Dissipative Transport in Quantum Hall Ferromagnets by Spinwave Scattering

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We report on a study of the effect upon electrical transport of spinwave scattering from charged quasiparticles in $\nu = 1$ quantum Hall ferromagnets (QHFs), including both Heisenberg (single layer) and easy-plane (bilayer) cases. We derive a quantum Langevin equation to describe the resulting diffusive motion of the charged particle and use this to calculate the contribution to low temperature conductivity from a density of charged particles. This conductivity has a power law dependence upon temperature. The contribution is small at low temperatures increasing to a large value at relatively modest temperatures. We comment upon high temperature transport and upon the contribution of scattering to the width of the zero bias peak in inter-layer tunneling conductance.

I. INTRODUCTION

In quantum Hall ferromagnets (QHFs) [1] there are two important energy gaps: the spinwave gap, describing the minimum energy spinwave excitation; and the quasiparticle gap, describing the energy of a widely-separated quasiparticle/quasihole pair. The spinwave gap is due to the Zeeman energy in the single layer, real spin QHF and to the inter-layer tunnelling in the bilayer pseudo-spin QHF. The quasiparticle gap is determined ultimately by the Coulomb energy. These energies are independent and widely separated. There is, therefore, a regime of temperatures between these two energies, where a large population of neutral spin-waves exists, yet where charged quasi-particles are rather dilute.

These spin-waves have a direct effect upon a number of experimental observables. The temperature dependence of magnetisation and nuclear relaxation rates have been studied both experimentally and theoretically [2–7]. In addition to these magnetic properties, one can expect *transport* properties of the system to be affected by the thermally-excited spin-waves. The heat flow carried by the spin-waves will lead to a thermal conductivity that follows a power-law of temperature (rather than being exponentially suppressed as in a standard quantum Hall state). One also expects a power-law contribution to the diffusion thermopower, since for weak disorder this a measure of the entropy per particle [8]. In the present paper we consider the consequences of a large thermal population of spin-waves on the electrical transport properties, motivated in part by the observation that a surprisingly low temperature is required for a good quantum Hall effect in the QHFs [9,10].

Much of the truly novel physics of QHFs stems from the nature of the underlying quantum Hall state. In particular, spin and charge fluctuations are intimately linked so that the magnetic vorticity and charge density are proportional to one another [11]. One consequence of this relationship is that spin-waves, while electrically neutral, carry a dipole moment and thus interact electrostatically with any charged excitations. Here, we study how the scattering of spin-waves off charged excitations, assumed present either by activation or by slight departures from $\nu = 1$, affects the diffusion of the charged excitations. Can it lead to a significant, finite-temperature enhancement of the longitudinal conductivity of these quasiparticles? We shall consider both Heisenberg and easy-plane QHFs at $\nu = 1$, relevant for single layer and bilayer quantum Hall systems, respectively.

The scattering of spinwaves from charged quasiparticles in the QHF has been considered previously in Ref. [7]. The focus in that work was on the temperature dependence of magnetisation and spectral properties of the electronic Green's function and its effect upon tunnelling conductance. In contrast, here we consider the consequences of quasi-particle/spinwave scattering upon in-plane transport.

Section II contains the principal results of the paper. We begin by describing the model we use, and in II A provide a simple derivation of the quasiparticle diffusion constant in terms of a force-force correlation function. Sections II B and II C provide a systematic derivation of this result, based upon the use of collective co-ordinates and an influence functional. In section II D we use the formula derived to calculate the diffusion constants for Heisenberg and easy-plane QHFs at low temperature. The results are discussed in section II E. We find that the contribution to longitudinal conductivity is small at temperatures much less than the spin-stiffness. The temperature dependence is strong, particularly in the bilayer QHF. We present arguments why the conductivity may be expected to become large at relatively modest temperatures. In III we comment on the effects of the spin-wave scattering in our model on the tunnelling conductance of a bilayer system QHF.

II. QUASI-PARTICLE DIFFUSION CONSTANT

We shall study the longitudinal conductivity of disorder-free QHFs at $\nu = 1$ containing a dilute gas of charged excitations. These may be present due either to thermal activation or to (local) deviations of density that cause (local) departures from precisely $\nu = 1$. The diffusion of these mobile charges will determine the longitudinal conductivity, σ_{xx}. Since we treat the charges as independent, positive and negative charges will contribute in equal measure to σ_{xx} (we consider the strong field limit of the lowest Landau level, for which there is a particle-hole symmetry at $\nu = 1$). For simplicity, we represent the total concentration of mobile charges (be they positive or negative) in terms of a single filling fraction $\delta\nu$, such that the number density of charges is $\delta\nu\bar{\rho}$, where $\bar{\rho} \equiv eB/h$ is the density of states in a Landau level. We shall also refer to these mobile charges as "quasi-particles" – independent of their internal spin structure – except when it is important to make a distinction.

What is the motion of a quasi-particle under an applied electrical field? At zero temperature, the quasi-particle moves perpendicular to the applied electric field and contributes to the Hall conductivity, but not to the longitudinal conductivity. This may be appreciated on various grounds. Firstly, translation invariance allows the electric field to be removed by a Lorentz transformation to a reference frame moving perpendicular to the electric field. In this frame the quasi-particle will be stationary. Its motion in the lab frame is, therefore, perpendicular to the electric field. A second way to appreciate this motion, and one that will prove useful in understanding the processes that we consider here, is to note that the kinetic energy of a particle in the lowest Landau level is quenched. In moving parallel to the electrical field, the quasi-particle would absorb energy from this field. Since there are no states in the lowest Landau level that have different energy, the quasi-particle is constrained energetically to move on contours of equipotential.

At a finite temperature, the quasi-particle moves in a heat bath of spin-waves. The heat bath defines a rest frame; translational invariance is broken and the longitudinal conductivity is not zero. [We assume that the spin-waves are in thermal equilibrium in the rest frame. Ultimately this is due to equilibration of spin-waves with the lattice by interaction with phonons.] The scattering of spin-waves induces a diffusive motion of the quasi-particle. It provides a mechanism by which the quasi-particle may lose energy to spin-waves and so move down an applied potential gradient.

A. Simplified derivation

We first provide a simple calculation of the diffusion constant, which illustrates how spin-wave scattering leads to quasi-particle diffusion. We treat the spin-waves as free particles in the absence of the quasi-particle, and study the scattering of these modes from the perturbing quasi-particle. This scattering induces motions of the quasi-particle. Were we to treat the spin-waves as the linearised excitations in the presence of the quasi-particle, there would be no scattering and hence no motions of the quasi-particle. Although we offer here no formal derivation of the approach we use, we expect it to capture the nonlinearities that arise from the fact that the displacement of the particle cannot be treated as a small fluctuation, and thus the spin-waves cannot be viewed as decoupled quadratic fluctuations. This is confirmed by the agreement of the results of this approach with the systematic derivations of sections II B and II C.

The first step in our calculation is to write down the rate of a process where a single spin-wave scatters off a quasi-particle:

$$\Gamma_{\delta R,i,f} = \frac{2\pi}{\hbar}\left|\langle f|\Delta\hat{\mathcal{H}}|i\rangle\right|^2 \delta\left(E_f^0 - E_i^0\right) \times\delta\left[\delta\boldsymbol{R} - \ell^2\hat{\boldsymbol{z}}\times(\boldsymbol{k}^f - \boldsymbol{k}^i)/\hbar\right] \quad (1)$$

In this expression, $|i\rangle, |f\rangle$ are initial and final states of the (unperturbed) spin-wave system with total momenta $\boldsymbol{k}^{i,f}$ and total energies $E_{i,f}^0$, which are coupled by the perturbation $\Delta\hat{\mathcal{H}}$ arising from the presence of a quasi-particle, whose position is displaced by $\delta\boldsymbol{R}$ under the scattering. $\ell \equiv \sqrt{\hbar/eB}$ is the magnetic length. The second delta function embodies the important physics of the lowest Landau level. The two components of position in the lowest Landau level are conjugate to one another. A change in the momentum of the quasi-particle, $\delta\boldsymbol{k}$, is equivalent to a change $\ell^2(\hat{\boldsymbol{z}}\times\delta\boldsymbol{k})/\hbar$ in its position. The rate of diffusion of the particle arising from this scattering process is obtained from Eq.(1) by averaging $|\delta\boldsymbol{R}|^2$ over a thermal distribution of initial spin-wave states. This determines the rate of increase of the mean square displacement of the quasi-particle, and hence the diffusion constant $D \equiv \frac{1}{4}d\langle|\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2\rangle/dt$:

$$D = \frac{\pi\ell^4}{2\hbar^3}\sum_{if}\rho_i^0\left|\langle f|\Delta\hat{\mathcal{H}}|i\rangle\right|^2|\boldsymbol{k}^i - \boldsymbol{k}^f|^2\delta\left(E_f^0 - E_i^0\right), \quad (2)$$

where $\rho_i^0 = e^{-E_i^0/k_BT}/Z$. The contribution to conductivity from a dilute (non-degenerate) gas of such quasi-particles may be deduced from Eq.(2) using the Einstein relation;

$$\sigma_{xx} = e^2 D\frac{dn}{d\mu} = \frac{e^2 D\delta\nu\bar{\rho}}{k_BT}. \quad (3)$$

$\delta\nu\bar{\rho}$ is the average number density of quasi-particles. Eq.(2) and the resulting expression for the conductivity may be rewritten in terms of a force-force correlation function as follows:

$$\sigma_{xx} = \delta\nu\frac{e^2}{h} \times\lim_{\omega\to 0}\left.\frac{\mathcal{I}m\langle[\hat{\boldsymbol{k}},\Delta\hat{\mathcal{H}}](\tilde{\omega})\cdot[\hat{\boldsymbol{k}},\Delta\hat{\mathcal{H}}](-\tilde{\omega})\rangle}{4\pi\bar{\rho}\hbar^4\omega}\right|_{i\tilde{\omega}\to\omega+i\delta}, \quad (4)$$

where we have used the fluctuation dissipation relation to express our result in terms of a retarded correlation function. The diffusion and conductivity of the quasi-particle are related to the forces exerted upon the quasi-particle by spin-waves from the heat-bath. Before going on to calculate Eq.(4) in various experimental regimes, we first give a more rigorous derivation using the collective coordinate technique.

B. Collective Coordinates

The collective coordinate technique [12] provides a systematic way of obtaining an effective theory for the interaction of a Skyrmion with the heat bath of spin-waves. This method has been fruitfully applied to the study of polaron transport in Ref. [13]. We follow the methods of this paper quite closely. The starting point is the sigma-model effective action for the QHF [11];

$$\mathcal{S} = \int dt d\boldsymbol{r} \left[\frac{\bar{\rho}}{2} \boldsymbol{A}[\boldsymbol{n}] \cdot \partial_t \boldsymbol{n} - \frac{\rho_s}{2} (\nabla \boldsymbol{n})^2 + \bar{\rho} g n_z \right] - \int dt \, V[\rho]. \tag{5}$$

$\boldsymbol{n}$ is an $O(3)$ vector field giving a coherent-state representation of the spin. The first term in this action is the Berry phase term describing the spin dynamics. It embodies the commutation relations of the spin operators. The second and third terms describe the exchange and Zeeman energy of the QHF (ρ_s is the spin stiffness and g the Zeeman energy per electron). For simplicity, we choose to study only the case of an Heisenberg ferromagnet in this section; the approach we use can easily be adapted for the easy-plane case.

It is the final term in Eq.(5) that distinguishes the QHF from a conventional ferromagnet; it describes the identity between charge and magnetic vorticity [11] discussed in the introduction. The charge density associated with a spin distortion, for $\nu = 1$, is given by

$$\rho = \frac{e}{8\pi} \epsilon_{ij} \boldsymbol{n} \cdot \partial_i \boldsymbol{n} \times \partial_j \boldsymbol{n}. \tag{6}$$

The final term in Eq.(5) indicates the Coulomb self interaction of the spin field. The foremost consequence of Eq.(6) is that magnetic vortices or Skyrmions in the QHF carry unit charge. The static Skyrmion distribution, $\boldsymbol{n}_0(\boldsymbol{r})$, is found by minimising the energy (minus the time independent part of Eq.(5)) in the single Skyrmion sector. This analysis was carried out in Ref. [11]. We do not require any details here except for the existence of $\boldsymbol{n}_0$. The next step is to expand in small fluctuations about the Skyrmion groundstate. Care must be taken with this expansion. If it is carried out for a static Skyrmion, some of the normal modes are found to have zero energy. They correspond to translation and rotation of the Skyrmion spin distribution. We use the collective coordinate technique in order to handle these zero modes. The basic idea is to exclude the zero modes from the spin-wave field and to elevate the Skyrmion position – its collective coordinates – to be a dynamical variable, $\boldsymbol{R}(t)$. The spin-wave expansion about the moving Skyrmion spin distribution is given by

$$\boldsymbol{n}(\boldsymbol{r}, t) = \boldsymbol{n}_0(\boldsymbol{r} - \boldsymbol{R}(t)) \sqrt{1 - |\boldsymbol{l}(\boldsymbol{r} - \boldsymbol{R}(t), t)|^2} + \boldsymbol{l}(\boldsymbol{r} - \boldsymbol{R}(t), t). \tag{7}$$

$\boldsymbol{l}(\boldsymbol{r} - \boldsymbol{R}(t), t)$ is the spin-wave field in the presence of the Skyrmion at the point $\boldsymbol{R}(t)$. It may be expanded in terms of spin-wave eigenfunctions as follows:

$$l(\boldsymbol{r} - \boldsymbol{R}(t), t) = \sum_{n=1}^{\infty} q_n(t) \eta_n(\boldsymbol{r} - \boldsymbol{R}(t), t), \tag{8}$$

where $\eta_n(\boldsymbol{r} - \boldsymbol{R}(t), t)$ is a spin-wave eigenstate in the presence of the Skyrmion and $q_n(t)$ is a time dependent occupation of this mode. Since the Skyrmion spin distribution changes in time as the Skyrmion moves, the eigenmodes themselves change. It is this additional time dependence that induces transitions between the spin-wave eigenmodes; although $\langle n(t) | m(t) \rangle = 0$ for $m \neq n$, $\langle n(t) | m(t + \delta t) \rangle \neq 0$ allowing transitions between them. These effects are encoded in the Berry phase term of Eq.(5). Upon substituting the collective coordinate expansion, Eq.(8), into the first term of Eq.(5), we find

$$\int dt d\boldsymbol{r} \frac{\bar{\rho}}{2} \boldsymbol{A}[\boldsymbol{n}] \cdot \partial_t \boldsymbol{n} = \int dt \, \hbar \pi \bar{\rho} \hat{\boldsymbol{z}} \cdot \boldsymbol{R} \times \dot{\boldsymbol{R}} + \int dt d\boldsymbol{r} \left[\frac{\hbar \bar{\rho}}{4} \dot{\boldsymbol{R}} \cdot i \bar{l} \nabla l + i \frac{\hbar \bar{\rho}}{4} \bar{l} \partial_t l \right] \tag{9}$$

We have adopted the complex notation $l = l_1 + i l_2$, $\bar{l} = l_1 - i l_2$. The first term describes the bare dynamics of the Skyrmion [14]. It is the usual action for a particle with Magnus force dynamics. The third term describes the spin-wave dynamics. The second term describes the time dependence of the spin-wave field arising from the motion of the Skyrmion ($d_t l = \partial_t l - \dot{\boldsymbol{R}} \cdot \nabla l$). It is this term that gives rise to the non-orthogonality of spin-wave eigenstates at different times and permits scattering between them. Notice that it consists of the coupling of the Skyrmion velocity to the total spin-wave momentum;

$$k_j = i \frac{\hbar \bar{\rho}}{4} \int d\boldsymbol{r} \, \bar{l} \nabla_j l. \tag{10}$$

The remaining terms in the joint spin-wave/Skyrmion effective action are obtained by substituting Eq.(8) into the time independent part of Eq.(5). The resulting expressions include terms describing the exchange and Zeeman energies of the spin-wave distortion and terms describing the interaction of the spin-waves with the Skyrmion. This interaction may be divided into two parts; exchange interactions and Coulomb interactions. The exchange interactions are local in space, whereas the Coulomb interactions are spatially non-local, due to the long range of the Coulomb potential. We neglect local interactions in our treatment of the interaction of quasi-particles with spin-waves. This approximation is justified provided the typical spin-wave wavelength is large compared to the size of the Skyrmion, in which limit the exchange interaction is suppressed relative to the non-local Coulomb interaction. Retaining only Coulomb coupling, and adding the spin-wave energy to Eq.(9), we find the following joint spin-wave/Skyrmion effective action:

$$\begin{aligned}
\mathcal{S}[\boldsymbol{R}, l, \bar{l}] &= \hbar\pi\bar{\rho}\int dt\hat{\boldsymbol{z}}\cdot\boldsymbol{R}\times\dot{\boldsymbol{R}} + \frac{\hbar\bar{\rho}}{4}\int dtd\boldsymbol{r}\dot{\boldsymbol{R}}\cdot i\bar{l}\nabla l \\
&+\frac{1}{2}\int dtd\boldsymbol{r}\bar{l}\left[i\frac{\hbar\bar{\rho}}{2}\partial_t - \rho_s\nabla^2 - \bar{\rho}g\right] l \\
&-\int dtd\boldsymbol{r}d\boldsymbol{r}' V(\boldsymbol{r}-\boldsymbol{r}')\rho_{\boldsymbol{n}_0}(\boldsymbol{r}-\boldsymbol{R}(t))\rho_{\boldsymbol{l}}(\boldsymbol{r}'-\boldsymbol{R}(t),t),
\end{aligned} \quad (11)$$

where $\rho_{\boldsymbol{n}_0}$ is the charge density of the Skyrmion and $\rho_{\boldsymbol{l}}$ is the charge density associated with spin-waves. $\rho_{\boldsymbol{l}}$ is given by

$$\rho_{\boldsymbol{l}} = -i\frac{e}{8\pi}\epsilon_{ij}\partial_i\bar{l}\partial_j l. \quad (12)$$

Before proceeding to calculate the Skyrmion dynamics from Eq.(11), let us make a few comments about the comparison of the Skyrmion/spin-wave problem with the polaron/phonon problem [13]. The dynamics of spins is entirely determined by their commutation relations and, importantly, there is no kinetic term in their Hamiltonian. This results in different dynamics for the Skyrmion and polaron. In the former case, one finds Magnus-force dynamics describing the motion of a Skyrmion perpendicular to an applied force. In the latter case, however, the dynamics have a conventional ballistic form. The second consequence is the absence of (multiple) spin-wave Cherenkov processes. Such terms are found in the phonon/polaron case through the collective coordinate expansion of the kinetic terms in the Hamiltonian. They are forbidden by energy conservation in the Skyrmion case, unless one allows for internal modes of the Skyrmion [15] (which we neglect here, under the assumption that the drift velocity of the Skyrmion is less than the critical velocity derived in Ref. [15]). These facts were missed in a previous analysis of the Skyrmion problem by Villares Ferrer and Caldeira [16]. Despite these key differences, when Cherenkov processes are neglected, we find that the coherent state representation of the Skyrmion and polaron problems are very similar and that the damping and diffusion of Skyrmions is very similar to that of polarons.

C. Feynman-Vernon Influence Functional

Our goal in this subsection is to use the Skyrmion/spin-wave effective action, Eq.(11), to study the Skyrmion dynamics in the presence of the heat bath of spin-waves. In order to carry out this analysis one may use the Feynman-Vernon influence functional approach [17,18]. The application of this approach to the present problem is very similar to its application in the polaron case [13]. The calculation proceeds through a number of steps, but the basic idea is the following: the reduced density matrix for the Skyrmion is found by tracing the total system density matrix over the spin-wave degrees of freedom, *i.e.* by 'integrating out' the spin-waves. The time evolution of this density matrix may be expressed in terms of a superpropagator, which is in turn expressed in terms of influence functionals that encode the effect of the spin-waves on the Skyrmion propagation. The result of such a calculation is to express the damping and diffusion of Skyrmions in terms of momentum-momentum correlation functions of the spin-wave heat bath in the presence of the Skyrmion potential. Full details of such a calculation may be found in Ref. [13]. A brief summary of an equivalent calculation using Keldysh techniques [19] is given in AppendixA. The main approximation in carrying out this procedure is that the Skyrmion is displaced by a distance less than the spinwave wavelength at each scattering process.

The above analysis results in the following Langevin equation describing the motion of the Skyrmion in the presence of the spin-wave heat-bath:

$$\begin{aligned}
2\pi\hbar\bar{\rho}\hat{\boldsymbol{z}}\times\dot{\boldsymbol{R}} + 2\bar{\gamma}\dot{\boldsymbol{R}} - e\boldsymbol{E} &= \boldsymbol{\zeta}(t) \\
\langle\zeta_i(t)\zeta_j(t')\rangle &= 2\overline{D}\delta_{ij}\delta(t-t').
\end{aligned} \quad (13)$$

The first term in this equation describes the transverse motion of the Skyrmion in response to an applied force. The second term describes dissipation of the Skyrmion motion due to the scattering of spin-waves. $\boldsymbol{E}$ is an applied electric field. The term on the right hand side describes diffusive motion of the Skyrmion due to the scattering of spin-waves. The dissipation and diffusion constants are related to the spin-wave momentum-momentum correlator *via*

$$\begin{aligned}
\bar{\gamma} &= \lim_{\omega\to 0}\omega\mathcal{I}m\Gamma(\omega) \\
\overline{D} &= 2\overline{\gamma}T \\
\Gamma(t) &= -i\frac{\theta(t)}{4\hbar}\langle[\hat{k}_i(t),\hat{k}_i(0)]\rangle.
\end{aligned} \quad (14)$$

These correlation functions account for both thermal and zero-point fluctuations of the background spin-field. Although zero-point fluctuations make important contributions to the renormalization of the Skyrmion polarization and energy [20,21], our results show that they do not contribute significantly to dissipation in either the single layer Heisenberg QHF or the ordered phase of the bilayer, easy-plane QHF.

The contribution of a dilute gas of Skyrmions with number density $\delta\nu\bar{\rho}$ to the longitudinal conductivity may be deduced from Eq.(13). In the limit of $2\bar{\gamma}/h\bar{\rho} \ll 1$ it is given by

$$\sigma_{xx} = \delta\nu \frac{e^2}{h} \frac{\bar{\gamma}}{\pi\hbar\bar{\rho}}. \tag{15}$$

This result may also be derived from the Einstein relation, Eq.(3). The appropriate diffusion constant for use in Eq.(3) is $D = \overline{D}/(2\pi\hbar\bar{\rho})^2$. This can be seen by using Eq.(13) to calculate $\langle|\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2\rangle = 4Dt = 4\overline{D}t/(2\pi\hbar\bar{\rho})^2$.

Eq.(15) represents the Skyrmion conductivity in terms of the spin-wave momentum-momentum correlation function in the presence of a static Skyrmion, Eq.(14). This is our primary result. We have used the very general Feynman-Vernon/Keldysh techniques in order to derive this result, however, it may also be obtained quite straightforwardly from the effective action Eq.(11) using the Kubo formula [22]. After expanding the momentum-momentum correlation function over spin-wave states in the presence of the Skyrmion and then expressing these states in terms of free spin-wave states using the lowest order perturbation theory, Eq.(15) may be expressed in terms of an average over free spin-waves:

$$\sigma_{xx} = \delta\nu \frac{e^2}{h} \times \lim_{\omega\to 0} \left. \frac{\mathcal{I}m\langle[\hat{\boldsymbol{k}}, \Delta\hat{\mathcal{H}}_{\boldsymbol{R}}](\tilde{\omega}) \cdot [\hat{\boldsymbol{k}}, \Delta\hat{\mathcal{H}}_{\boldsymbol{R}}](-\tilde{\omega})\rangle}{4\pi\bar{\rho}\hbar^4\omega} \right|_{i\tilde{\omega}\to\omega+i\delta}. \tag{16}$$

This recovers the result obtained in II A by simple Fermi's Golden rule arguments. There were two key approximations in the derivation of Eq.(16). The first is the requirement that the recoil of the Skyrmion after any particular scattering event is much less than the wavelength of the spin-waves involved. Secondly, we have expanded perturbatively in the interaction between spin-waves and the Skyrmion. This is equivalent to the Born approximation for the scattering of spin-waves. Notice that we have made no assumptions about the nature of the interaction between the spin-waves and Skyrmion in our derivation.

D. Low temperature conductivity.

We are now in a position to calculate the contribution to conductivity from spin-wave scattering. First we consider the *Heisenberg* case, for which the long-wavelength spin-wave dispersion is

$$E^0(k) = g + 2\rho_s k^2/\bar{\rho}. \tag{17}$$

At the lowest temperatures, the dominant interaction between spin-waves and charged excitations is through the Coulomb interaction. The perturbation in the spin-wave Hamiltonian due to the presence of a Skyrmion at point $\boldsymbol{R}$ is given by

$$\Delta\hat{\mathcal{H}}_{\boldsymbol{R}} = \int d\boldsymbol{r} V(\boldsymbol{R} - \boldsymbol{r}) \rho_{\boldsymbol{l}}(\boldsymbol{r}, t), \tag{18}$$

where the Skyrmion has been treated as a point charge on the lengthscale of the scattered spin-waves. The commutator of this Hamiltonian with the spin-wave momentum operator is given by $[\hat{k}_i, \Delta\hat{\mathcal{H}}_{\boldsymbol{R}}] = i\hbar\partial_{\boldsymbol{R}}\Delta\hat{\mathcal{H}}_{\boldsymbol{R}}$, where we have used translational invariance to express in terms of the derivative with respect to the Skyrmion co-ordinate. Substituting this into Eq.(16) and using the Coulomb interaction potential, $V(\boldsymbol{q}) = e^2/2\epsilon|\boldsymbol{q}|$, the conductivity of a dilute Skyrmion gas may be expressed in terms of a correlation function of the free spin-wave topological density;

$$\sigma_{xx} = \delta\nu \frac{e^2}{h} \frac{e^2}{16\pi\epsilon^2\hbar^2\bar{\rho}} \times \lim_{\omega\to 0} \int \frac{d\boldsymbol{q}}{(2\pi)^2} \left. \frac{\mathcal{I}m\langle\rho_{\boldsymbol{l}}(\boldsymbol{q}, \tilde{\omega})\rho_{\boldsymbol{l}}(-\boldsymbol{q}, -\tilde{\omega})\rangle}{\hbar\omega} \right|_{i\tilde{\omega}+i\delta}. \tag{19}$$

Calculating with the free spin-wave part of the effective action, Eq.(11), and the spin-wave charge density, Eq.(12), we find

$$\sigma_{xx} = \delta\nu \frac{e^2}{h} \frac{1}{6 \times 2^{11}\pi} \frac{E_C^2 (k_B T)^2}{\rho_s^4}. \tag{20}$$

at temperatures above the Zeeman gap and exponential suppression with a factor $e^{-2g/k_B T}$ at temperatures below the gap. $E_C = e^2/4\pi\epsilon\ell$ is the characteristic Coulomb energy.

The case of the *easy-plane* pseudo-spin ferromagnet is a little more subtle. The effective action in this case is obtained by replacing the Zeeman term, $\bar{\rho}gn_z$, in Eq.(5) by an easy-plane anisotropy or capacitance energy, γn_z^2. The pseudo-spin lies in the plane in the groundstate and the topological defects are vortices of the in-plane pseudo-spin orientation. The cores of these vortices are non-singular due to the pseudo-spin rising up or below the xy-plane. Depending upon the vorticity and orientation in the core, these vortices may carry $\pm 1/2$ charge in addition to their $\pm$ vorticity. These charged vortices are known as merons [23,1]. The exchange interaction energy between vortices varies logarithmically with their separation. At low temperatures this binds vortices into pairs of opposite vorticity. The charge carriers at low temperature are, therefore, bound pairs of merons with charge ± 1 [1]. These bound pairs behave as Skyrmions in the Heisenberg case. On lengthscales large compared with the meron separation, the meron pair may be viewed as a point charge. At low temperatures, therefore, we may use the interaction, Eq.(18), to model the scattering of spin-waves and Eq.(19) to calculate the conductivity.

The calculation is a little different to that of the Heisenberg spin-waves. With an easy-plane anisotropy, γn_z^2, the effective action Eq.(5) has a spin-wave dispersion given by [24]

$$E^0(k) = 2\sqrt{\rho_s \boldsymbol{k}^2(\rho_s \boldsymbol{k}^2 + 2\gamma)}/\bar{\rho}. \tag{21}$$

This dispersion is linear at low momentum crossing over to a quadratic behaviour at a momentum $\boldsymbol{k}_c = \sqrt{2\gamma/\rho_s}$ An effective theory for the linearly dispersing modes may be obtained in terms of the in-plane orientation of the pseudo-spin field by integrating out n_z from the effective action, Eq.(5). The result is a quantum XY-model;

$$\mathcal{S} = \int dt d\boldsymbol{r} \frac{\rho_s}{2}\left[\dot{\phi}^2 v^{-2} - (\nabla\phi)^2\right], \tag{22}$$

where $v = \sqrt{8\gamma\rho_s}/(\hbar\bar{\rho})$ is the velocity of the linearly dispersing modes. The spin-wave charge density may also be expressed in terms of ϕ. It is given by

$$\rho_\phi = i\frac{\hbar\bar{\rho}}{16\pi\gamma}\epsilon_{ij}\partial_i\phi\partial_j\dot{\phi}. \tag{23}$$

The interaction between the pseudo-spinwaves and the meron pair is given by Eq.(18), replacing $\rho_{\boldsymbol{l}}$ with ρ_ϕ. The conductivity is given by Eq.(19) with a similar replacement. Calculating the conductivity at low temperatures using Eqs.(22) and (23), we find

$$\sigma_{xx} = \delta\nu\frac{e^2}{h}\frac{\pi^3}{84}\frac{E_C^2(k_BT)^6}{(\hbar v)^8\bar{\rho}^4}. \tag{24}$$

This is suppressed by a factor of $(\pi\bar{\rho}k_BT/\gamma)^4/28$ relative to the Heisenberg ferromagnet. Turning on the anisotropy has stiffened up the low momentum spin-waves so that fewer are thermally excited at low temperatures leading to a corresponding reduction in quasi-particle scattering and conductivity. At temperatures above $k_BT \approx \gamma/\bar{\rho}$, significant numbers of thermally excited spin-waves are in the quadratic part of the dispersion, Eq.(21). These spinwaves also have sufficiently short wavelength to probe the structure of the meron pair. The conductivity is expected to cross over to the form given by Eq.(20), with a modified pre-factor. However, the temperature $k_BT \approx \gamma/\bar{\rho}$ is typically rather large and the system is likely to undergo a Kosterlitz-Thouless transition before this cross-over becomes apparent. Note that for a bilayer QHF at $\nu = 1$, at temperatures larger than both the tunnelling gap and the Zeeman energy there will be scattering of *both* easy-plane pseudo-spin waves, and Heisenberg "real" spin-waves. The present discussion indicates that in the low temperature regime, the scattering of the "real" spin-waves will give the dominant contribution to quasiparticle diffusion.

E. Discussion

The diffusion constants we calculate are strongly increasing functions of temperature. However, even for reasonably high temperatures, within the range of applicability of the spin-wave expansion, the diffusion constants (20,24) remain rather small. As an illustration, we consider the Heisenberg case, with typical parameters of $B = 4$T and a temperature $T = 3$K that is comparable to ρ_s (for a narrow 2DEG with $\epsilon_r = 12.5$). The above formula results in a longitudinal conductivity of only $0.06\delta\nu(e^2/h)$ for a concentration of $\delta\nu$ quasi-particles (the conductivity for easy-plane anisotropy is always smaller than that for the Heisenberg magnet). It may be difficult to observe this intrinsic diffusion owing to the effects of disorder.

Disorder can have a dramatic effect on quasi-particle diffusion, even if the rms disorder potential ϕ_{rms} is much smaller than temperature, $e\phi_{rms} \ll k_BT$. The additional $\boldsymbol{E} \times \boldsymbol{B}$ drift that the disordered potential introduces to the classical dynamics of the quasi-particle leads to [25] an effective diffusion constant D^* that is enhanced over the intrinsic diffusion D. The extent of this enhancement depends on the ratio $P = \phi/(BD)$. The intrinsic diffusion dominates ($D^* \simeq D$) provided $P \lesssim 1$. For the above parameters, this sets an upper limit of $e\phi_{rms}/k_B \lesssim 0.17K$ on the disorder strength. For stronger disorder, $P \gg 1$, the effective diffusion constant is enhanced, $D^* \sim DP^{10/13}$. [25] At temperatures much less than the disorder strength, $k_BT \ll e\phi$, the transport mechanism of quasi-particles will involve thermal activation or variable range hopping [26].

Even in the absence of disorder, we may ask what is the conductivity at high temperatures when the density of thermally generated charges is large and the assumption of independent quasi-particles used above breaks down. At a temperature $k_BT_{KT} = \pi\rho_s^R/2$, the easy-plane QHF is expected to undergo a Kosterlitz-Thouless transition where vortices unbind due to thermal fluctuations (ρ_s^R is the thermally renormalised spin stiffness). This has a profound effect upon the nature of transport. Above the KT transition, the charge is carried by merons. In addition to carrying charge in units of $\pm 1/2$, merons have a vortex configuration of in-plane spin. The interaction between unpaired merons and pseudo-spin-waves is, therefore, dominated by exchange. At high temperatures, above the KT-transition, we expect exchange scattering of thermally generated merons to lead to a conductance near to $\sigma = e^2/4h$. The reason for this is the following: the quantum XY-model displays a zero temperature phase transition at $v = 2\pi\rho_s/\Lambda$ (Λ is the ultra-violet momentum cut-off.), where zero point fluctuations destroy long-range order. One may use duality [27] at this point between the two zero temperature phases to argue that the vortex number conductivity takes a universal value

$\tilde{\sigma} = 1/h$ at the transition point and at temperatures above this critical point, in the quantum-critical regime. Since each meron carries a charge $e/2$, we expect a charge conductivity of $\sigma \approx e^2/4h$ provided that the Coulomb interaction may be ignored. [Of course, the conductivity $\tilde{\sigma}$ gives the response to a field that couples to vorticity and σ the response to a field that couples to charge. The charge and vorticity of a meron are independent; a meron of a particular vorticity may carry either charge. However, the conductivity in both cases is proportional to the density of merons and inversely proportional to the resistance to motion of an individual meron. We, therefore, anticipate the simple relationship $\sigma = e^2\tilde{\sigma}/4$.] This supports the suggestion in Ref. [10,28] that a rapid increase in longitudinal conductivity occurs at the KT-transition.

A similar analysis of the high temperature conductivity of the Heisenberg QHF is not possible. The Heisenberg QHF does not undergo a zero temperature phase transition as in the easy-plane case. The duality arguments that lead to the prediction of a universal conductivity in the case of the easy-plane QHF cannot be used. Recall, however, that in the case of low temperature response, the easy-plane QHF always has a lower conductivity than the corresponding Heisenberg magnet. This is due to the stiffening of the low-energy spinwaves in the easy-plane QHF leading to a reduction in their thermal population. Features in the behaviour of the easy-plane QHF due to pseudo-spin fluctuations are expected to be stronger in the Heisenberg QHF. We expect, therefore, that although it does not display a KT-transition, the Heisenberg QHF should show a crossover to dramatically enhanced (and possibly universal) conductivity at temperatures around $k_BT = \pi\rho_s^R/2$. Some circumstantial evidence for this is found in numerical simulations of the classical 2-dimensional O(3)-sigma model, where the topological compressibility is found to rise rapidly at around $k_BT \sim \rho_s$ [29]. These considerations may explain the longstanding puzzle in the IQHE that, although activated transport measurements at low temperatures indicate a large gap ($\approx 4\pi\rho_s$), one must go to much lower temperatures ($\approx \pi\rho_s/2?$) than the measured gap in order to see a well formed QH state and accompanying minimum in longitudinal conductivity [9].

III. BILAYER TUNNELLING: SPIN-WAVE LIFETIME

Our discussion has focused upon the transport properties of the QHF. Of late, however, much of the focus in the study of bilayer pseudo-spin QHFs has been on the tunnelling conductance between layers. This conductivity shows a dramatic enhancement at zero bias [10,28]. This is thought to be a direct consequence of interlayer coherence and the existence of the pseudo-spin-wave Goldstone mode [10,28,30–33]. An outstanding problem is to understand the height and width of the zero bias peak in the data of Ref. [10,28]. Within a perturbative treatment [31–33], interlayer tunnelling probes the spectral function of the pseudo-spin-waves, such that the height and width of the zero-bias peak are set at low temperature by the spin-wave lifetime in the limit of zero momentum. In Refs. [31,32] it has been suggested that this lifetime may be due to a finite density of merons. We can calculate this lifetime within our model of spin-wave scattering off dilute isolated quasi-particles (meron pairs) which feel no disorder, by taking Eq.(1) and integrating over final spin-wave states and particle displacements. The resulting scattering rate is [34]

$$\Gamma_k = \delta\nu\pi \frac{E_C^2 k^3 \ell^4}{\hbar^2 v} \qquad (25)$$

The scattering rate vanishes in the limit of small momentum. This is because the zero momentum pseudo-spin-wave is a Goldstone mode both in the free system and in the presence of a finite density of quasi-particles. This scattering does not appear to be a suitable mechanism by which the zero-bias peak may be broadened.

At high temperatures above the KT-transition, the interaction between pseudo-spinwaves and thermally generated merons is dominated by exchange. The resulting $\phi\phi$-correlation function may be deduced on phenomenological grounds. A finite correlation length develops in the QC regime due to the proliferation of unbound vortices. Pseudo-spinwaves are strongly scattered by these unbound vortices and their response is over-damped as a consequence. The only energy scale in the QC regime is provided by the temperature. This sets both the correlation length, $\xi(T) \propto T^{-1}$, and the damping rate. The resulting pseudo-spin correlator takes the form [35]

$$\begin{aligned} &\langle \phi(\boldsymbol{k},\tilde{\omega})\phi(-\boldsymbol{k},-\tilde{\omega})\rangle|_{i\tilde{\omega}\to\omega+i\delta} \\ &= \frac{1}{\rho_s}\left(\boldsymbol{k}^2 + \xi(T)^{-2} - v^{-2}\omega^2 - i\alpha v^{-2}\omega T\right)^{-1} \end{aligned} \qquad (26)$$

for $\hbar\omega \ll kT$, where α is a number of order 1. Calculating the tunnelling current as in Refs. [31–33] using the pseudo-spin-wave response function, Eq.(26), gives

$$I \propto \frac{2}{e^{eV/T} - 1} VT\xi(T)^4 \mathrm{sign}(V). \qquad (27)$$

The Josephson singularity in differential conductance, dI/dV, is not suppressed. The physics that leads to Eq.(26) is the scattering of pseudo-spin-waves from thermally excited merons. As in the case of Coulomb scattering, this does not lead to decay of the zero-momentum pseudo-spinwave and so does not introduce a finite width to the zero-bias peak.

IV. SUMMARY

We have studied the scattering of spinwaves from charged excitations in the $\nu = 1$ Heisenberg and easy-plane QHFs. This scattering leads to a diffusive motion of the charged quasiparticles, described by a quantum Langevin equation. The resulting contribution to low temperature conductivity follows characteristic power-laws for a given density of charge carriers. This contribution to conductivity is small.

We have argued on the basis of duality that the conductivity of the easy-plane QHF at temperatures above the KT-transition crosses over to a universal value. Such arguments do not apply in the Heisenberg case. However, from a comparison of the low temperature behaviour of the easy-plane and Heisenberg QHF we tentatively suggest a similar crossover at high temperatures for the Heisenberg QHF.

Finally, we have considered the dissipation of pseudo-spinwaves due to scattering from merons in the easy-plane QHF. This scattering gives rise to a pseudo-spinwave relaxation rate that goes to zero at zero pseudo-spinwave momentum. It cannot, therefore, give rise to a finite width of the zero bias interlayer tunnelling peak.

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APPENDIX A: DERIVATION OF THE LANGEVIN EQUATION

In Ref. [13], Castro Neto and Caldeira used collective coordinates and the Feynman-Vernon influence functional technique to derive the reduced density matrix describing the motion of a polaron in the presence of a heat bath of phonons. A Langevin equation for the polaron motion may be deduced from this density matrix. Precisely the same method may be applied to determine the reduced density matrix describing the motion of a Skyrmion in the presence of a heat bath of spinwaves. In this appendix, we use the alternative, but completely equivalent technique of Keldysh field theory [19] to determine the Langevin equation describing Skyrmion motion.

The Skyrmion position and the spinwave heat bath are described by $\boldsymbol{x}(\boldsymbol{y})$ and $l_+(l_-)$ on the forward (backwards) part of the Keldysh time contour [19]. The joint spinwave/Skyrmion action on the forward part of the contour is given by Eq.(11) with $\boldsymbol{R} \to \boldsymbol{x}$;

$\mathcal{S}[\boldsymbol{x}, l_+, \bar{l}_+]$

$$= \int dt \left[\hbar\pi\bar{\rho}\hat{\boldsymbol{z}}\cdot\boldsymbol{x}\times\dot{\boldsymbol{x}} - V(\boldsymbol{x}) + \boldsymbol{k}_+\cdot\dot{\boldsymbol{x}}\right] + \mathcal{S}_{\boldsymbol{x}}[l_+,\bar{l}_+] \quad \text{(A1)}$$

where $\mathcal{S}_{\boldsymbol{x}}[l_+,\bar{l}_+]$ is the action for spinwaves in the presence of a static Skyrmion at point $\boldsymbol{x}$ and $\boldsymbol{k}_+ = i\hbar\bar{\rho}\int d\boldsymbol{x}\bar{l}_+\nabla l_+/4$ is the total spinwave momentum. A similar action, $\mathcal{S}[\boldsymbol{y},l_-,\bar{l}_-]$ describes the motion on the return part of the Keldysh contour.

The next step is to make a Keldysh rotation to classical and quantum components of the fields l_+, l_- and the coordinates $\boldsymbol{x}$, $\boldsymbol{y}$;

$$\begin{aligned} l_{cl/q} &= (l_+ \pm l_-)/2 \\ \boldsymbol{R}/\boldsymbol{r} &= (\boldsymbol{x}\pm\boldsymbol{y})/2. \end{aligned} \quad \text{(A2)}$$

The classical and quantum coordinates, $\boldsymbol{R}$ and $\boldsymbol{r}$ may be interpreted as the centre of mass of the Skyrmion wavefunction and its spatial extent, respectively. Integrating out the spinwave fluctuations and retaining terms to quadratic order in $\boldsymbol{R}$ and $\boldsymbol{r}$ [This is called the Born approximation in Ref. [13]. It requires that the displacement of the Skyrmion in a single scattering process is less than the wavelength of the spinwaves involved] we obtain the following effective Keldysh action for the Skyrmion coordinates:

$$\begin{aligned} \mathcal{S}[\boldsymbol{R},\boldsymbol{r}] &= \int dt \left[4\pi\hbar\boldsymbol{r}\cdot\hat{\boldsymbol{z}}\times\dot{\boldsymbol{R}} + 2\boldsymbol{r}\cdot\nabla V(\boldsymbol{R})\right] \\ &+ \int dt_1 dt_2 \dot{\boldsymbol{r}}(t_1)\cdot\dot{\boldsymbol{R}}(t_2)\langle\boldsymbol{k}(t_1)\cdot\boldsymbol{k}(t_2)\rangle^R_{\boldsymbol{R}} \\ &+ \int dt_1 dt_2 \dot{\boldsymbol{R}}(t_1)\cdot\dot{\boldsymbol{r}}(t_2)\langle\boldsymbol{k}(t_1)\cdot\boldsymbol{k}(t_2)\rangle^A_{\boldsymbol{R}} \\ &+ \int dt_1 dt_2 \dot{\boldsymbol{r}}(t_1)\cdot\dot{\boldsymbol{r}}(t_2)\langle\boldsymbol{k}(t_1)\cdot\boldsymbol{k}(t_2)\rangle^K_{\boldsymbol{R}} \end{aligned} \quad \text{(A3)}$$

where $\langle\boldsymbol{k}(t_1)\cdot\boldsymbol{k}(t_2)\rangle^{A,R,K}_{\boldsymbol{R}}$ are the advanced, retarded and Keldysh components of the spinwave momentum-momentum correlator in the presence of a Skyrmion at point $\boldsymbol{R}$;

$$\begin{aligned} \langle\boldsymbol{k}(t).\boldsymbol{k}(0)\rangle^R_{\boldsymbol{R}} &= -i\frac{\theta(t)}{\hbar}\langle[\hat{k}_i(t),\hat{k}_i(0)]\rangle = 2\Gamma(t) \\ \langle\boldsymbol{k}(t).\boldsymbol{k}(0)\rangle^A_{\boldsymbol{R}} &= i\frac{\theta(-t)}{\hbar}\langle[\hat{k}_i(t),\hat{k}_i(0)]\rangle = 2\Gamma(-t) \\ \langle\boldsymbol{k}(t).\boldsymbol{k}(0)\rangle^K_{\boldsymbol{R}} &= \frac{1}{\hbar}\left[\langle\hat{k}_i(t)\hat{k}_i(0)\rangle + \langle\hat{k}_i(0)\hat{k}_i(t)\rangle\right]. \end{aligned} \quad \text{(A4)}$$

The Keldysh component of the spinwave momentum-momentum correlator is related to the advanced and retarded components by a fluctuation dissipation relation. The potential term has been expanded for small $\boldsymbol{r}$; $V(\boldsymbol{R}+\boldsymbol{r}) - V(\boldsymbol{R}-\boldsymbol{r}) \approx 2\boldsymbol{r}\cdot\nabla V(\boldsymbol{R})$. This expansion will be justified later.

Eq.(A3) is analogous to the effective action obtained in Ref. [13] using the Feynman-Vernon approach. To put Eq.(A3) into the same form as that used in Ref. [13], the spinwave momentum-momentum correlation function must be expanded over a basis of spin-wave states in the presence of the Skyrmion, making the identification $g^i_{nm} = \bar{\rho}\int d\boldsymbol{r}\eta^*_n\nabla_i\eta_m = \langle n|\hat{p}_i|m\rangle$.

At this stage, it is convenient to rearrange some terms in Eq.(A3). The second and third terms are integrated by parts with respect to t_1 and t_2 respectively and the fourth term is integrated by parts with respect to both t_1 and t_2. The result is

$$\begin{aligned} \mathcal{S}[\boldsymbol{R},\boldsymbol{r}] &= \int dt \left[4\pi\hbar\boldsymbol{r}\cdot\hat{\boldsymbol{z}}\times\dot{\boldsymbol{R}} + 2\boldsymbol{r}\cdot V(\boldsymbol{R})\right] \\ &- 4\int dt_1 dt_2 \boldsymbol{r}(t_1)\cdot\dot{\boldsymbol{R}}(t_2)\gamma(t_1-t_2) \\ &+ i\int dt_1 dt_2 \boldsymbol{r}(t_1)\cdot\boldsymbol{r}(t_2)D(t_1-t_2) \end{aligned} \quad \text{(A5)}$$

with

$$\begin{aligned} 2\gamma(t) &= \frac{d}{dt}\left(\Gamma(t)-\Gamma(-t)\right) \\ D(\omega) &= \omega\coth\left(\frac{\omega}{2T}\right)\gamma(\omega). \end{aligned} \quad \text{(A6)}$$

Our next few manipulations use Eq.(A5) to derive a Langevin equation for the Skyrmion motion. A similar calculation is carried out for a simpler system in Ref. [18]. The diffusion and dissipation coefficients, $D(t)$ and $\gamma(t)$ are in principle non-local in time. This implies that the Skyrmion motion may display memory effects. We are interested in the Markovian limit where these memory effect are negligible. In this case $\gamma(t) = \bar{\gamma}\delta(t)$ and $D(t) = \overline{D}\delta(t)$. Making this approximation in Eq.(A5) we find

$$\begin{aligned} \mathcal{S}[\boldsymbol{R},\boldsymbol{r}] = \int dt 2\boldsymbol{r}\cdot&\left[2\pi\hbar\bar{\rho}\hat{\boldsymbol{z}}\times\dot{\boldsymbol{R}} + 2\bar{\gamma}\dot{\boldsymbol{R}} + \nabla V(\boldsymbol{R})\right] \\ &+ i\int dt 4\overline{D}\boldsymbol{r}^2 \end{aligned} \quad \text{(A7)}$$

with

$$\begin{aligned} \bar{\gamma} &= \lim_{\omega\to 0}\omega\mathcal{I}m\Gamma(\omega) \\ \overline{D} &= 2\bar{\gamma}T \end{aligned} \quad \text{(A8)}$$

The term in the action $i4\overline{D}\boldsymbol{r}^2$ restricts $\boldsymbol{r}$ to have small amplitude, $\langle\boldsymbol{r}^2\rangle = 1/8\overline{D}$. This justifies the gradient expansion of the potential term that was made previously in going from Eq.(A3) to Eq.(A5). The final step in the derivation of the Langevin equation is to integrate out the quantum/relative coordinate, $\boldsymbol{r}$. The result of this simple Gaussian integration is

$$\mathcal{S}[\boldsymbol{R}] = i\int_0^t dt' \frac{\left[2\pi\hbar\bar{\rho}\hat{\boldsymbol{z}}\times\dot{\boldsymbol{R}} + 2\bar{\gamma}\dot{\boldsymbol{R}} + \nabla V(\boldsymbol{R})\right]^2}{4\overline{D}}. \quad \text{(A9)}$$

The Skyrmion motion described by Eq.(A9) is equivalent to that described by the Langevin equation Eq.(13)