \lambda_1 - 2\delta^2 \int_\Gamma \frac{\partial v_1}{\partial \nu} \frac{\partial v_0}{\partial \nu} \, ds + \mathcal{O}(\delta^3).
\]
Finally we have all the necessary estimates to reach the conclusion. Plugging the estimates obtained in Steps 1-4 into (4.28) leads to the desired final estimate:

\[ \lambda_1^3 = \lambda_0 + \delta \lambda_1 + \delta^2 \lambda_2 + O(\delta^3). \]

**Remark 4.3** Although we stop at the order two the analysis of Section 4.3 is constructive and can in principle be carried through iteratively to any order of approximation. Note that in order to prove the \( k \) order of convergence for the transmission eigenvalue we need to prove the \( (k-1) \) order of convergence for the corresponding eigenfunctions. Also the convergence procedure is not limited to only the first eigenvalue. All these generalizations rely upon the ability to compute explicitly the terms in the asymptotic expansion of the transmission eigenvalues and on the estimate for the zero order approximation of the eigenvalue.

**Remark 4.4** In principle the second order asymptotic expansion of the first transmission eigenvalue can be used to estimate the thickness of the layer provided that \( \Omega \) is known. In particular

\[ \delta \approx \frac{\lambda_1^1 - \lambda_0}{\lambda_1} \]

where \( \lambda_3^1 \) can be computed from the scattering data (see Cakoni et al. (2010a)) and \( \lambda_0 \) and \( \lambda_1 \) can be computed.

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A Auxiliary Regularity Estimates

We start by establishing some crucial elliptic regularity estimates with explicit dependence of the constants on $\delta$. In the following, we denote by $C$ a generic constant independent of $\delta$. In the next Lemma we adopt the notations and the definitions of (McLean, 2000, Section 4), which we recall here in a simplified setting. Let $\mathcal{O}$ be a connected Lipschitz domain of $\mathbb{R}^2$ and denote by $(x_1, x_2)$ the coordinates of a point $x$ in some given basis. We define the operator $\mathcal{P}$ in $\mathcal{O}$ by

$$\mathcal{P} := -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j},$$

where for $(i, j) \in \{1, 2\}^2 a_{ij} \in C^1(\mathcal{O})$. We say that $\mathcal{P}$ is coercive if there exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}^2$ and $x \in \mathcal{O}$

$$\sum_{i,j} a_{ij}(x)\xi_i \xi_j \geq C|\xi|^2.$$

We call the conormal derivative the operator $B_{\nu}$ given by

$$B_{\nu} := \sum_i \nu_i \gamma_{\partial \mathcal{O}} \left( \sum_j a_{ij} \frac{\partial}{\partial x_j} \right),$$

where $\gamma_{\partial \mathcal{O}}$ is the trace operator on $\partial \mathcal{O}$ and $\nu_i$ for $i = 1, 2$ is the $i$th component of the inward normal vector to $\partial \mathcal{O}$.

**Lemma A.1** Consider $\delta > 0$, $g \in H^{1/2}(\mathbb{R})$ and $f \in L^2(\mathbb{R} \times (0, \delta))$. Let $\mathcal{P}$ be a coercive operator with coercivity constant independent of $\delta$ and $B_{\nu}$ the associated conormal derivative. If $w \in H^1_0 := \{ w \in H^1(\mathbb{R} \times (0, \delta)), w(x_1, \delta) = 0 \text{ for all } x_1 \in \mathbb{R} \}$ solves

$$\begin{cases}
\mathcal{P} w = f \text{ in } \mathbb{R} \times (0, \delta), \\
B_{\nu} w(x_1, 0) = g(x_1) \text{ for all } x_1 \in \mathbb{R}, \\
w(x_1, \delta) = 0 \text{ for all } x_1 \in \mathbb{R}, 
\end{cases}$$

then

$$w \in C^{0, \alpha}(\mathbb{R} \times (0, \delta)) \cap H^1(\mathbb{R} \times (0, \delta)).$$

**References**


then there exists a constant $C > 0$ independent of $\delta$ such that
\[ \|w\|_{H^2(\mathbb{R} \times (0, \delta))} \leq C \left( \|f\|_{L^2(\mathbb{R} \times (0, \delta))} + \|g\|_{H^{1/2}(\mathbb{R})} \right). \]

**Remark A.1** The novel important aspect in the above a priori estimate is to show that the constant is independent of $\delta$, and to our knowledge such a result was not available in the literature.

**Proof.** Using Green’s formula we have that
\[ \int_{\mathbb{R} \times (0, \delta)} \mathcal{P}w \psi \, dx \, dx_2 = \Phi(w, \psi) + \int_{\mathbb{R}} B_v w(x_1, 0) \psi(x_1, 0) \, dx_1 \]
or
\[ \int_{\mathbb{R} \times (0, \delta)} f \psi \, dx \, dx_2 = \Phi(w, \psi) + \int_{\mathbb{R}} g(x_1) \psi(x_1, 0) \, dx_1 \] \tag{A.1}
for all $\psi \in H_0$, where the bilinear form $\Phi(\cdot, \cdot)$ is defined by
\[ \Phi(u, v) = \sum_{i,j} \int_{\mathbb{R} \times (0, \delta)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx \, dx_2 \quad \text{for all } (u, v) \in H_0. \]

In order to obtain the desired regularity result we apply the approach of the difference quotient in the direction $x_1$. To this end for $h \in \mathbb{R}$ and all $(x_1, x_2) \in \mathbb{R} \times [0, \delta]$ we define the difference quotient by
\[ \Delta_h u := \frac{u(x_1 + h, x_2) - u(x_1, x_2)}{h}. \]

Straightforward algebraic calculations show that the following formulas hold true
\[ \int_{\mathbb{R} \times (0, \delta)} (\Delta_h u) v \, dx \, dx_2 = -\int_{\mathbb{R} \times (0, \delta)} u(\Delta_{-h} v) \, dx \, dx_2, \] \tag{A.2}
\[ |\Phi(\Delta_h u, v) + \Phi(u, \Delta_{-h} v)| \leq C \|v\|_{H^1(\mathbb{R} \times (0, \delta))} \|u\|_{H^1(\mathbb{R} \times (0, \delta))}, \] \tag{A.3}
for all $u$ and $v$ in $H_0$, and moreover there exists a constant $C > 0$ independent of $\delta$ such that for all $u \in H_0$ and $h$ sufficiently small
\[ C \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2(\mathbb{R} \times (0, \delta))} \leq \|\Delta_h u\|_{L^2(\mathbb{R} \times (0, \delta))} \leq \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2(\mathbb{R} \times (0, \delta))}, \] \tag{A.4}
(see McLean (2000), Lemma 4.13 for the proof of this last result). Substituting (A.2) and (A.3) in (A.1) we obtain that for $\psi = \Delta_{-h} \Delta_h w \in H_0$ and all $h$ sufficiently small
\[ |\Phi(\Delta_h w, \Delta_h w)| \leq C \|w\|_{H^1(\mathbb{R} \times (0, \delta))} \|\Delta_h w\|_{H^1(\mathbb{R} \times (0, \delta))} \]
\[ + \int_{\mathbb{R}} g(x_1)(\Delta_{-h} \Delta_h w)(x_1, 0) \, dx_1 + \int_{\mathbb{R} \times (0, \delta)} f(\Delta_{-h} \Delta_h w) \, dx \, dx_2. \] \tag{A.5}
From McLean (2000), Exercise 4.4, we have that or all $s \in \mathbb{R}$
\[ \|\Delta_h u\|_{H^s(\mathbb{R})} \leq \left\| \frac{\partial u}{\partial x_1} \right\|_{H^s(\mathbb{R})}. \]
provided it is known that $\frac{\partial u}{\partial x_1} \in H^s(\mathbb{R})$. On the other hand for the boundary term in (A.5) we have

$$\int_{\mathbb{R}} |g \Delta_{-h} \Delta_h w| \, dx_1 \leq \|g\|_{H^{1/2}(\mathbb{R})} \|\Delta_{-h} \Delta_h w\|_{H^{-1/2}(\mathbb{R})}$$

$$\leq \|g\|_{H^{1/2}(\mathbb{R})} \|\Delta_h w\|_{H^{1/2}(\mathbb{R})} \leq C \|g\|_{H^{1/2}(\mathbb{R})} \|\Delta_h w\|_{H^{1}((\mathbb{R} \times (0,\delta)))} \quad (A.6)$$

with a constant $C > 0$ independent of $\delta$ and $h$ (see the trace Lemma A.4 for the last inequality). Finally, the coercivity of $\Phi$ together with (A.4), (A.5) and (A.6) give

$$\|\Delta_h w\|_{H^1((\mathbb{R} \times (0,\delta)))} \leq C \left( \|g\|_{H^{1/2}(\mathbb{R})} + \|w\|_{H^1((\mathbb{R} \times (0,\delta)))} + \|f\|_{L^2((\mathbb{R} \times (0,\delta)))} \right)$$

which in view of McLean (2000), Lemma 4.13, gives

$$\left\| \frac{\partial w}{\partial x_1} \right\|_{H^1((\mathbb{R} \times (0,\delta)))} \leq C \left( \|g\|_{H^{1/2}(\mathbb{R})} + \|w\|_{H^1((\mathbb{R} \times (0,\delta)))} + \|f\|_{L^2((\mathbb{R} \times (0,\delta)))} \right).$$

To estimate the second order derivative with respect to $x_2$, we recall that $Pw = f$ and since $P$ is coercive there exists a constant $C > 0$ that depends on the coefficients $a_{ij}$ but not on $\delta$ such that

$$\left\| \frac{\partial^2 w}{\partial x_2^2} \right\|_{L^2((\mathbb{R} \times (0,\delta)))} \leq C \left( \|f\|_{L^2((\mathbb{R} \times (0,\delta)))} + \left\| \frac{\partial w}{\partial x_1} \right\|_{H^1((\mathbb{R} \times (0,\delta)))} \right) \leq C \left( \|g\|_{H^{1/2}(\mathbb{R})} + \|w\|_{H^1((\mathbb{R} \times (0,\delta)))} + \|f\|_{L^2((\mathbb{R} \times (0,\delta)))} \right).$$

Then the result is a consequence of the standard a priori estimate for $\|w\|_{H^1((\mathbb{R} \times (0,\delta)))}$ making use of the coercivity of $P$. \qed

From the above regularity result in a strip it is now possible to obtain the same type of regularity result in $\Omega \setminus \overline{\Omega}_\delta$ first with homogeneous mixed boundary conditions which is stated in Lemma A.2 and then with inhomogeneous mixed boundary conditions which is stated in Lemma A.3.

**Lemma A.2** There exists a constant $C > 0$ independent of $\delta$ such that for all $f \in L^2(\Omega \setminus \overline{\Omega}_\delta)$ the unique solution $w \in H^1(\Omega \setminus \overline{\Omega}_\delta)$ of

$$\begin{cases}
-\Delta w = f \text{ in } \Omega \setminus \overline{\Omega}_\delta, \\
\frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma, \\
w = 0 \text{ on } \Gamma_\delta,
\end{cases}$$

satisfies the a priori estimate

$$\|w\|_{H^2(\Omega \setminus \overline{\Omega}_\delta)} \leq C \|f\|_{L^2(\Omega \setminus \overline{\Omega}_\delta)}.$$

**Proof.** First from the standard a priori estimate for the Laplacian we have that

$$\|w\|_{H^1(\Omega \setminus \overline{\Omega}_\delta)} \leq C \|f\|_{L^2(\Omega \setminus \overline{\Omega}_\delta)} \quad (A.7)$$

with a constant $C > 0$ independent of $\delta$. To obtain $H^2$ estimates our approach is based on first locally straighten the boundary and then apply Lemma A.1. To this end, since $\Omega \setminus \overline{\Omega}_\delta$ is a compact set, there exists an integer $n$ and a sequence $(\Omega_i)_{i=1,\ldots,n}$ of bounded and connected domains of $\mathbb{R}^2$ such that
\[ \Omega \setminus \overline{\Omega}_\delta \subset \bigcup_{i=1}^{n} \Omega_i \text{ for all } \delta \text{ sufficiently small. Moreover, we take } \Omega_i \text{ such that there exists } s_i > 0 \text{ and a } C^\infty([-s_i, s_i]), k \gg 0, \text{ function } x_{\Gamma} \text{ such that for all } \delta \text{ sufficiently small we have} \]

\[ (\Omega \setminus \overline{\Omega}_\delta) \cap \Omega_i = \{ x_{\Gamma}(s) + \eta(s, \nu), \forall (s, \nu) \in (-s_i, s_i) \times (0, \delta) \} \]

where \( x_{\Gamma}(s) \in \Gamma \) for \( s \in (-s_i, s_i) \) and \( i = 1, \cdots, n \) (see Figure 2). Thus for all \( \delta \) sufficiently small,

\[ \varphi_i : (\Omega \setminus \overline{\Omega}_\delta) \cap \Omega_i \to \overline{\Omega}_i 
\]

\[ x \mapsto (s, \nu) \]

is a C\(^2\)-diffeomorphism, where \( \Omega_i := (-s_i, s_i) \times (0, \delta) \). Let \((\phi_i)_{i=1, \cdots, n}\) be a partition of unity such that \( \phi_i \in C^\infty(\mathbb{R}^2) \), supp\((\phi_i) \subset \Omega_i \) and \( \sum_i \phi_i = 1 \) in \( \Omega \setminus \overline{\Omega}_\delta \). Hence if we define \( w_i := \phi_i w \in H^1(\Omega \setminus \overline{\Omega}_\delta) \), then

\[ \| w \|_{H^2(\Omega \setminus \overline{\Omega}_\delta)} \leq \sum_{i=1}^{N} \| w_i \|_{H^2(\Omega \setminus \overline{\Omega}_\delta)} \]  \hspace{1cm} (A.8)

and \( w_i \) is compactly supported in \( \Omega_i \). Furthermore, \( w_i \) solves

\[ \begin{align*}
-\Delta w_i &= f_i \text{ in } \Omega \setminus \overline{\Omega}_\delta, \\
\frac{\partial w_i}{\partial \nu} &= g_i \text{ on } \Gamma, \\
w_i &= 0 \text{ on } \Gamma_{\delta},
\end{align*} \]

\[ \text{with} \]

\[ f_i := f \phi_i + 2 \nabla w \nabla \phi_i + w \Delta \phi_i \quad \text{and} \quad g_i := \frac{\partial \phi_i}{\partial \nu}. \]

In the following, for \( U \in L^2((\Omega \setminus \overline{\Omega}_\delta) \cap \Omega_i) \) and \( G \in H^s(\Gamma \cap \partial \Omega_i) \) we denote by \( \tilde{U} \in L^2(\tilde{\Omega}_i) \) and \( \tilde{G} \in H^s((-s_i, s_i)) \) the functions defined by \( \tilde{U} := U \circ \varphi_i \) and \( \tilde{G} := G \circ \varphi_i \), respectively. Since the \( W^{2,\infty} \) norm of \( \varphi_i \) and \( \varphi_i^{-1} \) does not depend on \( \delta \) it is easy to see that there exists a constant \( C > 0 \) independent of \( \delta \) such that for \( U \in H^p((\Omega \setminus \overline{\Omega}_\delta) \cap \Omega_i) \)

\[ \frac{1}{C} \| \tilde{U} \|_{H^p(\tilde{\Omega}_i)} \leq \| U \|_{H^p((\Omega \setminus \overline{\Omega}_\delta) \cap \Omega_i)} \leq C \| \tilde{U} \|_{H^p(\tilde{\Omega}_i)} \] \hspace{1cm} (A.9)

and for \( G \in H^s(\Gamma \cap \partial \Omega_i) \)

\[ \frac{1}{C} \| \tilde{G} \|_{H^s((-s_i, s_i))} \leq \| G \|_{H^s(\Gamma \cap \partial \Omega_i)} \leq C \| \tilde{G} \|_{H^s((-s_i, s_i))} \] \hspace{1cm} (A.10)
With these notations, we can now prove using the calculations developed in Section 3.2 that \( \tilde{w}_i \in H^1(\Omega_i) \) solves

\[
\begin{align*}
\mathcal{P}\tilde{w}_i &= \tilde{f}_i \text{ in } \Omega_i, \\
(\mathcal{B}_w\tilde{w}_i)(s,0) &= \tilde{g}_i(s) \text{ for all } s \in (-s_i,s_i), \\
\tilde{w}_i(s,\delta) &= 0 \text{ for all } s \in (-s_i,s_i),
\end{align*}
\tag{A.11}
\]

where \( \tilde{f}_i(s,\eta) := (1+\eta s)(f_i \circ \varphi_i)(s,\eta), \tilde{g}_i(s,\eta) := -(1+\eta s)(g_i \circ \varphi_i)(s,\eta) \) and \( \mathcal{P} \) and \( \mathcal{B}_w \) are as in Lemma A.1 (note that the coercivity constant for \( \mathcal{P} \) does not depend on \( \delta \)). Furthermore since \( \tilde{w}_i \), \( \tilde{f}_i \) and \( \tilde{g}_i \) are equal to 0 in a vicinity of \(-s_i\) and \( s_i \) we can extend them by 0 into \( \mathbb{R} \times (0,\delta) \). For sake of simplicity, we do not change the notations for their extension and note that these extension also satisfy the system (A.11) for \( s \in \mathbb{R} \). Hence we can apply Lemma A.1 to \( \tilde{w}_i \) to obtain

\[
\|\tilde{w}_i\|_{H^2(\mathbb{R} \times (0,\delta))} \leq C \left( \|\tilde{g}_i\|_{H^{1/2}(\mathbb{R})} + \|\tilde{f}_i\|_{L^2(\mathbb{R} \times (-s_i,s_i))} \right)
\]

where \( C \) is independent of \( \delta \). Using (A.9) and (A.10) in (A.8) together with our first a priori estimate (A.7) finally proves the lemma. \( \square \)

Next we obtain the same type of estimates as in Lemma A.2 for inhomogeneous boundary condition on \( \Gamma \). The challenge is to show that the lifting function is bounded independently of \( \delta \) in appropriate norm.

**Lemma A.3** Let \( g \in H^{1/2}(\Gamma) \) and \( w \in H^1(\Omega \setminus \overline{\Omega}_\delta) \) be the unique solution of

\[
\begin{align*}
\Delta w &= 0 \text{ in } \Omega \setminus \overline{\Omega}_\delta, \\
\frac{\partial w}{\partial \nu} &= g \text{ on } \Gamma, \\
w &= 0 \text{ on } \Gamma_\delta.
\end{align*}
\]

Then there exists a constant \( C > 0 \) such that for all \( \delta > 0 \)

\[
\|w\|_{H^2(\Omega \setminus \overline{\Omega}_\delta)} \leq C \|g\|_{H^{1/2}(\Gamma)}.
\]

In addition, if \( \|g\|_{H^{1/2}(\Gamma)} \leq C_g \) for some \( C_g > 0 \) and all \( \delta > 0 \), then

\[
\|w\|_{H^2(\Omega \setminus \overline{\Omega}_\delta)} \xrightarrow{\delta \to 0} 0.
\]

**Proof.** We start by building an appropriate lifting of \( g \) which equals to 0 on \( \Gamma_\delta \). To this end, let us define \( g_i = \phi_i g \) and its local counterpart \( \bar{g}_i \) using the same partition of unity and local parameterization as in Lemma A.2. Then we can define an extension of \( \bar{g}_i \) to \( \mathbb{R} \) denoted by \( \bar{g}_i \) by

\[
\bar{g}_i(s) := \begin{cases} 
\bar{g}_i(s) \text{ if } s \in [-s_i,s_i], \\
0 \text{ elsewhere.}
\end{cases}
\]

For any function \( G \in L^1(\mathbb{R}) \) let

\[
\mathcal{F}G(\xi) := \int_{\mathbb{R}} G(s)e^{-i2\pi s\xi} \, ds \quad \text{and} \quad \mathcal{F}^{-1}G(s) := \int_{\mathbb{R}} G(\xi)e^{i2\pi s\xi} \, d\xi
\]
be its Fourier transform and its inverse Fourier transform, respectively. Since $\Gamma$ is of class $C^2$ and $g \in H^{1/2}(\Gamma)$, Plancherel’s Theorem ensure the existence of a constant $C$ independent of $\delta$ such that

$$
\|(1 + \xi^2)^{1/4} \mathcal{F}g\|_{L^2(\mathbb{R})} \leq C\|g\|_{H^{1/2}(\Gamma)}.
$$

(A.12)

For all $\xi \in \mathbb{R}$ and $\eta \in [0, \delta]$ let us define

$$
\xi w_i(\xi, \eta) := \frac{\mathcal{F}\mathcal{G}_i(\xi)}{\xi} \left( \frac{\sinh(|\xi|(\eta - \delta))}{\cosh(|\xi|\delta)} \right)
$$

and

$$
\mathcal{H}_i(s, \eta) := (\mathcal{F}^{-1}w_i)(s, \eta).
$$

Then $\mathcal{H}_i$ satisfies $\mathcal{H}_i(s, \delta) = 0$ and $\frac{\partial \mathcal{H}_i}{\partial s}(s, 0) = \mathcal{G}_i(s)$ in $\mathbb{R}$. Moreover, there exists a constant $C > 0$ independent of $\delta$ such that for all $(\xi, \eta) \in \mathbb{R} \times (0, \delta)$ we have

$$
|w_i(\xi, \eta)|^2 \leq C(\mathcal{F}\mathcal{G}_i(\xi))^2, \quad |\xi w_i(\xi, \eta)|^2 \leq C(\mathcal{F}\mathcal{G}_i(\xi))^2, \quad \left| \frac{\partial w_i}{\partial \eta}(\xi, \eta) \right|^2 \leq C(\mathcal{F}\mathcal{G}_i(\xi))^2
$$

Integrating the above inequalities over $\mathbb{R} \times (0, \delta)$ and using (A.12) and the Plancherel’s Theorem we have that there exists a constant $C > 0$ independent of $\delta$ such that

$$
\|\mathcal{H}_i\|_{H^1(\mathbb{R} \times (0, \delta))}^2 \leq C\|g\|_{H^{1/2}(\Gamma)}^2.
$$

Moreover, for all $(\xi, \eta) \in \mathbb{R} \times (0, \delta)$ we also have

$$
\xi^2 w_i(\xi, \eta) = |\xi| \mathcal{F}\mathcal{G}_i(\xi) \frac{\sinh(|\xi|(\eta - \delta))}{\cosh(|\xi|\delta)},
$$

$$
\xi \frac{\partial w_i}{\partial \eta}(\xi, \eta) = |\xi| \mathcal{F}\mathcal{G}_i(\xi) \frac{\cosh(|\xi|(\eta - \delta))}{\cosh(|\xi|\delta)},
$$

$$
\frac{\partial^2 w_i}{\partial \eta^2}(\xi, \eta) = |\xi| \mathcal{F}\mathcal{G}_i(\xi) \frac{\sinh(|\xi|(\eta - \delta))}{\cosh(|\xi|\delta)}.
$$

Therefore

$$
\int_{\mathbb{R}} \int_{0}^{\delta} |\xi|^4 |w_i(\xi, \eta)|^2 d\eta d\xi = \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}\mathcal{G}_i(\xi)|^2 \int_{0}^{\delta} \frac{\sinh^2(|\xi|(\eta - \delta))}{\cosh^2(|\xi|\delta)} d\eta d\xi
$$

$$
= \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}\mathcal{G}_i(\xi)|^2 \int_{0}^{\delta} \frac{\cosh(2|\xi|(\eta - \delta)) - 1}{2 \cosh(|\xi|\delta)^2} d\eta d\xi
$$

$$
= \frac{1}{2} \int_{\mathbb{R}} |\xi|^2 |\mathcal{F}\mathcal{G}_i(\xi)|^2 \left( \frac{\sinh(|\xi|\delta)}{|\xi| \cosh(|\xi|\delta)} - \frac{\delta}{\cosh(|\xi|\delta)^2} \right) d\xi
$$

$$
\leq \frac{1}{2} \int_{\mathbb{R}} |\xi||\mathcal{F}\mathcal{G}_i(\xi)|^2 \frac{\sinh(|\xi|\delta)}{\cosh(|\xi|\delta)} d\xi
$$

$$
\leq \frac{1}{2} \int_{\mathbb{R}} |\xi||\mathcal{F}\mathcal{G}_i(\xi)|^2 d\xi,
$$

(A.13)
and hence an application of (A.12) yields
\[ \int_{\mathbb{R}} \int_{0}^{\delta} |\xi|^4 |w_i(\xi, \eta)|^2 \, d\eta d\xi \leq C \|g\|^2_{H^{1/2}(\Gamma)}. \]

In a similar way we obtain that
\[ \int_{\mathbb{R}} \int_{0}^{\delta} \left| \xi \frac{\partial w_i}{\partial \eta}(\xi, \eta) \right| \, d\eta d\xi \leq C \|g\|^2_{H^{1/2}(\Gamma)} \quad \text{and} \quad \int_{\mathbb{R}} \int_{0}^{\delta} \left| \frac{\partial^2 w_i}{\partial \eta^2}(\xi, \eta) \right| \, d\eta d\xi \leq C \|g\|^2_{H^{1/2}(\Gamma)} \]

with a different constant \( C > 0 \) independent of \( \delta \). Hence, once more application of Plancherel’s Theorem implies
\[ \|w_i\|_{H^2(\mathbb{R} \times (0, \delta))} \leq C \|g\|_{H^{1/2}(\Gamma)}. \quad (A.14) \]

Next assume in addition that there exists \( C_g > 0 \) such that \( \|g\|_{H^{1/2}(\Gamma)} \leq C_g \) for all \( \delta > 0 \), then for almost every \( \xi \in \mathbb{R} \),
\[ |\xi| |Fg_i(\xi)|^2 \frac{\sinh(|\xi|\delta)}{\cosh(|\xi|\delta)} \to 0 \quad \delta \to 0 \]
and since
\[ |\xi| |Fg_i(\xi)|^2 \frac{\sinh(|\xi|\delta)}{\cosh(|\xi|\delta)} \leq |\xi|(Fg_i(\xi))^2, \]
the Lebesgue dominated convergence theorem with (A.13) implies that
\[ \int_{\mathbb{R}} \int_{0}^{\delta} |\xi|^4 |w_i(\xi, \eta)|^2 \, d\eta d\xi \to 0 \quad \delta \to 0 \]

In a similar way we obtain that
\[ \int_{\mathbb{R}} \int_{0}^{\delta} \left| \xi \frac{\partial w_i}{\partial \eta}(\xi, \eta) \right| \, d\eta d\xi \to 0 \quad \delta \to 0 \quad \text{and} \quad \int_{\mathbb{R}} \int_{0}^{\delta} \left| \frac{\partial^2 w_i}{\partial \eta^2}(\xi, \eta) \right| \, d\eta d\xi \to 0, \]
whence, again from the Plancherel’s Theorem,
\[ \|w_i\|_{H^2(\mathbb{R} \times (0, \delta))} \to 0. \quad (A.15) \]

Now we go back to the physical domain and define for \( x \in \Omega \setminus \Gamma_\delta \)
\[ w_g(x) := \sum_{j \text{ s.t. } x \in \Omega_j} (\overline{w}_j \circ \varphi_j^{-1})(x). \]

Then \( w_g \) satisfies \( \frac{\partial w_g}{\partial \nu} |_\Gamma = g \) together with \( w_g |_{\Gamma_\delta} = 0 \) and using the fact that \( \Gamma \) is of class \( C^2 \) and that \((\overline{w}_i)_i \) satisfy (A.14), we can claim that there exists a constant \( C > 0 \) independent of \( \delta \) such that
\[ \|w_g\|_{H^2(\Omega \setminus \Gamma_\delta)} \leq C \|g\|_{H^{1/2}(\Gamma)}. \quad (A.16) \]

In addition, if there exists \( C_g > 0 \) such that \( \|g\|_{H^{1/2}(\Gamma)} \leq C_g \) for all \( \delta > 0 \), then from (A.15) we also have that
\[ \|w_g\|_{H^2(\Omega \setminus \Gamma_\delta)} \to 0. \quad (A.17) \]
Finally we consider $W := w - w_g$ which solves
\[
\begin{cases}
\Delta W = -\Delta w_g & \text{in } \Omega \setminus \overline{T_{\delta}}, \\
\frac{\partial W}{\partial \nu} = 0 & \text{on } \Gamma, \\
W = 0 & \text{on } \Gamma_{\delta},
\end{cases}
\]

Lemma A.2 states that there exists a constant $C$ independent of $\delta$ such that
\[
\|W\|_{H^2(\Omega \setminus T_{\delta})} \leq C\|w_g\|_{H^2(\Omega \setminus T_{\delta})}^2.
\]

This last estimate together with (A.16) imply the existence of a constant $C$ independent of $\delta$ such that
\[
\|w\|_{H^2(\Omega \setminus T_{\delta})} \leq C\|g\|_{H^{1/2}(\Gamma)}.
\]

and, if in addition there exists $C_g > 0$ such that $\|g\|_{H^{1/2}(\Gamma)} \leq C_g$ for all $\delta > 0$, (A.17) implies
\[
\|w\|_{H^2(\Omega \setminus T_{\delta})} \to 0
\]
as $\delta \to 0$, which ends the proof.

Next we prove a trace theorem which displays explicit dependence on $\delta$ of the constant.

**Lemma A.4** For $k = 1, 2$, there exists a constant $C > 0$ independent of $\delta$ such that for all $v \in H^k(\Omega \setminus T_{\delta})$ with $v|_{T_{\delta}} = 0$
\[
\|v\|_{H^{k-1/2}(\Gamma)} \leq C\|v\|_{H^k(\Omega \setminus T_{\delta})}.
\]

**Proof.** We prove the result for the domain $\mathbb{R} \times (0, \delta)$ and then use the partition of unity and change of variable introduced in the proof of Lemma A.2 to obtain the desired result. To this end, we consider an arbitrary $v \in C^\infty(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$ and denote as before by $\mathcal{F}v$ its Fourier transform with respect to the first variable where $\xi$ denotes the dual variable. Integrating by parts, we obtain that
\[
(1 + \xi^2)^{2(k-1/2)}|\mathcal{F}v(\xi, 0)|^2 = (1 + \xi^2)^{2(k-1/2)}|\mathcal{F}v(\xi, \delta)|^2
\]
\[
- 2\Re \left( \int_0^\delta (1 + \xi^2)^{k-1} \frac{\partial \mathcal{F}v}{\partial \eta}(\xi, \eta)(1 + \xi^2)^{k} \mathcal{F}v(\xi, \eta) d\eta \right)
\]
for all $\xi \in \mathbb{R}$. Integrating this equality along $\mathbb{R}$, and using the Cauchy-Schwartz inequality and Plancherel’s Theorem imply the existence of $C > 0$ independent of $\delta$ such that
\[
\|v(s, 0)\|_{H^{k-1/2}(\mathbb{R})} \leq C \left( \|v\|_{H^k(\mathbb{R} \times (0, \delta))} + \|v(s, \delta)\|_{H^{k-1/2}(\mathbb{R})} \right).
\]

A density argument ensures that the above estimate holds for all $v \in H^2(\mathbb{R} \times (0, \delta))$, and thus we obtain the result for $v$ whose trace is 0 on $\mathbb{R} \times \{\delta\}$.

We conclude this section with the following technical trace lemma, which was already used in the proof of Lemma A.1.

**Lemma A.5** For any $w \in H^1(\Omega \setminus T_{\delta})$ we have that
\[
\|w\|_{L^2(\Omega \setminus T_{\delta})} \leq C(\delta^{1/2}\|w\|_{L^2(\Gamma)} + \delta\|w\|_{H^1(\Omega \setminus T_{\delta})}),
\]
and for any function \( w \in H^2(\Omega \setminus \overline{\Omega}_\delta) \) such that \( w|_{\Gamma_\delta} = 0 \) we have that

\[
\|w\|_{H^{1/2}(\Gamma)} \leq C \delta^{1/2} \|w\|_{H^2(\Omega \setminus \overline{\Omega}_\delta)},
\]

(A.19)

where the constant \( C > 0 \) is independent of \( \delta \).

Proof. To prove (A.5) we take \( \phi \in C^\infty_0(\mathbb{R}^2) \) and using local variables in the layer to obtain

\[
\tilde{\phi}(s, \eta) = \tilde{\phi}(s, 0) + \int_0^\eta \frac{\partial \tilde{\phi}}{\partial t}(s, t) \, dt \, ds
\]

for every \( \eta \leq \delta \), where \( \tilde{\phi} \) denotes the function in the new variables \((s, \eta)\) (see Lemma A.2 for the definition). The Cauchy-Schwartz inequality implies

\[
|\tilde{\phi}(s, \eta)|^2 \leq C \left( |\tilde{\phi}(s, 0)|^2 + \delta \|\phi\|_{H^1(\Omega \setminus \overline{\Omega}_\delta)}^2 \right),
\]

whence we obtain the desired estimate for \( \phi \in C^\infty_0(\mathbb{R}^2) \) by integrating this inequality over \( \Gamma \times [0, \delta] \). A density argument gives the result for all \( w \in H^1(\Omega \setminus \overline{\Omega}_\delta) \).

To prove (A.19) we again take \( \phi \in C^\infty_0(\mathbb{R}^2) \) and in a similar way as above use local coordinates in the layer. Thus we obtain that for \( s \in [0, s_0] \)

\[
|\tilde{\phi}(s, 0)| \leq |\tilde{\phi}(s, \delta)| + \int_0^\delta \frac{\partial \tilde{\phi}}{\partial \eta}(s, \eta) \, d\eta \leq |\tilde{\phi}(s, \delta)| + \delta^{1/2} \|\phi\|_{H^1(\Omega \setminus \overline{\Omega}_\delta)}
\]

which leads to

\[
\|\phi\|_{L^2(\Gamma)} \leq C \left( \|\phi\|_{L^2(\Gamma_\delta)} + \delta^{1/2} \|\phi\|_{H^1(\Omega \setminus \overline{\Omega}_\delta)} \right)
\]

for some positive \( C > 0 \) independent of \( \delta \). Similarly we also have

\[
\|\nabla \Gamma \phi\|_{L^2(\Gamma)} \leq C \left( \|\nabla \Gamma \phi\|_{L^2(\Gamma_\delta)} + \delta^{1/2} \|\phi\|_{H^2(\Omega \setminus \overline{\Omega}_\delta)} \right).
\]

By density, the above two inequalities remain true for all \( w \in H^2(\Omega \setminus \overline{\Omega}_\delta) \) and by interpolation between \( L^2(\Gamma) \) and \( H^1(\Gamma) \) we obtain the desired result noting that \( w|_{\Gamma_\delta} = 0 \). \( \square \)

B Perturbation of an Eigenvalue Problem

We recall here some known results about the convergence of eigenvalues for self adjoint, positive and compact operators. The proof of the following fundamental result can be found in Section 3 of Oleinik et al. (1992).

**Theorem B.1** Assume that \( A : H \rightarrow H \) is a linear self-adjoint positive and compact operator on an Hilbert space \( H \). Let \( u \in H \) be such that \( \|u\|_H = 1 \) and \( \lambda, \, r > 0 \) such that

\[
\|Au - \lambda u\|_H \leq r.
\]

Then there is an eigenvalue \( \lambda_i \) of the operator \( A \) satisfying

\[
|\lambda - \lambda_i| \leq r.
\]
Furthermore, for any \( r^* > r \) there exists \( u^* \in H \) with \( \|u^*\|_H = 1 \) belonging to the eigenspace associated with all the eigenvalues of the operator \( A \) lying in \( [\lambda - r^*, \lambda + r^*] \) that satisfies

\[
\|u - u^*\|_H \leq \frac{2r}{r^*}.
\]

Based on this general result, we can obtain the following lemma for the Laplace operator with Dirichlet boundary conditions which is used in our asymptotical analysis in the main body of the paper.

**Lemma B.1** Let \( \lambda_i \) be a simple eigenvalue of the negative Laplacian with Dirichlet boundary conditions in \( \Omega \). Assume that it exists \( u \in H^1_0(\Omega) \) and \( r \) such that

\[
\left| \int_{\Omega} \nabla u \cdot \nabla v - \lambda_i uv \, dx \right| \leq r \|v\|_{H^1_0(\Omega)} \forall v \in H^1_0(\Omega), \tag{B.1}
\]

and

\[
\|u\|_{L^2(\Omega)} - 1 \leq r. \tag{B.2}
\]

Then it exists an eigenfunction \( u_i \) associated with the eigenvalue \( \lambda_i \) and normalized as \( \|u_i\|_{L^2(\Omega)} = 1 \) such that

\[
\|u - u_i\|_{H^1(\Omega)} \leq C r.
\]

for some constant \( C > 0 \) independent of \( r \) and \( u \).

**Proof.** The proof is essentially based on Theorem B.1. We define the operator \( A : H^1_0(\Omega) \to H^1_0(\Omega) \) by

\[
(Au, v)_{H^1_0(\Omega)} := (u, v)_{L^2(\Omega)} \forall (u, v) \in H^1_0(\Omega),
\]

where for all \( (u, v) \in H^1_0(\Omega) \),

\[
(u, v)_{H^1_0(\Omega)} := \int_{\Omega} \nabla u \cdot \nabla v \, dx
\]

and \((u, v)_{L^2(\Omega)}\) is the usual \( L^2 \) scalar product. Obviously \( A \) is a self-adjoint positive and compact operator on \( H^1_0(\Omega) \), and hence it has a discrete spectrum \((1/\lambda_i)_{i=1, \ldots, \infty} \) such that

\[
0 < \lambda_1 \leq \cdots \leq \lambda_i \leq \cdots \xrightarrow{i \to \infty} \infty,
\]

and their associated eigenfunctions \( u_i \in H^1_0(\Omega) \) satisfy

\[-\Delta u_i = \lambda_i u_i.\]

By hypothesis, the function \( \bar{u} := u/\|u\|_{H^1_0(\Omega)} \) satisfies

\[
\left\| A\bar{u} - \frac{1}{\lambda_i} \bar{u} \right\|_{H^1_0(\Omega)} \leq \frac{r}{\lambda_i \|u\|_{H^1_0(\Omega)}}.
\]

Therefore, since \( \lambda_i \) is supposed to be simple, the second part of Theorem B.1 ensures the existence of an eigenfunction \( \bar{u}_i \in H^1_0(\Omega) \) of \( A \) associated with \( 1/\lambda_i \) and normalized as \( \|\bar{u}_i\|_{H^1_0(\Omega)} = 1 \), such that

\[
\|\bar{u} - \bar{u}_i\|_{H^1_0(\Omega)} \leq C_1 \frac{r}{\|u\|_{H^1_0(\Omega)}}, \tag{B.3}
\]
for some constant $C_1 > 0$ that only depends on $\lambda_i$ and on the distance between $1/\lambda_i$ and the closest eigenvalue of $A$. To end the proof we must renormalize this last inequality.

Let us introduce $u_i := \tilde{u}_i / \|\tilde{u}_i\|_{L^2(\Omega)}$, then (B.3) gives

$$\|u - u_i\|_{H^1_0(\Omega)} \leq C_1 r + \|u - H^1_0(\Omega) \tilde{u}_i\|_{H^1_0(\Omega)}. \tag{B.4}$$

By using the definition of $u_i$, the second term in this expression becomes

$$\|u_i - \|u\|_{H^1_0(\Omega)} \tilde{u}_i\|_{H^1_0(\Omega)} \leq \sqrt{\lambda_i} - \|u\|_{H^1_0(\Omega)} \leq \lambda_i - \|u\|_{H^1_0(\Omega)}^2 \frac{1}{\sqrt{\lambda_i}} \tag{B.5}$$

but from (B.1) and (B.2) we have

$$\left| \lambda_i - \|u\|_{H^1_0(\Omega)}^2 \right| \leq \lambda_i - \lambda_i \|u\|_{L^2(\Omega)}^2 + r \|u\|_{H^1_0(\Omega)} \leq \lambda_i \left| 1 - \|u\|_{L^2(\Omega)} \right| + r \|u\|_{L^2(\Omega)} + r \|u\|_{H^1_0(\Omega)} \leq \lambda_i r (2 + r) + r \|u\|_{H^1_0(\Omega)}. \tag{B.6}$$

The $H^1_0$-norm of $u$ can be controlled by using (B.1) and Poincaré's inequality:

$$\|u\|_{H^1_0(\Omega)} \leq r + \lambda_i \sqrt{\lambda_0} (1 + r).$$

This last inequality together with (B.4), (B.5) and (B.6) give the result for a constant $C$ that only depends on $\lambda_0$ and $\lambda_i$. \qed