Universal non-linear conductivity near to an itinerant-electron ferromagnetic quantum critical point

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We study the conductivity in itinerant-electron systems near to a magnetic quantum critical point. We show that, for a class of geometries, the universal power-law dependence of resistivity upon temperature may be reflected in a universal non-linear conductivity; when a strong electric field is applied, the resulting current has a universal power-law dependence upon the applied electric field. For a system with thermal equilibrium current proportional to T^α and dynamical exponent z, we find a non-linear resistivity proportional to $E^{\frac{z-1}{z(1+\alpha)-1}}$.

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The notion of quantum criticality provides one of the few unifying theoretical principles of strongly correlated electrons [1, 2, 3]. It describes a range of phenomena in systems that are near to continuous, zero-temperature phase transitions; phase transitions that are driven by quantum rather than thermal fluctuations. In thermal equilibrium, quantum critical systems show characteristic spatial and temporal scaling in their response to external probes. For example, the conductivity of an itinerant-electron system near to a magnetic quantum phase transition has a power-law dependence upon temperature [4, 5, 6].

The behaviour of quantum critical systems out of thermal equilibrium has begun to attract growing attention over the past few years. Near to a quantum phase transition all of the intrinsic energy scales of a system, other than the Fermi energy, renormalize to zero. In thermal equilibrium, the only remaining energy scale is the temperature itself. Because of this, quantum critical systems are particularly susceptible to being driven out of equilibrium by external probes. In certain situations the universal temporal scaling near to the quantum critical point may reveal itself in universal features of the steady-state adopted out of equilibrium; the out-of-equilibrium state being largely determined by a system's dynamics.

Several recent works have addressed the question of whether universality persists when a quantum critical system is driven out of thermal equilibrium by the application of a strong electric field. In particular, Refs. [7] and [8, 9] considered two-dimensional superconductor-insulator transitions [10, 11], the former in the case where the quantum dynamics and phase transition were controlled by coupling to a Caldeira-Leggett bath and the latter in the case of intrinsic superconducting dynamics. These systems can indeed display universality out of equilibrium [12] in both their current response [7, 8] and their current noise statistics[9]. The triumph of Refs. [7] and [8] was to provide field-theoretical derivations of the scaling predicted by naïve dimensional analysis. Numerical studies of related one-dimensional systems produced similar results [13].

Whilst these works provide interesting proofs of principle — and indeed, may yet be compared with experiment — most quantum critical systems that are studied experimentally are of a rather different type. The critical modes at the superconductor-insulator transition are charged and couple directly to the electric field. A more typical situation has critical modes without a charge — often magnetic — which affect transport by scattering from electrons. Here we address the question of whether universal non-linear response in transport occurs in this more general setting.

We find that given certain conditions on size and geometry, quantum critical itinerant magnets show a universal non-linear current response. For a long, narrow sample, with an electric field applied along its length, we predict a universal non-linear scaling of current with field given by

$$j \propto E^{\frac{z-1}{z(1+\alpha)-1}}, \tag{1}$$

where the thermal equilibrium resistivity is proportional to T^α and z is the dynamical exponent. In the case of the Moriya-Hertz-Millis [4, 5, 6] model of the critical ferromagnet, $\alpha = (d + z - 1)/z$. Provided that certain constraints upon the dimension of the system are satisfied, this result does not depend further upon the system dimensions. In the following, we will take some time to discuss this matter and compare our results to those of related works.

We hope that these results will provide an alternative experimental window upon quantum criticality. Despite its successes, the theory of itinerant-electron quantum criticality has some puzzling problems; although power law dependencies upon temperature are seen experimentally, there is often a discrepancy between the observed and predicted powers in transport. Non-linear response may help to resolve this issue by providing two consistency checks: whether the equilibrium exponents are consistent with the out-of-equilibrium exponents through Eq.(1); and whether the out-of-equilibrium exponent is consistent with the Moriya-Hertz-Millis theory.

Our paper is outlined as follows: we begin in Section 1 with a general description of our scheme, paying particular attention to matters of geometry and heat flows within the system. This will enable a heuristic derivation of our main results and a detailed comparison with the complementary work of Mitra *et al.* [15]. In Section 2, we will begin with a survey of the Boltzmann treatment of the linear response of the itinerant-electron quantum critical system in thermal equilibrium. This will allow us to introduce some notation and familiarize the reader with its application in this context. We follow this in Section 3 by applying the Boltzmann transport formalism to the out of equilibrium system. This section contains a formal derivation of our main results. Finally, in Section 4, we turn to a discussion of the limitations of our analysis and of the prospects for seeing the effects that we predict in experiment.

GENERAL SCHEME

Geometry: We consider a long, narrow, quantum-critical, itinerant electron system with an electric field applied along its length. The system is longer than its transport length so that an electron traversing the sample scatters from magnons many times. The system must also be wide enough that it displays bulk behaviour, but narrow enough that heat generated within the sample can be transported to the boundary.

Critical Fields: Starting from some low base temperature, T_0, and gradually increasing the electric field one may anticipate two fields at which the response may become non-linear:

i. When the energy gained by an electron from the electric field between scattering events exceeds the temperature;

$$E_1 \sim \frac{T_0}{l_{\rm tr}},$$

where $l_{\rm tr}$ is the transport scattering length.

ii. When the Joule heating rate exceeds the rate at which heat may be transported from the sample by a transverse heat flow;

$$E_2^2 \sigma \sim \kappa T_0/W^2 \quad \Rightarrow \quad E_2 \sim \frac{T_0}{W}\sqrt{l_{\rm th}/l_{\rm tr}}$$

where σ and κ are the electrical and thermal conductivities. The latter result has been obtained using $\kappa/\sigma T_0 = l_{\rm th}/l_{\rm tr}$. $l_{\rm th}$ is the thermal scattering length and W is the sample width.

In this work, we will be primarily concerned with the former case. In order for a system to be in this regime, we require that $E_1 \ll E_2$, so that we hit the field E_1 first when increasing the electric field from zero; *i.e.* we require that $W \ll \sqrt{l_{\rm tr} l_{\rm th}}$. In addition, for the system to exhibit bulk behaviour requires that it be wider than its correlation length, $W \gg l_{\rm th}$. Combining these two conditions upon the sample width yields

$$l_{\rm th} \ll W \ll \sqrt{l_{\rm tr} l_{\rm th}}. \qquad (2)$$

Due to additional angular factors, the transport length is substantially greater than the thermal scattering length, $l_{\rm tr} \gg l_{\rm th}$, so that there is a large window of sample widths over which the type of non-linear response that we envisage can occur. In the high-temperature limit (with $T \ll \epsilon_{\rm F}$ nevertheless) in which experimental investigations of itinerant electron quantum criticality are usually carried out, $l_{\rm tr} \sim l_{\rm th}/\theta^2$ where $\theta \sim q/k_{\rm F} \sim (T/\epsilon_{\rm F})^{1/z}$ is the angle of scattering.

This regime is somewhat delicately balanced between the macro- and microscopic. In a truly macroscopic sample where $W \to \infty$, non-linearity always occurs due to the failure to conduct away excess Joule heat. In our case, the transverse size of the system must be small enough that $W \ll \sqrt{l_{\rm tr} l_{\rm th}}$, but the system inherits its behaviour from macroscopic equilibrium properties since it is larger than the correlation length[14]. If the above constraints are satisfied, the non-linear transport properties depend only upon bulk properties and are independent of the dimensions of the system.

Thermal Coupling: Determining the non-linear response requires keeping careful track of the various heat flows. We consider a simplified scheme of thermal couplings in our sample:

i. The electrons couple to a heat sink at the boundaries of the sample and scatter from magnons. We do not consider electron-electron scattering since this is higher order in temperature or electric field than electron-magnon scattering and so sub-leading at low temperatures and fields.

ii. The magnons may scatter both from one another and from the electrons. We do not consider coupling between magnons and the heat sink. Our reason is that magnon-phonon relaxation is higher order in temperature or field than magnon-electron scattering and therefore weaker at low temperatures and field.

Heuristic Treatment: Given these descriptions of the geometry of our system and the various microscopic couplings, we are now in a position to give a heuristic derivation of our main results. Heat enters the system via Joule heating and ultimately leaves through a transverse heat flow maintained by a transverse variation in temperature. In the absence of scattering between electrons, this energy must pass through the magnons: Joule heating pumps energy into higher moments of the electron distribution. This is ultimately carried away by a transverse heat flow maintained by a gradient in the symmetrical part of the electron distribution. Energy can only pass into the symmetrical part of the distribution due to scattering *via* magnons. In leading approximation, the magnons are raised to an effective temperature $T_{\rm eff}(E)$ determined by a balance between the Joule heating rate and the rate at which the magnons at $T_{\rm eff}(E)$ lose energy to the electrons at $T_0 \sim 0$. Equating these two rates of change of energy leads to a self-consistency equation for $T_{\rm eff}$.

In the high-temperature limit, the scattering time τ, the transport scattering time $\tau_{\rm tr}$ and the magnon energy decay rate $d\xi/dt$ are related as follows:

$$\frac{1}{\tau_{\rm tr}} \sim \frac{T^{2/z}}{\tau}, \qquad \frac{d\mathcal{E}}{dt} \sim \frac{T^2}{\tau}.$$

Using these relations for a system whose thermal equilibrium resistivity scales as T^{α}, we find

$$\begin{aligned} \text{Joule Heating} &\propto \sigma E^2 \propto E^2 T_{\rm eff}^{-\alpha}, \\ \text{Energy Relaxation} &\propto T_{\rm eff}^{2(1-1/z+\alpha)}, \\ \frac{d\mathcal{E}}{dt} &= \sigma E^2. \end{aligned} \tag{3}$$

By equating these two rates leads we deduce that $T_{\rm eff} \propto E^{z/(z(1+\alpha)-1)}$ and $j \propto E^{(z-1)/(z(1+\alpha)-1)}$. The following sections will flesh out these ideas with particular reference to the Moriya-Hertz-Millis [4, 5, 6] model of the critical ferromagnet.

Comparison with Mitra et al.: A recent work of Mitra *et al* has considered the same system as studied here, but in a different geometry. The results obtained in Ref.[15] are different from ours because of this geometry. In order to allay any confusion, it is worth spending a moment to note the main distinction between our two works. Mitra *et al*[15] consider an itinerant electron system with essentially two-dimensional geometry and an electric field applied in the third short direction. In this case, an electron traversing the sample from one lead to another does not scatter appreciably from magnons— the electron distribution is determined to leading order by the distributions in the leads and may be written directly in terms of them using a Keldysh formalism. Mitra *et al* present an appealing derivation of this zeroth order distribution and show, using a renormalization group analysis, that an effective temperature proportional to the applied voltage results. In our case, by contrast, an electron traversing the sample between the two leads scatters many times off magnons and the electron distribution must be calculated self-consistently from the start[16].

In the rest of this paper, we will spend some time fleshing out the mathematical details of this general scheme. We begin in the next section by reviewing the Boltzmann approach to thermal equilibrium transport in quantum critical metals.

BOLTZMANN APPROACH IN THERMAL EQUILIBRIUM

We will use Boltzmann transport techniques to analyse the out-of-equilibrium response of a quantum critical system to an electric field. Although this approach is familiar in other contexts, itinerant-electron quantum-critical transport is usually analysed by other means. Therefore, in this section, we will spend a little time summarising quantum critical transport in thermal equilibrium and how this may be described using a Boltzmann equation approach. This exposition will also serve as a useful way of defining the notation that we will use later in our analysis of the non-linear response.

Our first step will be to describe the thermal-equilibrium magnon propagator. We follow this by writing down the electron Boltzmann equation and construct its linear response solution. Finally, we quote a number of relaxation rates that will useful in our non-equilibrium analysis. The details of the calculation of these within a Boltzmann framework is somewhat similar to that of the relaxation rates due to phonon scattering. We sketch these calculations in Appendix A.

Magnon Propagator

We work within the Moriya-Hertz-Millis [4, 5, 6] approach to itinerant electron quantum criticality. The bosonic, magnetic critical modes — the magnons — are treated separately from the electrons (although they are, of course, made from electrons). The effects of scattering between the magnons and electrons are treated self-consistently; the magnon dynamics being determined by Landau damping and the electronic transport being determined by scattering from the magnon fluctuations.

The first step in the Moriya-Hertz-Millis approach to itinerant electron quantum criticality, is to determine the critical properties of the magnons. These critical properties are the combined result of the magnons' self-interaction and their overdamped dynamics due to Landau damping. The simplest way to do this is through the self-consistent renormalization group[4]. Alternatively, one may use a more rigorous application of the renormalization group[5, 6] in order to obtain essentially the same results. In either case, the critical magnon propagator takes on the following form in the equilibrium quantum critical state:

$$D^{\rm R}(\boldsymbol{q},\omega) = \left[{\rm i}\frac{|\omega|}{\Gamma_{\boldsymbol{q}}} + \boldsymbol{q}^2 + r(T) \right]^{-1}, \tag{4}$$

where $\Gamma_{\boldsymbol{q}}$ describes the Landau damping. $\Gamma_{\boldsymbol{q}}$ is proportional to $|\boldsymbol{q}|$ in the ferromagnet and constant in an antiferromagnet (or $\Gamma_{\boldsymbol{q}} = \Gamma|\boldsymbol{q}|^{z-2}$ in general). The magnon gap $r(T)$ takes on characteristic power-law forms in temperature in the quantum critical system;

$$r(T) \propto T^{\frac{d+z-2}{z}}, \tag{5}$$

where d is the dimension and z is the dynamical exponent ($z = 3$ in the ferromagnet and 2 in the antiferromagnet). Through most of our subsequent analysis, we shall concentrate upon the situation in three dimensions. This is readily extended to other dimensions. The overdamping of magnons has important consequences. Unlike their phonon counterparts, magnon excitations do not have a well-defined energy for a particular wave-vector. This does not have enormous consequences for a Boltzmann analysis in thermal equilibrium, but it does necessitate modification of the magnon Boltzmann equation when we consider the out-of-equilibrium situation.

The Boltzmann Equation

The electronic Boltzmann equation for scattering from an uncharged auxiliary mode may be written

$$\begin{aligned} &[\partial_t + (e\boldsymbol{E}/\hbar)\cdot\partial_{\boldsymbol{k}}]\, f_{\boldsymbol{k}} \\ &= -\int \frac{d\boldsymbol{q}}{(2\pi)^3}\, [\gamma_{\boldsymbol{kq}} f_{\boldsymbol{k}}\,(1-f_{\boldsymbol{q}}) - \gamma_{\boldsymbol{qk}} f_{\boldsymbol{q}}\,(1-f_{\boldsymbol{k}})]\,. \end{aligned} \tag{6}$$

The matrices $\gamma_{\boldsymbol{kq}}$ describe scattering from the auxiliary modes— magnons in our case. Quite generally, for scattering from auxiliary modes that have a thermal distribution, the scattering matrices satisfy the detailed balance relationship

$$\gamma_{\boldsymbol{kq}} = \gamma_{\boldsymbol{qk}} \exp[(\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{q}})/T]. \tag{7}$$

In the case of magnon scattering, the scattering matrices take the form

$$\gamma_{\boldsymbol{pq}} = |g_{\boldsymbol{q}-\boldsymbol{p}}|^2 \left[\begin{array}{r} [1+n(\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{q}})]\,\rho(\boldsymbol{p}-\boldsymbol{q},\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{q}}) \\ +n(\epsilon_{\boldsymbol{q}}-\epsilon_{\boldsymbol{p}})\rho(\boldsymbol{q}-\boldsymbol{p},\epsilon_{\boldsymbol{q}}-\epsilon_{\boldsymbol{p}}) \end{array} \right] \tag{8}$$

where $g_{\boldsymbol{q}-\boldsymbol{p}}$ is the matrix element for electron-magnon scattering and $\rho(\boldsymbol{q},\omega)$ is the magnon spectral function. In antiferromagnets, the matrix element $g_{\boldsymbol{q}-\boldsymbol{p}}$ has significant momentum dependence, with scattering hot spots corresponding to resonance of the magnetic ordering wave-vector with the Fermi surface. For simplicity, we restrict our analysis to the case of ferromagnets or long wavelength helimagnets where the momentum dependence of $g_{\boldsymbol{q}-\boldsymbol{p}}$ is weak and can be neglected. The magnon spectral function is given by

$$\rho(\boldsymbol{q},\omega) = -\frac{1}{\pi}\mathcal{I}m D^{\mathrm{R}}(\boldsymbol{q},\omega) = \frac{\omega/\Gamma_{\boldsymbol{q}}}{(r+\boldsymbol{q}^2)^2 + (\omega/\Gamma_{\boldsymbol{q}})^2}. \tag{9}$$

It is determined by the magnon propagator given in Eq. (4). It contains all of the information about how dynamics is incorporated into the critical behaviour through the relative scaling of frequency and momentum: $\omega \sim q^z \sim q^2\Gamma_{\boldsymbol{q}}$. In what follows, it will prove very useful to work with the general form of the Boltzmann equation (6) rather than the form obtained after explicit substitution of $\gamma_{\boldsymbol{pq}}$.

Linear Response Solution of the Boltzmann Equation

The generic notation of Eq. (6) allows us to construct a formal linear response solution of the Boltzmann equation both in thermal equilibrium and, ultimately, out of thermal equilibrium. In order to orient ourselves for the latter more involved calculation, let us first construct the conventional linear response solution with this general notation. Identifying [9]

$$\mathsf{M}_{\boldsymbol{kq}} = \frac{\gamma_{\boldsymbol{qk}}}{\gamma_{\boldsymbol{k}}}\frac{1-f_{\boldsymbol{k}}}{1-f_{\boldsymbol{q}}} \qquad \gamma_{\boldsymbol{k}} = \int d\boldsymbol{q}\,\gamma_{\boldsymbol{kq}}\frac{1-f_{\boldsymbol{q}}}{1-f_{\boldsymbol{k}}} \tag{10}$$

and adopting an Einstein convention with implied integration over the momentum $\boldsymbol{q}$, but not $\boldsymbol{k}$, we may write the Boltzmann equation in the form

$$[\partial_t + (e\boldsymbol{E}/\hbar)\cdot\partial_{\boldsymbol{k}}]\, f_{\boldsymbol{k}} = -\gamma_{\boldsymbol{k}}[1-\mathsf{M}]_{\boldsymbol{kq}} f_{\boldsymbol{q}}(1-f_{\boldsymbol{q}}). \tag{11}$$

Let us consider an initial thermal distribution of electrons and auxiliary modes at the same temperature. The deviation in the electron distribution from its initial thermal distribution, $f^0_{\boldsymbol{k}}$, in response to an electric field is given by a solution of the linearised equation

$$(e\boldsymbol{E}/\hbar)\cdot\partial_{\boldsymbol{k}}\left[f^0_{\boldsymbol{k}} + \delta f_{\boldsymbol{k}}\right] = -\gamma_{\boldsymbol{k}}[1-\mathsf{M}]_{\boldsymbol{kq}}\delta f_{\boldsymbol{q}}, \tag{12}$$

where an Einstein convention has again been adopted. There are a couple of steps required in deriving this equation. Firstly, we have used the fact that the scattering integral is zero when the electrons and magnons are in thermal distributions at the same temperature. One must also allow for the dependence of $[1-\mathsf{M}]_{\boldsymbol{kq}}$ upon $f_{\boldsymbol{q}}$ in obtaining the first functional derivative of the scattering integral.

A formal solution to the linearised Boltzmann equation (12) is readily obtained. Expanding to linear order in the electrical field we find

$$\delta f_{\boldsymbol{k}} = [1-\mathsf{M}]^{-1}_{\boldsymbol{kq}}\frac{1}{\gamma_{\boldsymbol{q}}}\boldsymbol{E}\cdot\partial_{\boldsymbol{q}} f_{\boldsymbol{q}}. \tag{13}$$

This result may be integrated to obtain the current that flows in response to the application of the electric field;

$$\begin{aligned} \boldsymbol{j} &= \int \frac{d\boldsymbol{k}}{(2\pi)^d}\boldsymbol{k}\delta f_{\boldsymbol{k}} \\ &= \int \frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\boldsymbol{q}}{(2\pi)^3}\boldsymbol{k}[1-\mathsf{M}]^{-1}_{\boldsymbol{kq}}\frac{1}{\gamma_{\boldsymbol{q}}}\boldsymbol{E}\cdot\partial_{\boldsymbol{q}} f_{\boldsymbol{q}}, \end{aligned} \tag{14}$$

where an explicit integral over $\boldsymbol{k}$ has been restored. In the next section, we will turn to a consideration of the non-linear response of the electron-magnon system using a very similar Boltzmann transport analysis. Before this, we identify a number of different time-scales of relevance to our problem and calculate them in the case of magnon scattering.

A Compendium of Relaxation Rates

The electronic scattering integral is given by the right-hand side of Eq. (6) or Eq.(12),

$$\begin{aligned}\left(\frac{\partial f_{\boldsymbol{q}}}{\partial t}\right)_{\rm Scatt} &= -\int \frac{d\boldsymbol{p}}{(2\pi)^3}\left[\begin{array}{l}\gamma_{\boldsymbol{qp}} f_{\boldsymbol{q}}(1-f_{\boldsymbol{p}}) \\ \quad -\gamma_{\boldsymbol{pq}} f_{\boldsymbol{p}}(1-f_{\boldsymbol{q}})\end{array}\right] \\ &= -\gamma_{\boldsymbol{q}}[1-\mathsf{M}]_{\boldsymbol{qp}} f_{\boldsymbol{p}}(1-f_{\boldsymbol{p}}) \\ &= -\gamma_{\boldsymbol{q}}[1-\mathsf{M}]_{\boldsymbol{qp}}\delta f_{\boldsymbol{p}}, \end{aligned} \tag{15}$$

where $\gamma_{\boldsymbol{q}}$ and $\mathsf{M}_{\boldsymbol{qp}}$ are given by Eq. (10). Integration over $\boldsymbol{p}$ has been suppressed in the final two expressions, which are drawn from Eqs. (11) and (12), respectively.

We may identify several different time-scales from this scattering integral that will appear in our later study of the non-equilibrium response;

$$\begin{aligned}\frac{1}{\tau_{\boldsymbol{q}}} &= \gamma_{\boldsymbol{q}} = \int \frac{d\boldsymbol{p}}{(2\pi)^3}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}, \\ \frac{1}{\tau^{\rm tr}_{\boldsymbol{q}}} &= \gamma^{\rm tr}_{\boldsymbol{q}} = \int \frac{d\boldsymbol{p}}{(2\pi)^3}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}\left[1-\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{q^2}\frac{\gamma^{\rm tr}_{\boldsymbol{q}}}{\gamma^{\rm tr}_{\boldsymbol{p}}}\right], \\ \frac{d\mathcal{E}}{dt} &= \frac{1}{2}\int \frac{d\boldsymbol{p}}{(2\pi)^3}\frac{d\boldsymbol{q}}{(2\pi)^3}(\epsilon_{\boldsymbol{q}}-\epsilon_{\boldsymbol{p}})\left[\begin{array}{l}\gamma_{\boldsymbol{qp}} f^0_{\boldsymbol{q}}(1-f^0_{\boldsymbol{p}}) \\ \quad -\gamma_{\boldsymbol{pq}} f^0_{\boldsymbol{p}}(1-f^0_{\boldsymbol{q}})\end{array}\right].\end{aligned} \tag{16}$$

The first of these scattering rates is simply the inverse time between collisions. The second is the transport scattering rate. This has the usual additional geometrical factor arising since large angle scattering has more effect upon transport than small angle scattering[17]. The ratio $\gamma^{\rm tr}_{\boldsymbol{q}}/\gamma^{\rm tr}_{\boldsymbol{p}}$ is conventionally set to 1 since we are interested in the scattering of fermions near to the Fermi surface. The final expression is the rate of flow of energy from auxiliary modes at a temperature T (at which $\gamma_{\boldsymbol{pq}}$ is evaluated) to electrons at temperature T_0 (indicated by the superscript 0 on the electron distribution functions).

In the high-temperature limit, these relaxation rates have the following temperature dependence within the Moriya-Hertz-Millis [4, 5, 6] theory in d dimensions:

$$\begin{aligned}1/\tau &\propto T^{(d+z-3)/z}, \\ 1/\tau_{\rm tr} &\propto T^{(d+z-1)/z}, \\ d\mathcal{E}/dt &\propto T^{(d+3z-3)/z}.\end{aligned} \tag{17}$$

Details of how to get these results from Eqs.(16) are given in Appendix A. After these preliminaries, we are now in a position to adapt our Boltzmann equation to describe the out-of-equilibrium behaviour of our system.

NON-EQUILIBRIUM RESPONSE

In this section we will turn the machinery of Boltzmann transport to the question of non-equilibrium behaviour in the itinerant critical ferromagnet. As discussed earlier, for a system to have an out-of-equilibrium steady state under the application of an electric field, it must be coupled to a heat sink that can dissipate the excess energy generated by Joule heating. We must pay careful attention to the various thermal couplings. The nature of these bears repetition at this juncture.

As described in the introduction, we consider a long, narrow sample in which the excess Joule heat is carried away by a transverse heat current to a heat sink at the edge. Heat entering the electrons *via* Joule heating passes to the magnons and then back to the electrons *via* mutual scattering and leaves the electrons *via* coupling to a heat sink at the boundary. The magnons themselves interact both with the electrons and with one another. Electron-electron scattering is neglected in our analysis— it is higher order in temperature and hence field — as is coupling of magnons directly to the heat sink.

Our analysis is divided into three parts. We begin by writing down the Boltzmann equations for the electron-magnon system. These equations embody the various thermal couplings and interactions in our system. The only subtlety enters through the form of the magnons' Boltzmann equation: the overdamped nature of the magnon excitations leads to a slightly more complicated equation than the comparable case of phonon scattering. In fact, the details of the magnon Boltzmann equation will not have a huge effect upon our main result. Next, we will present formal solutions for the electron and magnon distribution functions. Our main results follow from consideration of these solutions in the limit where magnon-magnon scattering leads to a thermal distribution of magnons with temperature determined by the electric field. We will end with an argument why correction to this thermal distribution of magnons do not change the scaling of our results.

The Boltzmann Equation

The electron Boltzmann equation is given by a minimal modification of the thermal equilibrium Boltzmann equation (6):

$$\begin{aligned}&\left[\partial_t + (e\boldsymbol{E}/\hbar)\cdot\partial_{\boldsymbol{k}} + \boldsymbol{v}_{\boldsymbol{k}}\cdot\nabla\right] f_{\boldsymbol{k}} \\ &= \mathsf{I}^{em}_{\boldsymbol{k}}[f,n] + \text{Scattering to heat sink} \\ &= -\int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\begin{array}{l}\gamma_{\boldsymbol{kq}} f_{\boldsymbol{k}}(1-f_{\boldsymbol{q}}) \\ \quad -\gamma_{\boldsymbol{qk}} f_{\boldsymbol{q}}(1-f_{\boldsymbol{k}})\end{array}\right] + \text{heat sink}\end{aligned} \tag{18}$$

where $\mathsf{I}^{em}_{\boldsymbol{k}}[f,n]$ indicates the scattering integral for electrons of momentum $\boldsymbol{k}$ scattering from magnons. The gradient term $\boldsymbol{v_k}.\nabla$ has been added to allow for the possibility of transverse heat flow. The electron-magnon scattering matrices now take a slightly modified form compared with that in thermal equilibrium (8). Since the magnons are overdamped and as a result do not have a definite relationship between their energy and momentum, the magnon distribution is a function of both energy and momentum. Taking this into account, the scattering matrices take the form

$$\gamma_{\boldsymbol{pq}} = |g_{\boldsymbol{q}-\boldsymbol{p}}|^2 \left[\begin{array}{c} [1+n_{\boldsymbol{p}-\boldsymbol{q}}(\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{q}})]\,\rho(\boldsymbol{p}-\boldsymbol{q},\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{q}}) \\ +n_{\boldsymbol{q}-\boldsymbol{p}}(\epsilon_{\boldsymbol{q}}-\epsilon_{\boldsymbol{p}})\rho(\boldsymbol{q}-\boldsymbol{p},\epsilon_{\boldsymbol{q}}-\epsilon_{\boldsymbol{p}}) \end{array} \right]. \tag{19}$$

The linearised expansion about the zero-field, base-temperature distribution $f^0_{\boldsymbol{q}}$ takes the form

$$\begin{aligned} & (e\boldsymbol{E}/\hbar)\cdot\partial_{\boldsymbol{k}}\left[f^0_{\boldsymbol{k}}+\delta f_{\boldsymbol{k}}\right] \\ = & -\int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\begin{array}{c} \gamma_{\boldsymbol{kq}} f_{\boldsymbol{k}}\,(1-f_{\boldsymbol{q}}) \\ -\gamma_{\boldsymbol{qk}} f_{\boldsymbol{q}}\,(1-f_{\boldsymbol{k}}) \end{array}\right] \\ & -\gamma_{\boldsymbol{k}}[1-\mathsf{M}]_{\boldsymbol{kq}}\delta f_{\boldsymbol{q}} + \text{heat sink} \end{aligned} \tag{20}$$

where $\gamma_{\boldsymbol{k}}$ and $\mathsf{M}_{\boldsymbol{kq}}$ take a slightly modified form out of equilibrium given by

$$\begin{aligned} \gamma_{\boldsymbol{k}} &= \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left[\gamma_{\boldsymbol{kq}}\left(1-f^0_{\boldsymbol{q}}\right)+\gamma_{\boldsymbol{qk}} f^0_{\boldsymbol{q}}\right] \\ \gamma_{\boldsymbol{k}}\mathsf{M}_{\boldsymbol{kq}} &= \gamma_{\boldsymbol{kq}} f^0_{\boldsymbol{k}} + \gamma_{\boldsymbol{qk}}\left(1-f^0_{\boldsymbol{k}}\right). \end{aligned} \tag{21}$$

These reduce to our previous expressions for $\gamma_{\boldsymbol{k}}$ and $\mathsf{M}_{\boldsymbol{kq}}$ in thermal equilibrium (10). The simplified forms given by (10) can be obtained by making use of the detailed balance condition, which is not satisfied out of equilibrium. A couple of points are worth making about Eq.(20). Firstly, there is a zeroth order term on the right-hand side. This term is not present in thermal equilibrium (it is zero upon applying the detailed balance condition). This term has a different symmetry in momentum space than the first order term in δf and we will use this in our analysis shortly.

The added complication due to the magnons being overdamped is compounded when we come to write down the magnon Boltzmann equation in a moment. Luckily, this does not affect the bulk of our calculation. We will use the formal notation $\mathsf{I}^{em}_{\boldsymbol{k}}[f,n]$ through as much of our analysis as possible in order to keep algebra to a minimum. When we eventually substitute the particular form of the scattering integrals near to the end of the calculation, we will find that most of the integration of these scattering integrals carries over directly from the thermal equilibrium calculation.

The Magnon Boltzmann Equation takes the form

$$\begin{aligned} \partial_t n_{\boldsymbol{k}}(\epsilon) &= \mathsf{I}^{me}_{\boldsymbol{k}}[f,n]+\mathsf{I}^{mm}_{\boldsymbol{k}}[n] \\ &= \int \frac{d\boldsymbol{p}}{(2\pi)^3}\frac{d\boldsymbol{q}}{(2\pi)^3}|g_{\boldsymbol{k}}|^2 \left[\begin{array}{c} -n_{\boldsymbol{k}}(\epsilon) f_{\boldsymbol{p}}(1-f_{\boldsymbol{q}}) \\ +[1+n_{\boldsymbol{k}}(\epsilon)](1-f_{\boldsymbol{p}})f_{\boldsymbol{q}} \end{array}\right]\delta(\epsilon+\epsilon_{\boldsymbol{p}}-\epsilon_{\boldsymbol{q}})\delta(\boldsymbol{p}-\boldsymbol{q}+\boldsymbol{k}) \\ &\quad +\lambda\int d\epsilon_1 d\epsilon_2 d\epsilon_3 \frac{d\boldsymbol{p}_1}{(2\pi)^3}\frac{d\boldsymbol{p}_2}{(2\pi)^3}\frac{d\boldsymbol{p}_3}{(2\pi)^3}\rho(\epsilon_1,\boldsymbol{p}_1)\rho(\epsilon_2,\boldsymbol{p}_2)\rho(\epsilon_3,\boldsymbol{p}_3) \\ &\qquad \times\left[\begin{array}{c} -n_{\boldsymbol{k}}(\epsilon)n_{\boldsymbol{p}_1}(\epsilon_1)[1+n_{\boldsymbol{p}_2}(\epsilon_2)][1+n_{\boldsymbol{p}_3}(\epsilon_3)] \\ +[1+n_{\boldsymbol{k}}(\epsilon)][1+n_{\boldsymbol{p}_1}(\epsilon_1)]n_{\boldsymbol{p}_2}(\epsilon_2)n_{\boldsymbol{p}_3}(\epsilon_3) \end{array}\right]\delta(\boldsymbol{k}+\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{p}_3)\delta(\epsilon+\epsilon_1-\epsilon_2-\epsilon_3). \end{aligned} \tag{22}$$

The easiest way to see the origins of the various terms in this equation is to momentarily treat the magnons as if they had a definite relationship between energy and momentum. In this case, the magnon spectral function becomes a delta-function and the scattering integrals reduce to the same form as those for electron-phonon scattering. As for the electron Boltzmann equation, we will carry out as much of our analysis as possible using the formal expressions $\mathsf{I}^{me}_{\boldsymbol{k}}[f,n]$ and $\mathsf{I}^{mm}_{\boldsymbol{k}}[n]$ for the magnon-electron and magnon-magnon scattering integrals.

Solving the Boltzmann Equations

Our analysis of the Boltzmann equations (18) and (22) derived above proceeds as follows: We begin by dividing the electron distribution function into two parts— a spherically symmetric part and a non-symmetric part. The magnon Boltzmann equation is divided similarly. After this division, the resulting Boltzmann equations have simple interpretations. The equation for the symmetric part of the distribution function describes the balance between the transverse heat flow and the flow of energy out of the magnons into the symmetric part of the electron distribution. The equation for the remaining part describes a balance between the flow of energy from the electrons into the magnons and the Joule heating rate. In order to obtain useful results from these equations, we will go to a limit where the magnon distribution is assumed to be thermalised at some temperature $T_{\mathrm{eff}}(E)$. The final step in our analysis will be to show that corrections to the thermal distribution of magnons do not change the way in which the current response scales with field.

Expanding the Boltzmann Equation:
The electron distribution function is divided into its symmetric part f^0 (assumed to be a Fermi distribution at a low base temperature that varies slowly across the sample) and the remainder δf. With this notation and after expanding to linear order in δf, the Boltzmann equations may be written in the form

$$\boldsymbol{v}_{\boldsymbol{q}} \cdot \nabla f^0_{\boldsymbol{q}} = \left[\mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n]\right]^S, \tag{23}$$

$$\boldsymbol{E} \cdot \partial_{\boldsymbol{q}} \left(f^0_{\boldsymbol{q}} + \delta f_{\boldsymbol{q}}\right) = \left[\mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n]\right]^A + \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}}{\delta f_{\boldsymbol{k}}} \delta f_{\boldsymbol{k}}, \tag{24}$$

$$0 = \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0, n] + \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}}{\delta f_{\boldsymbol{q}}} \delta f_{\boldsymbol{q}} + \mathsf{I}^{mm}_{\boldsymbol{k},\epsilon}[n]. \tag{25}$$

We have adopted an Einstein convention where terms like $(\delta \mathsf{I}^{em}_{\boldsymbol{q}}/\delta f_{\boldsymbol{k}})\delta f_{\boldsymbol{k}}$ are implicitly integrated over $\boldsymbol{k}$. The superscripts S and A refer to symmetric and non-symmetric parts of the scattering integrals in $\boldsymbol{q}$. We have allowed for a transverse gradient in f^0 which supports a transverse heat flow.

One might question how, given the fact that we are interested in the non-linear response, we can use a linear analysis in δf. In a linear-response, relaxation-time approximation, the Fermi surface is effectively shifted a distance $\tau_{\mathrm{tr}} eE/\hbar$ in momentum space. Provided this is much less than the Fermi wavevector ($\tau_{\mathrm{tr}} eE/\hbar \ll k_{\mathrm{F}}$) a linear response analysis may be applied. In the present case, it turns out that the transport relaxation time, τ_{tr}, self-consistently becomes a power of E so that the resultant current is non-linear in E.

Heat flows

The physical content of the equations (23), (24) and (25) is most readily appreciated by considering the energy transfers that they represent. In the case of (23) and (24) we multiply by the electron energy $\epsilon_{\boldsymbol{q}}$ and integrate over $\boldsymbol{q}$. In the case of (25), we multiply by the magnon energy ϵ and the spectral density $\rho(\boldsymbol{k}, \epsilon)$ and integrate over $\boldsymbol{k}$ and ϵ. After doing this, Eqs.(23-25) reduce to

$$0 = \int \frac{d\boldsymbol{q}}{(2\pi)^3} \epsilon_{\boldsymbol{q}} \left[\boldsymbol{v}_{\boldsymbol{q}} \cdot \nabla f^0_{\boldsymbol{q}} - \left[\mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n]\right]^S\right], \tag{26}$$

$$0 = \int \frac{d\boldsymbol{q}}{(2\pi)^3} \epsilon_{\boldsymbol{q}} \left[\boldsymbol{E} \cdot \partial_{\boldsymbol{q}} \left(f^0_{\boldsymbol{q}} + \delta f_{\boldsymbol{q}}\right) - \left[\mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n]\right]^A - \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}}{\delta f_{\boldsymbol{k}}}[f^0, n] \delta f_{\boldsymbol{k}}\right], \tag{27}$$

$$0 = \int \frac{d\boldsymbol{k}}{(2\pi)^3} d\epsilon \rho(\epsilon, \boldsymbol{k}) \epsilon \left[\mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0, n] + \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}}{\delta f_{\boldsymbol{q}}}[f^0, n] \delta f_{\boldsymbol{q}} + \mathsf{I}^{mm}_{\boldsymbol{k},\epsilon}[n]\right]. \tag{28}$$

Eq.(26) may be interpreted as a balance between the transverse heat flow — described by the first term on the right-hand side — and the energy flowing into the symmetrical part of the electron distribution — described by the second term on the right hand side. The flow of heat into the heat sink has been treated as a boundary condition in writing down this equation. Solving this equation leads to the explicit limit on the sample width discussed in the introduction and worked out in detail in Ref.[18]. We will not concentrate upon it further here.

Eq.(27) can be interpreted as a balance between Joule heating — described by the first term on the right-hand-side — and the rate at which energy flows from the non-symmetrical part of the electron distribution into the magnons — described by the second and third terms. To see this requires a little manipulation. The first term may be written explicitly as Joule heating after integrating by parts with respect to $\boldsymbol{q}$.

Eq.(28) corresponds to a balance between the rate at which energy flows into the magnons from the symmetrical and non-symmetrical parts of the electron distribution — described by the first and second terms respectively. The net flow of energy into the magnons from the electrons is zero in a steady state, if we neglect heat flow directly from the magnons to the heat sink. The latter process is ignored since it is much slower than magnon-electron scattering. The third term is identically zero, since magnon-magnon scattering conserves energy. This fact is extremely useful. By considering this integrated equation, one can avoid having to deal explicitly with the magnon-magnon scattering integral.

We can transform this final equation into a more useful form by using the fact that electron-magnon scattering is energy conserving. This implies that

$$\int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}} \mathsf{I}^{em}_{\boldsymbol{q}}[f^0,n] = \int \frac{d\boldsymbol{k}}{(2\pi)^3} d\epsilon \;\; \epsilon\rho(\epsilon,\boldsymbol{k})\mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0,n]$$

$$\int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}} \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0,n]}{\delta f_{\boldsymbol{k}}}\delta f_{\boldsymbol{k}} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} d\epsilon \;\; \epsilon\rho(\epsilon,\boldsymbol{k})\frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0,n]}{\delta f_{\boldsymbol{q}}}\delta f_{\boldsymbol{q}}$$

i.e the energy entering the electrons from the magnons is equal to the energy entering the magnons from the electrons. Using these results reduces Eq.(28) to the form

$$0 = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}} \left[\mathsf{I}^{em}_{\boldsymbol{q}}[f^0,n] + \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0,n]}{\delta f_{\boldsymbol{k}}}\delta f_{\boldsymbol{k}} \right]. \quad (29)$$

To make further progress, we must solve the Boltzmann equations explicitly. We will do this to leading order through an approximation that the magnon distribution is thermal. We will then argue that corrections to this do not alter the scaling.

Thermal Magnon Approximation

Our leading approximation is to assume a thermal distribution of magnons, $n_{\boldsymbol{k}}(\epsilon) = n^0(\epsilon) = (\mathrm{e}^{\epsilon/T_{\mathrm{eff}}}-1)^{-1}$, where the effective temperature is to be determined shortly. In this case, the linearised Boltzmann equations (23), (24) and (25) reduce to

$$\boldsymbol{v}_{\boldsymbol{q}} \cdot \nabla f^0_{\boldsymbol{q}} = \mathsf{I}^{em}_{\boldsymbol{q}}[f^0,n^0], \quad (30)$$

$$\boldsymbol{E}\cdot\partial_{\boldsymbol{q}}\left(f^0_{\boldsymbol{q}} + \delta f_{\boldsymbol{q}}\right) = \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0,n^0]}{\delta f_{\boldsymbol{k}}}\delta f_{\boldsymbol{k}}, \quad (31)$$

$$0 = \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0,n^0] + \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0,n^0]}{\delta f_{\boldsymbol{q}}}\delta f_{\boldsymbol{q}}. \quad (32)$$

We have used the fact that the magnon-magnon scattering is identically zero for a thermal distribution of magnons. The second equation may be formally solved for δf to obtain,

$$\delta f_{\boldsymbol{q}} = \left[1 - \left(\frac{\delta \mathsf{I}^{em}}{\delta f}\right)^{-1}\boldsymbol{E}.\partial_{\boldsymbol{q}}\right]^{-1}\left(\frac{\delta \mathsf{I}^{em}}{\delta f}\right)^{-1}\boldsymbol{E}.\partial_{\boldsymbol{q}} f^0, \quad (33)$$

where we have suppressed momentum labels and integrals over momentum for brevity. Expressions such as $(\delta \mathsf{I}^{em}/\delta f)^{-1}$ are to be understood as matrix inverses with appropriate integrations over momentum in their products. In order to determine the effective temperature, we substitute this solution for δf into the energy-integrated form of Eq.(28) or (31) given by Eq.(29) and expand to leading order in $\boldsymbol{E}$. The result of this substitution is

$$0 = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}} \left[\mathsf{I}^{em}_{\boldsymbol{q}} + \boldsymbol{E}\cdot\partial_{\boldsymbol{q}}\left[\left(\frac{\delta \mathsf{I}^{em}}{\delta f}\right)^{-1}\boldsymbol{E}\cdot\partial_{\boldsymbol{q}}\delta f\right]\right]. \quad (34)$$

Integrating the second term by parts reduces it to the form

$$0 = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}} \left[\mathsf{I}^{em}_{\boldsymbol{q}} - \frac{1}{3}E^2 \boldsymbol{v}_{\boldsymbol{q}}\cdot\left(\frac{\delta \mathsf{I}^{em}}{\delta f}\right)^{-1}\partial_{\boldsymbol{q}}\delta f\right]. \quad (35)$$

The second term is now explicitly the leading order contribution to the Joule heating rate. This is balanced against the first term which describes the decay of energy from a thermal distribution of magnons at temperature $T_{\mathrm{eff}}(E)$. Since both the electron and magnon distributions involved in the expressions are thermal distributions — at $T=0$ and $T=T_{\mathrm{eff}}(E)$ respectively — we may evaluate Eq.(35) using the results of Section 3. Writing Eq.(35) in terms of the scattering matrices of Section 3, it can be reduced to

$$\int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}}\gamma_{\boldsymbol{q}}[1-\mathsf{M}]_{\boldsymbol{qp}} f_{\boldsymbol{p}}(T_0)\left(1-f_{\boldsymbol{p}}(T_0)\right),$$
$$= \int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}}\gamma_{\boldsymbol{q}}[1-M]_{\boldsymbol{qp}}\delta f_{\boldsymbol{p}}. \quad (36)$$

Substituting for δf to leading order in $\boldsymbol{E}$ from Eq. (31) into Eq. (36), we obtain

$$\underbrace{\int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}}.\partial_{\boldsymbol{q}}\left[[1-\mathsf{M}]^{-1}_{\boldsymbol{qp}}\gamma^{-1}_{\boldsymbol{p}}\boldsymbol{E}.\partial_{\boldsymbol{q}} f_{\boldsymbol{q}}(T_0)\right]}_{\text{Joule Heating}}$$
$$= \underbrace{\int \frac{d\boldsymbol{q}}{(2\pi)^3}\epsilon_{\boldsymbol{q}}\gamma_{\boldsymbol{q}}[1-\mathsf{M}]_{\boldsymbol{qp}} f_{\boldsymbol{p}}(T_0)\left[1-f_{\boldsymbol{p}}(T_0)\right]}_{\text{Energy decay from } T_{\mathrm{eff}} \text{ to } T_0} \quad (37)$$

This equation may be written in the form $d\mathcal{E}/dt \propto E^2\tau_{\mathrm{tr}}$ as before in Eq,(3). Since the magnon distribution is thermal at temperature T_{eff}, using the temperature scaling of the various relaxation rates given in Eq.(17), we find

$$T_{\mathrm{eff}} \propto E^{z/(d+2z-2)}$$

in the high-field/temperature limit and d dimensions, implying a non-linear current

$$j \propto E^{(z-1)/(d+2z-2)}$$

as suggested in Section 1.

Corrections to Thermal Magnon Approximation

Corrections to a thermal distribution of magnons may be rather large, since in the absence of the electric field the magnons are essentially in a zero-temperature distribution. We argue, nevertheless, that corrections to the thermal distribution of magnons considered above do not change the scaling of current response. The analysis is similar to the calculation of phonon drag in thermal equilibrium (which similarly does not change the scaling with temperature).

We expand the magnon distribution to linear order about the effective thermal distribution; $n = n^0 + \delta n$. Substituting into the Boltzmann Eqs.(23,24) and (25), we obtain

$$\begin{aligned} \boldsymbol{v_q} \cdot \nabla f^0_{\boldsymbol{q}} &= \mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n^0] + \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n^0]}{\delta n_{\boldsymbol{k},\epsilon}} \delta n^S_{\boldsymbol{k},\epsilon}, \quad (38) \\ \boldsymbol{E} \cdot \partial_{\boldsymbol{q}} \left(f^0_{\boldsymbol{q}} + \delta f_{\boldsymbol{q}} \right) &= \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n^0]}{\delta f_{\boldsymbol{k}}} \delta f_{\boldsymbol{k}} + \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n^0]}{\delta n_{\boldsymbol{k},\epsilon}} \delta n^A_{\boldsymbol{k},\epsilon}, \quad (39) \\ 0 &= \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0, n^0] + \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0, n^0]}{\delta f_{\boldsymbol{q}}} \delta f_{\boldsymbol{q}} \quad (40) \\ &\quad + \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}[f^0, n^0]}{\delta n_{\boldsymbol{q},\xi}} \delta n^S_{\boldsymbol{q},\xi} + \frac{\delta \mathsf{I}^{mm}_{\boldsymbol{k},\epsilon}[n^0]}{\delta n_{\boldsymbol{q},\xi}} \delta n_{\boldsymbol{q},\xi}. \quad (41) \end{aligned}$$

In Eqs.(38) and (39), δn has been divided into symmetric and non-symmetric parts δn^S and δn^A. These contribute to the equations for the symmetric and non-symmetric parts of the electron distribution respectively. Eq.(41) can be solved formally for δn with the result

$$\begin{aligned} \delta n &= \underbrace{-\left(\frac{\delta \mathsf{I}^{me}}{\delta n} + \frac{\delta \mathsf{I}^{mm}}{\delta n} \right)^{-1} \mathsf{I}^{me}}_{\delta n^S} \quad (42) \\ &\quad \underbrace{-\left(\frac{\delta \mathsf{I}^{me}}{\delta n} + \frac{\delta \mathsf{I}^{mm}}{\delta n} \right)^{-1} \frac{\delta \mathsf{I}^{me}}{\delta f} \delta f}_{\delta n^A} \quad (43) \end{aligned}$$

We have identified the spherically symmetric and non-symmetric parts of δn. Substituting this back into Eq.(38) and (39) one obtains

$$\begin{aligned} \boldsymbol{v_q} \cdot \nabla f^0_{\boldsymbol{q}} &= \mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n^0] \\ &\quad - \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}}{\delta n} \left(\frac{\delta \mathsf{I}^{me}}{\delta n} + \frac{\delta \mathsf{I}^{mm}}{\delta n} \right)^{-1} \mathsf{I}^{me} \quad (44) \\ \boldsymbol{E} \cdot \partial_{\boldsymbol{q}} \left(f^0_{\boldsymbol{q}} + \delta f_{\boldsymbol{q}} \right) &= \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}[f^0, n^0]}{\delta f_{\boldsymbol{k}}} \delta f_{\boldsymbol{k}} \\ &\quad \underbrace{- \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}}{\delta n_{\boldsymbol{k},\epsilon}} \left(\frac{\delta \mathsf{I}^{me}}{\delta n} + \frac{\delta \mathsf{I}^{mm}}{\delta n} \right)^{-1} \frac{\delta \mathsf{I}^{me}}{\delta f}}_{\frac{\delta \mathsf{I}^{eme}}{\delta f}} \delta f \quad (45) \end{aligned}$$

In the second of these equations, we have adopted the notation of Lifshitz-Pitaevskii[19] identifying this as a term describing a magnon-mediated electron-electron interaction. The argument that magnon drag does not affect scaling is completed by showing that $\delta \mathsf{I}^{eme}/\delta f$ scales with at least as high a power of T as $\delta \mathsf{I}^{em}/\delta f$. This requires us to go beyond the generic form of the scattering integrals to use their explicit expressions for magnon-electron scattering given in the Boltzmann equations (18) and (22). We ignore the magnon-magnon scattering (it is higher order in T —and hence E— than the magnon-electron scattering) we may write

$$\frac{\delta \mathsf{I}^{eme}}{\delta f} = - \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}}{\delta n_{\boldsymbol{k},\epsilon}} \left(\frac{\delta \mathsf{I}^{me}}{\delta n} + \frac{\delta \mathsf{I}^{mm}}{\delta n} \right)^{-1} \frac{\delta \mathsf{I}^{me}}{\delta f}. \quad (46)$$

Taking the explicit form of the scattering integrals, the various functional derivatives may be written as

$$\begin{aligned} \frac{\delta \mathsf{I}^{em}_{\boldsymbol{q}}}{\delta n_{\boldsymbol{k},\epsilon}} &= -|g|^2 \rho(\boldsymbol{k}, \epsilon) \left[\begin{array}{l} (f_{\boldsymbol{q}} - f_{\boldsymbol{k}-\boldsymbol{q}}) \delta(\epsilon - \epsilon_{\boldsymbol{q}} + \epsilon_{\boldsymbol{k}-\boldsymbol{q}}) \\ \quad + (f_{\boldsymbol{q}} - f_{\boldsymbol{k}+\boldsymbol{q}}) \delta(\epsilon - \epsilon_{\boldsymbol{q}+\boldsymbol{k}} + \epsilon_{\boldsymbol{q}}) \end{array} \right] \\ \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k},\epsilon}}{\delta n_{\boldsymbol{l},\nu}} &= |g|^2 \delta(\boldsymbol{l} - \boldsymbol{k}) \delta(\nu - \epsilon) \\ &\quad \times \int \frac{d\boldsymbol{p}}{(2\pi)^3} (f_{\boldsymbol{p}+\boldsymbol{k}} - f_{\boldsymbol{p}}) \delta(\epsilon + \epsilon_{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}+\boldsymbol{k}}) \\ \frac{\delta \mathsf{I}^{me}_{\boldsymbol{k}\epsilon}}{\delta f_{\boldsymbol{l}}} &= |g|^2 \left[\begin{array}{l} -(n_{\boldsymbol{k}\epsilon} + f_{\boldsymbol{l}+\boldsymbol{k}}) \delta(\epsilon + \epsilon_{\boldsymbol{l}} - \epsilon_{\boldsymbol{l}+\boldsymbol{k}}) \\ \quad + (1 + n_{\boldsymbol{k}\epsilon} - f_{\boldsymbol{l}-\boldsymbol{k}}) \delta(\epsilon + \epsilon_{\boldsymbol{l}-\boldsymbol{k}} - \epsilon_{\boldsymbol{l}}) \end{array} \right] \end{aligned}$$

Substituting these equations back into Eq.(46) shows that the corrections due to magnon drag lead to contributions to the electron scattering integral that are at least of the same order in temperature as the direct contribution.

CONCLUSIONS AND PROSPECTS

We have considered non-linear transport near to an itinerant electron quantum critical point. Since the dynamics near to a quantum critical point are universal, and since steady-state, out-of-equilibrium distributions are determined by dynamics, we have argued that the universality present near to an equilibrium quantum critical point may be reflected in the out-of-equilibrium behaviour.

There are two ways in which a quantum critical itinerant electron system may be driven out of equilibrium by an electric field. At the highest fields, the rate of Joule heating overwhelms the rate at which heat may be transported out of the system by thermal conduction and the system heats up until the two balance. This mechanism leads to non-linear response in truly bulk samples. We have considered non-linear response in a restricted geometry where we anticipate that conductivity becomes non-linear at a lower field governed by the rate at which energy can scatter between electrons and magnons. The resulting conductivity is expected to be independent of sample size and geometry (provided that certain constraints are satisfied). At the lowest fields, the response will return to the linear, thermal equilibrium response.

The existence of the intermediate range of non-linearity requires restrictions upon the sample width so that thermal conduction can be maintained at a sufficient rate to transport away heat generated by Joule heating. The sample width must nevertheless be sufficient that the magnons demonstrate their bulk behaviour- *i.e* it must be larger than the magnon correlation length. Because of the rather different scaling of transport and thermal relaxation lengths with temperature (and hence field), there is a large window of fields within which the type of non-linearity that we investigate should exist.

What are the prospects for seeing these effects experimentally? We have described a particular experimental geometry in which the heat current is transverse to the electrical current. This enabled the algebra to be readily negotiated. In order to see these effects experimentally, we suggest a slightly different geometry [22]. One possibility is the following: take a bow tie shaped sample with current injected and removed along opposite wings of the bow tie. A current sent through this sample should demonstrate a non-linear steady-state of the type that we have described. The constriction at the centre of the bow tie will have enhanced field and current densities and will operate in a non-linear regime. The injection of current along the extended edge of the bow tie will reduce contact heating; performing the experiment in a pulsed manner will further mitigate these effects. The large heat capacity of the wings of the bow tie compared to the constriction will allow a relatively long pulse time before heat effects become significant; the wings will effectively act as a low-temperature heat sink. A sketch of this arrangement is shown in Fig. 1.

It remains to estimate what field strengths give rise to the non-linear effects that we anticipate. For a typical quantum-critical itinerant magnet (e.g. $Sr_3Ru_2O_7$, which has $n = 2 \times 10^{27}\,\mathrm{m}^{-3}$, $\sigma = 10^8\,\Omega^{-1}\mathrm{m}^{-1}$) and typical cryogenic temperatures of around $100\,\mathrm{mK}$, the electric field required to observe these non-linear effects is of the order

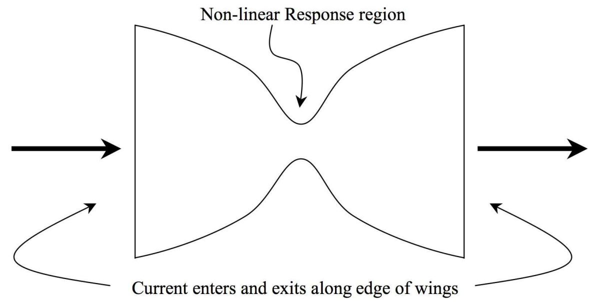

FIG. 1: **Schematic diagram of proposed experimental system:** i. Current enters and leaves the bow tie shaped sample along the edges of the wings, reducing contact heating. ii. Enhanced field and current density in the constriction leads to non-linear response in this regime. iii. The extended wings act as a low-temperature heat sink.

$$E = \frac{k_\mathrm{B} T}{e v_\mathrm{F} \tau} \approx 50\,\mathrm{Vm}^{-1}$$

(We have used the Drude formula $\sigma = ne^2\tau/m$ and $n = 4\pi k_\mathrm{F}^3/3$) or a voltage drop of $0.25\,\mathrm{V}$ over a typical sample length of $5\,\mathrm{mm}$.

In conclusion, the universal power-law scaling of conductivity near to an itinerant, magnetic, quantum critical point is reflected in a universal power law scaling with electric field in the non-linear conductivity regime. This provides a new way to investigate the consistency between theoretically predicted power-laws and those seen experimentally.

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APPENDIX A

In this appendix, we outline the evaluation of the scattering rates given in Eq.(16). The details of the calculation is somewhat similar to that of the same relaxation rate due to phonon scattering. Because of this similarity, our analysis follows quite closely — and makes explicit reference to — the calculation of electron-phonon relaxation rates presented in Chapter 8 of the textbook of Mahan[23]. We shall carry out the calculations in d-dimensions, where $d = 2$ or 3 in the physical system.

Our first task is to make a few manipulations of the energy relaxation rate to put it in a simplified form. The first step involves using the detailed balance relation expressed in the form

$$\gamma_{\boldsymbol{pq}} = \gamma_{\boldsymbol{qp}} \frac{f^T_{\boldsymbol{q}}(1-f^T_{\boldsymbol{p}})}{f^T_{\boldsymbol{p}}(1-f^T_{\boldsymbol{q}})} = \gamma_{\boldsymbol{qp}} \frac{n^T(\Delta\epsilon_{\boldsymbol{q}})}{n^T(\Delta\epsilon_{\boldsymbol{q}})+1},$$

where $n^T(\Delta\epsilon_{\boldsymbol{q}})$ is the Bose distribution at temperature T and $\Delta\epsilon_{\boldsymbol{q}} = \epsilon_{\boldsymbol{q}} - \epsilon_{\boldsymbol{p}}$. The next step is to integrate over $|\boldsymbol{q}|$ assuming that the $|\boldsymbol{q}|$-dependence of terms other than the Fermi-distribution functions is small and also that the density of electronic states is constant at the Fermi surface. In so doing, we encounter two integrals

$$\begin{aligned} I_1(\omega) &= \int d\epsilon f(\epsilon)(1-f(\epsilon-\omega)) = \omega n(\omega), \\ I_2(\omega) &= \int d\epsilon f(\epsilon-\omega)(1-f(\epsilon)) = \omega(n^0(\omega)+1). \end{aligned}$$

After these manipulations, the energy relaxation may be expressed as

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2}\int \frac{d^d\boldsymbol{p} d^d\hat{\boldsymbol{q}}}{(2\pi)^{2d}} \rho_{\mathrm{F}} \frac{\gamma_{\boldsymbol{qp}}\Delta\epsilon^2_{\boldsymbol{q}}}{n^T(\Delta\epsilon_{\boldsymbol{q}})+1}\left[n^0(\Delta\epsilon_{\boldsymbol{q}}) - n^T(\Delta\epsilon_{\boldsymbol{q}})\right],$$

where ρ_{F} is the electronic density of states at the Fermi surface and the remaining integral over $\boldsymbol{q}$ is just angular; functions of $\boldsymbol{q}$ are to be interpreted as having $|\boldsymbol{q}|$ equal to the Fermi wavevector. The superscripts 0 and T on the Bose-distribution function indicate that they are at the base temperature or the elevated temperature, T, respectively. Notice that this is automatically zero when $T^0 = T$.

In the limit $T^0 \to 0$, we may neglect $n^0(\Delta\epsilon_{\boldsymbol{q}})$. Also, in the limit where $\Delta\epsilon_{\boldsymbol{q}} \ll T$, we may write the energy relaxation in the same form as the other relaxation rates using $n^T(\Delta\epsilon_{\boldsymbol{q}})/(n^T(\Delta\epsilon_{\boldsymbol{q}})+1) \simeq (1-f^T_{\boldsymbol{p}})/(1-f^T_{\boldsymbol{q}})$. As all the terms in our scattering rates are now at the elevated temperature T, we will from now on omit this superscript for clarity.

So far, we have reduced our scattering rates to the forms

$$\begin{aligned} \frac{1}{\tau_{\boldsymbol{q}}} &= \gamma_{\boldsymbol{q}} = \int \frac{d^d\boldsymbol{p}}{(2\pi)^d}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}, \\ \frac{1}{\tau^{\mathrm{tr}}_{\boldsymbol{q}}} &= \gamma^{\mathrm{tr}}_{\boldsymbol{q}} = \int \frac{d^d\boldsymbol{p}}{(2\pi)^d}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}\left[1 - \frac{\boldsymbol{q}.\boldsymbol{p}}{\boldsymbol{q}^2}\frac{\gamma^{\mathrm{tr}}_{\boldsymbol{q}}}{\gamma^{\mathrm{tr}}_{\boldsymbol{p}}}\right], \\ \frac{d\mathcal{E}}{dt} &= -\frac{1}{2}\int \frac{d^d\boldsymbol{p}}{(2\pi)^d}\frac{d^d\hat{\boldsymbol{q}}}{(2\pi)^d}\rho_{\mathrm{F}}\Delta\epsilon^2_{\boldsymbol{q}}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}. \end{aligned} \quad (47)$$

In the present case, we are interested in scattering from critical magnons. $\gamma_{\boldsymbol{pq}}$ then takes the form given by Eq. (8). In order to calculate the various relaxation rates explicitly, it is useful to introduce a generalization of the McMillan function. In the discussion of electron-phonon scattering, this takes the form[23]

$$\alpha^2 F(E,\omega) = \frac{\hbar}{2\pi}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d}|g_{\boldsymbol{q}}|^2\delta(\omega-\omega_{\boldsymbol{q}})\delta(E-\epsilon_{\boldsymbol{k}+\boldsymbol{q}}), \quad (48)$$

where $\omega_{\boldsymbol{q}}$ is the frequency of a phonon with momentum $\boldsymbol{q}$ and $\epsilon_{\boldsymbol{p}}$ is the energy of an electron with momentum $\boldsymbol{p}$. The generalisation of this to scattering from overdamped modes is given by

$$\alpha^2 F(E,\omega) = \frac{\hbar}{2\pi}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d}|g_{\boldsymbol{q}}|^2\rho(\boldsymbol{q},\omega)\delta(E-\epsilon_{\boldsymbol{k}+\boldsymbol{q}}), \quad (49)$$

where $\rho(\boldsymbol{q},\omega) = \mathcal{I}m D^{\mathrm{R}}(\boldsymbol{q},\omega)$ is the magnon spectral function. The analogous McMillan function for transport is given by

$$\alpha^2_t F(E,\omega) = -\frac{\hbar}{2\pi}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d}|g_{\boldsymbol{q}}|^2\frac{\boldsymbol{q}.\boldsymbol{k}}{\boldsymbol{k}^2}\rho(\boldsymbol{q},\omega)\delta(E-\epsilon_{\boldsymbol{k}+\boldsymbol{q}}). \quad (50)$$

The additional angular factors weight scattering at different angles in the usual way. Eqs. (49,50) are the generalisations of Eqs.(8.145,8.146) in Ref.[23].

With this identification, expressions for the various scattering integrals may be obtained in the limit $\Delta\epsilon_{\boldsymbol{q}} \ll T$ as follows (see for example Section 8.3.1 of Ref.[23] — we use $(1-f_{\boldsymbol{p}})/(1-f_{\boldsymbol{q}}) \approx n(\Delta\epsilon_{\boldsymbol{q}})$):

$$\frac{1}{\tau_{\boldsymbol{k}}} = 2\frac{2\pi}{\hbar}\int_0^\infty d\omega n(\omega)\alpha^2F(\epsilon_{\boldsymbol{k}},\omega) = 2\int_0^\infty d\omega\frac{d^d\boldsymbol{q}}{(2\pi)^d}|g_{\boldsymbol{q}}|^2 n(\omega)\rho(\boldsymbol{q},\omega)\delta(\epsilon_{\boldsymbol{k}+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}}), \tag{51}$$

$$\frac{1}{\tau^{\rm tr}_{\boldsymbol{k}}} = 2\frac{2\pi}{\hbar}\int_0^\infty d\omega n(\omega)\alpha_t^2F(\epsilon_{\boldsymbol{k}},\omega) = -2\int_0^\infty d\omega\frac{d^d\boldsymbol{q}}{(2\pi)^d}|g_{\boldsymbol{q}}|^2 n(\omega)\frac{\boldsymbol{q}.\boldsymbol{k}}{\boldsymbol{k}^2}\rho(\boldsymbol{q},\omega)\delta(\epsilon_{\boldsymbol{k}+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}}), \tag{52}$$

$$\frac{d\mathcal{E}}{dt} = 4\pi\hbar^2\rho_{\rm F}\int_0^\infty d\omega n(\omega)\omega^2\alpha^2F(\epsilon_{\boldsymbol{k}},\omega) = 2\hbar^3\rho_{\rm F}\int_0^\infty d\omega\frac{d^d\boldsymbol{q}}{(2\pi)^d}|g_{\boldsymbol{q}}|^2 n(\omega)\omega^2\rho(\boldsymbol{q},\omega)\delta(\epsilon_{\boldsymbol{k}+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}}). \tag{53}$$

These integrals may be simplified by first linearising the electron energy near to the Fermi surface; $\epsilon_{\boldsymbol{k}+\boldsymbol{q}}-\epsilon_{\boldsymbol{k}} \approx \boldsymbol{v}_{\boldsymbol{k}}.\boldsymbol{q} = v_{\rm F}\cos\theta|\boldsymbol{q}|$, where $\boldsymbol{v}_{\boldsymbol{k}}$ is the Fermi velocity and θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$. Assuming that the matrix element does not have a significant angular dependence— an assumption that is only true for ferromagnets; antiferromagnets have hot lines of scattering on the Fermi surface where electron states are related by the ordering wave-vector that lead to complications in this latter case[24]— the angular integrals over $\boldsymbol{q}$ may then be carried out. One then obtains

$$\begin{aligned}
\frac{1}{\tau_{\boldsymbol{k}}} &= \frac{2g^2}{(2\pi)^2 v_{\rm F}}\int_0^\infty d\omega n(\omega) \\
&\qquad \times\int^{T/v_{\rm F}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\rho(\boldsymbol{q},\omega) \\
\frac{1}{\tau^{\rm tr}_{\boldsymbol{k}}} &= \frac{2g^2}{(2\pi)^2 v_{\rm F}}\frac{1}{2k_{\rm F}^2}\int_0^\infty d\omega n(\omega) \\
&\qquad \times\int^{T/v_{\rm F}} d|\boldsymbol{q}||\boldsymbol{q}|^{d}\rho(\boldsymbol{q},\omega) \\
\frac{d\mathcal{E}}{dt} &= \hbar^3\rho_{\rm F}\frac{2g^2}{(2\pi)^2 v_{\rm F}}\int_0^\infty d\omega n(\omega)\omega^2 \\
&\qquad \times\int^{T/v_{\rm F}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\rho(\boldsymbol{q},\omega)
\end{aligned} \tag{54}$$

The momentum integral has acquired an explicit cut-off at $T/v_{\rm F}$ after linearising the electron energy at the Fermi surface. The remaining integrals may be calculated after explicit substitution of the magnon spectral function.

We will evaluate these expressions in both high- and low-temperature limits. Which of these limits one is in is determined by a comparison of $r(T)$ with the typical value of $|\boldsymbol{q}|^2$. The former varies with temperature according to $r(T)\sim T^{\frac{d+z-2}{z}}$ and the latter as T^2. In the low-temperature limit, $r \gg \boldsymbol{q}^2$ and in the high-temperature limit $r \ll \boldsymbol{q}^2$. In both cases, we consider temperatures much less than the Fermi energy, $T \ll \epsilon_{\rm F}$.

Carrying out the integrals in the *low-temperature* limit, we find

$$\begin{aligned}
\frac{1}{\tau_{\boldsymbol{k}}} &= \frac{2g^2}{(2\pi)^2 v_{\rm F}}\int_0^\infty d\omega\, n(\omega)\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\rho(\boldsymbol{q},\omega) \\
&\sim \int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\int_0^\infty d\omega\, n(\omega)\frac{\omega/\Gamma_{\boldsymbol{q}}}{r^2+(\omega/\Gamma_{\boldsymbol{q}})^2} \\
&\sim \int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\Gamma_{\boldsymbol{q}}\int_0^\infty \frac{d\omega}{r\Gamma_{\boldsymbol{q}}}\, n(\omega)\frac{\omega/\Gamma_{\boldsymbol{q}}}{1+(\omega/r\Gamma_{\boldsymbol{q}})^2} \\
&\sim \int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\Gamma_{\boldsymbol{q}}\int_0^\infty du\, n(ur\Gamma_{\boldsymbol{q}})\frac{u}{1+u^2} \\
&\sim \frac{T}{r(T)}\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\int_0^\infty du\,\frac{1}{1+u^2} \\
&\sim \frac{T^d}{r(T)},
\end{aligned}$$

where we have used the fact that $r \gg \boldsymbol{q}^2$ at low temperatures. The frequency integral is dominated by the region where ω is up to order $r(T)\Gamma_{\boldsymbol{q}}$. For $\boldsymbol{q}\sim T$, at these frequencies $\omega/T \sim \Gamma r(T)T^{z-3} \ll 1$ and the Bose distribution can be approximated by its low-frequency limit $n(x)\sim T/x$. As a final consistency check, we need to make sure that the dominant momentum $\boldsymbol{q}$ is of the order of T as indeed it is.

A similar evaluation of the transport scattering rate yields the result

$$\frac{1}{\tau^{\rm tr}_{\boldsymbol{k}}} \sim \frac{T^{d+2}}{r(T)}. \tag{55}$$

The energy relaxation is given by

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \hbar^3\rho_{\rm F}\frac{2g^2}{(2\pi)^2 v_{\rm F}}\int_0^\infty d\omega\,\omega^2 n(\omega)\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\rho(\boldsymbol{q},\omega)\\
&\sim \int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\int_0^\infty d\omega\,\omega^2 n(\omega)\frac{\omega/\Gamma_{\boldsymbol{q}}}{r^2+(\omega/\Gamma_{\boldsymbol{q}})^2}\\
&\sim rT\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\Gamma_{\boldsymbol{q}}^2\int_0^{T/r\Gamma_{\boldsymbol{q}}} du\frac{u^2}{1+u^2}\\
&\sim rT\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\Gamma_{\boldsymbol{q}}^2\left[\frac{T}{r\Gamma_{\boldsymbol{q}}}-\frac{\pi}{2}\right]\\
&\sim T^2\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\Gamma_{\boldsymbol{q}}\\
&\sim T^2\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d+z-4}\\
&\sim T^{d+z-1}. \qquad (56)
\end{aligned}
$$

Carrying out the same integrations in the *high-temperature* limit, we find

$$
\begin{aligned}
\frac{1}{\tau_{\boldsymbol{k}}} &= \frac{2g^2}{(2\pi)^2 v_{\rm F}}\int_0^\infty d\omega n(\omega)\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\rho(\boldsymbol{q},\omega)\\
&\sim \int_0^\infty d\omega n(\omega)\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\frac{\omega/\Gamma|\boldsymbol{q}|^{z-2}}{\boldsymbol{q}^4+(\omega/\Gamma|\boldsymbol{q}|^{z-2})^2}\\
&\sim \int_0^\infty d\omega n(\omega)\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d+z-4}\frac{\omega}{\Gamma^2\boldsymbol{q}^{2z}+\omega^2}\\
&\sim \int_0^\infty d\omega\frac{n(\omega)}{\omega}\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}|\frac{|\boldsymbol{q}|^{d+z-4}}{\Gamma^2\boldsymbol{q}^{2z}/\omega^2+1}\\
&\sim \int_0^\infty d\omega n(\omega)\omega^{\frac{d-3}{z}}\int_0^{\frac{T}{v_{\rm F}}\left(\frac{\Gamma}{\omega}\right)^{1/z}} du\frac{u^{d+z-4}}{u^{2z}+1}\\
&\sim T^{\frac{d+z-3}{z}}\int_0^\infty dv\, n(vT)v^{\frac{d-3}{z}}\int_0^\infty du\frac{u^{d+z-4}}{u^{2z}+1}\\
&\sim T^{\frac{d+z-3}{z}} \qquad (57)
\end{aligned}
$$

In carrying out these manipulations we have used the fact that $\frac{T}{v_{\rm F}}\left(\frac{\Gamma}{\omega}\right)^{1/z}\to\infty$ at high temperatures, which is consistent since the dominant contribution to the frequency integral comes from $\omega\sim T$. We have rescaled the momentum and frequency integrals and then and used the explicit substitution $\Gamma_{\boldsymbol{q}}=\Gamma|\boldsymbol{q}|^{z-2}$. A similar evaluation of the transport scattering rate yields

$$
\frac{1}{\tau_{\boldsymbol{k}}^{\rm tr}}\sim T^{\frac{d+z-1}{z}}, \qquad (58)
$$

i.e. it carries an extra factor of $\boldsymbol{q}^2\sim T^{2/z}$ compared with the scattering rate. Finally, the energy relaxation rate is given by

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \hbar^3\rho_{\rm F}\frac{2g^2}{(2\pi)^2 v_{\rm F}}\int_0^\infty d\omega\, n(\omega)\omega^2\\
&\qquad\times\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d-2}\rho(\boldsymbol{q},\omega)\\
&\sim \int_0^\infty d\omega n(\omega)\omega^2\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}||\boldsymbol{q}|^{d+z-4}\frac{\omega}{\Gamma^2\boldsymbol{q}^{2z}+\omega^2}\\
&\sim \int_0^\infty d\omega n(\omega)\omega\int_0^{\frac{T}{v_{\rm F}}} d|\boldsymbol{q}|\frac{|\boldsymbol{q}|^{d+z-4}}{\Gamma^2\boldsymbol{q}^{2z}/\omega^2+1}\\
&\sim \int_0^\infty d\omega n(\omega)\omega^{\frac{d+2z-3}{z}}\int_0^{\frac{T}{v_{\rm F}}\left(\frac{\Gamma}{\omega}\right)^{1/z}} du\frac{u^{d+z-4}}{u^{2z}+1}\\
&\sim T^{\frac{d+3z-3}{z}}\int_0^\infty dv\, n(vT)v^{\frac{d+2z-3}{z}}\int_0^\infty du\frac{u^{d+z-4}}{u^{2z}+1}\\
&\sim T^{\frac{d+3z-3}{z}} \qquad (59)
\end{aligned}
$$

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scattering integral.

$$\begin{aligned}
&\gamma_{\boldsymbol{q}}[1-M]_{\boldsymbol{qp}}\delta f_{\boldsymbol{p}}\\
=&\int\frac{d\boldsymbol{p}}{(2\pi)^3}\left[\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}g(|\boldsymbol{q}|)\boldsymbol{q}.\boldsymbol{E}-\gamma_{\boldsymbol{pq}}\frac{1-f_{\boldsymbol{q}}}{1-f_{\boldsymbol{p}}}g(|\boldsymbol{p}|)\boldsymbol{p}.\boldsymbol{E}\right]\\
=&\int\frac{d\boldsymbol{p}}{(2\pi)^3}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}\left[g(|\boldsymbol{q}|)\boldsymbol{q}.\boldsymbol{E}-\frac{f_{\boldsymbol{q}}(1-f_{\boldsymbol{q}})}{f_{\boldsymbol{p}}(1-f_{\boldsymbol{p}})}g(|\boldsymbol{p}|)\boldsymbol{p}.\boldsymbol{E}\right]\\
=&\int\frac{d\boldsymbol{p}}{(2\pi)^3}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}\left[g(|\boldsymbol{q}|)\boldsymbol{q}.\boldsymbol{E}-\frac{\partial_\epsilon f_{\boldsymbol{q}}}{\partial_\epsilon f_{\boldsymbol{p}}}g(|\boldsymbol{p}|)\boldsymbol{p}.\boldsymbol{E}\right]\\
=&\ \delta f_{\boldsymbol{q}}\int\frac{d\boldsymbol{p}}{(2\pi)^3}\gamma_{\boldsymbol{qp}}\frac{1-f_{\boldsymbol{p}}}{1-f_{\boldsymbol{q}}}\left[1-\frac{\partial_\epsilon f_{\boldsymbol{q}}}{\partial_\epsilon f_{\boldsymbol{p}}}\frac{g(|\boldsymbol{q}|)\boldsymbol{q}.\boldsymbol{E}}{g(|\boldsymbol{p}|)\boldsymbol{p}.\boldsymbol{E}}\right]
\end{aligned}$$

The final manipulation to get this to the same form as above to us the fact that within the relaxation-time approximation $g(\boldsymbol{p})=\partial_\epsilon f_{\boldsymbol{p}}/m\gamma_{\boldsymbol{p}}^{\rm tr}$.

[18] The heat sink is strictly necessary to permit the formation of a steady state. It will not appear explicitly in our analysis, which will be essentially linear response for the electrons. The actual answer will turn out to be nonlinear in the electric field because of the field dependence of the magnon-electron scattering rate. For an interesting discussion of the role of the heat-sink in conductivity measurements see A.-M. Tremblay, B. Patton, P. C. Martin, and P. F. Maldague, Phys. Rev. A **19**, 1721 (1979).

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