BOUNDARY-LAYER EFFECTS
IN LIQUID-LAYER FLOWS

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ABSTRACT

In this thesis we describe various regimes of practical and theoretical significance that arise in the laminar two-dimensional flow of a layer of an incompressible viscous fluid over a solid surface at high Reynolds number. In Part I we consider steady flows over a distorted rigid surface. Almost uniform flows are considered first, when the distortion is sufficient to provoke a viscous-inviscid interaction, and therefore boundary-layer separation. The two cases of supercritical and subcritical flow have quite distinct features, and are discussed separately. The governing equations in each case require a numerical treatment in general, but analytical progress has been made in certain important regimes e.g. when the distortion is relatively small and linearisation of the problem is possible. Next, the grossly separated motion of fully-developed flows over large obstacles, with dimensions of the order of the depth of the liquid layer, is studied on the basis of inviscid Kirchhoff free-streamline theory. Some comparisons of the theory with recent experiments are also given. In Part II we discuss unsteady and instability aspects of two-dimensional flow over a flat surface. It is shown that viscous and mean flow effects can combine to give instability in some cases, whereas previous studies have only found viscous effects to be stabilising. Unsteadiness of a two-layer fluid flow, with fluids of different viscosity and density, and incorporating surface tension effects, is also discussed. In Part III, deviating from the above theme slightly, we discuss briefly the steady, high-Reynolds-number flow in an asymmetric branching channel, again in the context of viscous-inviscid interactions. The asymmetry is found to force a large-scale response both up- and downstream of the start of the bifurcation.
The aim is to find the pressure distributions on the channel walls and on the dividing body. This requires the use of a Wiener-Hopf technique in view of the mixed boundary conditions.
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CHAPTER ONE

Introduction.
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Introduction.

Liquid-layer flows are widespread in nature. Common examples involving water are found in rivers, along irrigation channels, down spillways, under sluice-gates, over weirs, and in the kitchen sink. The types of behaviour often exhibited in these flows are easily observed and are intriguingly diverse. These include the hydraulic jump, the undular bore, lee waves downstream of an immersed obstacle, breaking waves on a beach, turbulent motion, and the solitary wave, to name but a few. Such flows have an obvious practical importance in engineering problems, in hydraulics and in industrial chemistry, for example. The study of liquid-layer flows is therefore an important part of fluid dynamics; on the theoretical side, the diversity of phenomena presents an exciting and difficult mathematical challenge.

All the examples of water flows mentioned above, and many other liquid-layer flows, may be modelled to a high degree of accuracy as the motion of an incompressible Newtonian fluid, bounded below by a solid surface (the channel bed) and above by either a free surface or a fluid-fluid interface. Typically, we are dealing with an air-water interface, but other two-fluid systems arise in some contexts. The incompressible Navier-Stokes equations therefore govern the flow field, and all that is required is a mathematical solution of these equations in order to reveal the physical mechanisms behind the various flow features. But the complexity of the equations means that simplifying assumptions and approximations are inevitable if analytical progress is to be made.
In many practical examples, the Reynolds number of a liquid-layer flow is large. So a useful working assumption is to suppose that the fluid is effectively inviscid: see, for example, Lamb (1932), Lighthill (1978). An inviscid approximation to the flow solution is only appropriate, however, if it gives the limit of the actual viscous flow when the Reynolds number becomes asymptotically large. An assumption implicit in all inviscid flow studies is that the action of viscosity is only important in certain narrow regions, such as boundary layers or surface layers, which become vanishingly thin as the Reynolds number increases. It is argued that these viscous regions may therefore be neglected. A less stringent assumption is to suppose that the flow in the viscous regions determines a second-order correction to the predominantly inviscid fluid flow outside. That is the basis of the classical boundary-layer theory (Prandtl 1904). The only requirement for such an approach to work is an a priori knowledge of the location of the viscous regions. However, a central feature of the motion of a fluid with small viscosity, and one that ruins an inviscid approximation to the motion, is the phenomenon of boundary-layer separation. In the present context, separation may occur when, for example, the flow encounters irregularities on the channel bed, or it may be induced at either the free surface or the bed by the passage of waves on the free surface.

We turn next to laminar separation, then. A rational treatment of boundary-layer separation, that is, one which is consistent with the full Navier-Stokes equations, was first given independently by Stewartson & Williams (1969) and Neiland (1969), both related to the trailing-edge study of Messiter (1970). These important contributions to the subject were concerned with separation in a supersonic flow (essentially a compressible external flow), but the basic ideas are
applicable to other situations and have since been extended to subsonic and hypersonic flows, internal flows, and also liquid-layer flows, among others; see the reviews of Stewartson (1974), Messiter (1979) and Smith (1982). The crucial element in this "modern boundary-layer theory" is to account for a local interaction between the viscous flow near a solid boundary, say, and the essentially inviscid flow just outside the boundary layer. The interaction is crystalised in a relation between the pressure driving the flow in the viscous region and the displacement of the inviscid flow. The pressure can only be determined by solving the viscous problem. A sizeable adverse pressure gradient can then lead to regular separation of the boundary layer. This is quite unlike the classical non-interactive boundary-layer theory which almost invariably breaks down with the Goldstein (1948) singularity when the viscous flow is driven by an adverse pressure gradient (Stewartson 1970). Further, we may identify two types of separation: small-scale separation, which occurs in the vicinity of a small disturbance, such as a slight change in the wall conditions compatible with the scalings of the interactive structure (and therefore usually Reynolds-number dependent); and large-scale separation, which is provoked by a sizeable disturbance (e.g. an obstacle of finite dimensions) to the flow. The second of these is often the more relevant from a practical point of view, of course, because while the Reynolds number of a flow may easily be increased, the geometry is usually fixed.

The above is for steady flow. In unsteady flows, viscosity can play an important role even when separation does not take place. Usually, a laminar boundary layer is expected to dissipate energy, so that viscosity acts to stabilise a flow. But the action of viscosity is not always so easy to predict. For example, the boundary layer on a
flat plate is unstable to arbitrarily small disturbances, even though the equivalent inviscid flow is stable. The action of viscosity is clearly a mechanism for the transition to turbulence in many flows. In a liquid layer, the passage of waves on the surface provokes an oscillatory boundary layer on the channel bed, and this too may lead to an instability.

The concern in this thesis, therefore, is with aiming to understand more the effects of viscosity, and viscous-inviscid interactions in particular, in both steady and unsteady liquid-layer motions.

Our discussion of liquid-layer flows in the thesis is divided into two parts: Part I (chapters 2 to 4) concerns steady flows, and in particular addresses both small-scale and large-scale separation due to bottom topography, and also the laminar hydraulic jump; in Part II (chapters 5 and 6) we consider viscous instabilities in unsteady flows. (Part III of the thesis concerns a rather different topic; see below.)

We begin in Part I with a discussion of the small-scale separation of a liquid-layer flow over an obstacle on the channel bed (chapter 2), the motion in the liquid layer being assumed to be two-dimensional, laminar, and predominantly irrotational. The situation is then similar to the classical inviscid model of flow over bottom topography (see, for example, the books by Henderson (1966), Lamb (1932), Lighthill (1978), and others). The classical model explains the alternative elevation or dipping of the free surface over the immersed obstacle, according to whether the flow is supercritical or subcritical, that is, whether the Froude number is greater than or less than unity. However, according to the simple inviscid theory, upstream influence is only possible in subcritical flow, and not in
supercritical flow. One consequence of this is the implausible prediction that there is no adjustment of a supercritical flow upstream of even a very severe disturbance. The situation changes, however, when viscous effects are included, and indeed in some sense it reverses. Within an interactive framework, upstream influence can occur in a supercritical stream via the slower-moving layers of fluid near the channel bed, and hence throughout the fluid by means of the viscous-inviscid interaction. For sufficiently large obstacles, then, upstream separation of the boundary layer is a possibility in supercritical flow, according to the modern boundary-layer approach, and that is in line with our intuitive expectations. Moreover, contrary to the inviscid theory, there is more upstream influence in a supercritical than in a subcritical flow, due to the action of viscosity.

Mathematically, the problem of small-scale separation reduces to solving the interactive boundary-layer equations, in which the pressure is proportional to the local boundary-layer displacement, for flow over a prescribed wall indentation. (In the case of supercritical flow, these equations are identical with those of a limit of hypersonic flow (Brown, Stewartson & Williams 1974/75, 1975).) In §2 of chapter 2 we describe a numerical integration of the equations for the supercritical flow over a forward-facing step, for various values of the step height, and in §3 we do the same for the subcritical flow over an isolated hump. In both cases, we generate separated flow solutions. The ultimate aim is to find the trends of the separated flows as the disturbance size increases. In the supercritical case, we find that the point of separation is pushed increasingly far upstream as the height of the step increases, and the corresponding free interaction studied by Smith & Gajjar (1983) and Brown, Stewartson &
Williams (1975) seems to be emerging there.

The free interaction in a supercritical stream is believed to be relevant to the laminar hydraulic jump commonly observed in the kitchen sink, where there is a spontaneous jump from supercritical to subcritical flow conditions, and which therefore occurs without a change in wall conditions. The simple classical inviscid theory cannot give a rational account of this phenomenon. The associated energy loss across the jump is often attributed to turbulence. But not all hydraulic jumps are turbulent (although in rivers they usually are), and the energy loss in a laminar jump must be due to viscous dissipation in the boundary layer. In chapter 3 we give a quantitative comparison between the theory of Gajjar & Smith (1983) and the experiments of Craik, Latham, Fawkes & Gribbon (1981). The asymptotic theory turns out to give a surprisingly good prediction of the free-surface profile just beyond the start of the jump.

In long channels, such as rivers or aqueducts, the assumption that the flow is predominantly irrotational seems rather inappropriate. The effect of shear is considered in chapter 4, as well as in the unsteady flows considered in Part II. In chapter 4 we consider the large-scale separation of a steady laminar flow over a simple hump with dimensions comparable with the depth of the fluid. We give a description based on Kirchhoff free-streamline theory, which turns out to be consistent with boundary-layer separation. Separation occurs both upstream of the hump and on the surface of the hump, and the separating streamlines enclose regions of slowly moving fluid. The downstream region grows linearly with the Reynolds number $R$, and the upstream region is also long, of length $\ln(R)$. A similar structure holds in symmetrically constricted tubes and asymmetrically constricted (closed) channels (Smith 1979, Smith & Duck 1980).
comparison of the theory of separating flows over relatively long humps with the experiments of Huppert & Britter (1982) is also presented, and shows qualitative agreement.

In Part II we discuss some unsteady aspects of liquid-layer flows with either fully-developed or partly-developed velocity profiles. We suppose the bed of the channel to be flat, and consider the development of small amplitude "free waves" on the surface. In chapter 5 we extend the classical KdV theory to include both slight viscous effects and the non-zero mean flow. Solutions of the classical KdV equation, which governs the evolution of the displacement of the free surface in an inviscid fluid, include the solitary wave and the periodic cnoidal wave (Korteweg & de Vries, 1895). The stability of the inviscid solitary wave has been demonstrated by Benjamin (1972). The attenuation of the wave motion by viscous dissipation has been calculated by Keulegan (1948) and by Kakutani & Matsuuchi (1975). However, these authors suppose the fluid in the channel to be at rest, apart from the weak motion induced by the waves. We show that the interaction between a pre-existing mean flow and the small viscous effects can sometimes, but not always, cause growth of the solitary wave, leading to an eventual breakdown of the governing viscosity-modified KdV equation within a finite time. The discussions of the finite-time breakdown are also discussed.

In chapter 6 we analyse the possible viscous instability of a laminar two-fluid flow, which may represent a simple two-dimensional model of air blowing over shallow water, for instance. Viscous effects tend to be destabilising here, as in the case of Tollmien-Schlichting waves in boundary layers, whilst surface tension and density stratification tend to stabilise short waves and long waves respectively. In some circumstances, depending on the physical
properties of the two fluids and on the external "wind" speed, there may be only a narrow band of unstable waves. The nonlinear evolution of such a band, or wave packet, is examined, and it is found that there is usually a breakdown of the relevant evolution equation (a Ginzberg-Landau equation) within a finite time, with unstable waves becoming narrower and increasing in amplitude. Such nonlinear instabilities occur in an air-water system only for rather high wind velocities of about 15 mph (force 4 on the Beaufort scale), but would occur at much lower "wind" speeds when the two fluids are of nearly equal density.

We analyse rather a different flow in Part III of the thesis, viz. the high-Reynolds-number flow in a two-dimensional asymmetrically bifurcating (closed) channel, again in the context of viscous-inviscid interactions. Such a flow has a clear practical significance in physiological flows (Pedley 1980) and in many engineering problems, although, of course, the more realistic three-dimensional counterpart is likely to produce important effects, such as secondary motion, that the simple two-dimensional model cannot predict. The aim is to find the pressure distributions on the channel walls, the effects upstream of the bifurcation, and the flux of fluid in the downstream channels. The asymmetry of the geometry is found to force a response on a long $O(R^{1/7})$ length scale both up- and downstream of the start of the bifurcation, in common with other asymmetric disturbances in channels (Smith 1976a). Separation from one of the upstream channel walls is then a distinct possibility.

A short section summarizing the main conclusions of each chapter is presented at the end of the thesis.

Throughout this thesis we write the incompressible Navier-Stokes equations in the non-dimensional form
\[ u_t + uu_x + vu_y = -p_x + \frac{1}{R} (u_{xx} + u_{yy}) ; \]

\[ v_t + uv_x + vv_y = -p_y - \sigma + \frac{1}{R} (v_{xx} + v_{yy}) ; \]

the equation of continuity is

\[ u_x + v_y = 0 . \]

We will give the relevant non-dimensionalisation at the beginning of each chapter. In general the Reynolds number \( R = UL/\nu \) is based on a typical length scale \( L \), velocity \( U \), and the kinematic viscosity \( \nu \) of the fluid. The inverse Froude number \( \sigma = Fr^{-1} = gL/U^2 \) where \( g \) is the acceleration due to gravity. The coordinates \( x, y \) are perpendicular and parallel to \(-g\) respectively, and the corresponding velocity components are \( u, v \). We shall often work with the stream function \( \psi \), so that

\[ u = \psi_y , \quad v = -\psi_x . \]

The steady flows of Parts I and III have \( \psi_t = 0 \), of course. Also, in Part III we write the equations in terms of the modified pressure, so that \( \sigma = 0 \), and then \( x \) and \( y \) are parallel and perpendicular to the channel walls respectively. Finally, we give the free surface conditions relevant to the work in chapters 2 to 5: if the free surface is given by \( y = \eta(x,t) \), and if surface tension is negligible, then

\[ v = \eta_t + u\eta_x , \]

\[ (u_y + v_x)(1 - \eta_x^2) - 2(u_x - v_y)\eta_x = 0 , \]

\[ p - \frac{2}{R(1 + \eta_x^2)} (u_x\eta_x^2 - (u_y + v_x)\eta_x + v_y) = 0 \]

at \( y = \eta(x,t) \).
PART I

Steady liquid-layer flows.
CHAPTER TWO

Small-scale separation in an undeveloped stream.
CHAPTER TWO

§1 Introduction.

We consider the steady laminar two-dimensional high-Reynolds-number flow of a liquid layer past an obstacle on an otherwise flat horizontal wall, with almost zero vorticity everywhere except in a thin boundary layer adjacent to the wall: see the definition sketch in figure 1. Strictly speaking the boundary layer is thin as long as its development length is $o(t^*Re)$ in the limit as $Re \to \infty$; here

$$Re = \frac{U^* t^*/\nu}{(1.1)}$$

is the Reynolds number of the flow based on the liquid depth $t^*$ and the uniform stream $U^*$ outside the boundary layer, and $\nu$ is the kinematic viscosity of the fluid. When the obstacle is of a reasonably simple form, with only one streamwise (horizontal) length scale $t^*L$ and one height scale $t^*H$, and is sufficiently regular, there are four independent parameters affecting the flow in the high Reynolds number regime. These are the relative dimensions of the obstacle ($L$ and $H$), the oncoming boundary-layer thickness (a fraction $\varepsilon$, say, of the fluid depth), and the Froude number $Fr = U^2/g t^*$. In this discussion we confine our attention to obstacles which are long relative to the depth of the fluid, and which lie well within the boundary layer i.e. $L < 1$ and $H < \varepsilon$. Such obstacles are expected to force a response throughout the flow on an equally long length scale. The disturbance to the uniform stream outside the boundary layer is then of a simple inviscid kind, and in particular the pressure varies hydrostatically there, to leading order, due to the deformation of the free surface: see equation (1.7a). A complete analysis of the motion must take into account the non-parallel part of the basic flow, which
is due to the diffusion of vorticity from the wall, of course. If, however, we concern ourselves with zeroth-order solutions only these non-parallel effects are negligible as long as \( L < Re \), that is, if the length scale of the disturbance is much shorter than the development length of the boundary layer. So we concentrate on obstacles with

\[ 1 < L < Re. \]  \hspace{1cm} (1.2)

The flow may be analysed in terms of a three-layered structure in the vertical coordinate \( y \): see figure 2. The disturbances in the outer region I and the majority of the boundary layer II are governed by inviscid dynamics, the difference being that in I the flow is irrotational while in II it is rotational. Viscous forces come into play in a wall layer, region III, of (unknown) thickness \( \delta_w \), say, relative to the oncoming boundary layer. We wish to determine the relative magnitude of the obstacle height \( H \) that provokes a nonlinear response in III, for then boundary-layer separation is a distinct possibility, amongst other physically important effects. Due to the oncoming boundary layer we expect a streamwise velocity \( u \) of order \( \delta_w \) here, and hence inertia and viscous forces of order \( \delta_w^2 L^{-1} \) and \( (\delta_w \varepsilon^2 Re)^{-1} \) respectively. These are comparable when \( \delta_w \sim (L/\varepsilon^2 Re)^{1/3} \). So obstacles of height comparable with the wall layer thickness, \( H \sim \varepsilon \delta_w \), force a nonlinear-viscous response in III. Moreover, a viscous-inviscid interaction occurs between the regions I and III if the pressure perturbations are of the same size in both. In III the pressure force balances inertia and viscous forces when \( p \sim \delta_w^2 \). Region II is displaced vertically by an amount \( O(\varepsilon \delta_w) \) due to the thickness of region III. This displacement effect is transmitted to region I, where a pressure perturbation also of order \( \varepsilon \delta_w \) is produced due to the hydrostatic response there. Thus the two pressures in I and III are comparable when \( \varepsilon \delta_w \sim \delta_w^2 \) i.e. when
\[ \delta W \sim \varepsilon. \] In terms of \( \varepsilon \), then, we consider obstacles with dimensions
\[ L \sim \varepsilon^3 \text{Re}, \quad H \sim \varepsilon^2. \] (1.3)

The above restrictions on \( L \) require
\[ \text{Re}^{-1/3} \ll \varepsilon \ll 1. \] (1.4)

Suppose that the obstacle shape is given by
\[ y = \varepsilon^2 \bar{h} G(X) \] (1.5)
where \( x^2/\varepsilon^2 = LX, \ y^2/\varepsilon^2 = y \), and the coordinate system \( Ox^x'y^x \) is fixed in the vicinity of the obstacle. \( G(X) \) may represent an isolated hump, a step, or a ramp, for example; \( \bar{h} \) is an \( O(1) \) height (or slope) scale of the obstacle. According to the arguments given above the flow in I is given in non-dimensional form by
\[ \psi = y + \varepsilon^2 [A(X) - yP(X)] + o(\varepsilon^2), \] (1.6a)
\[ p = \varepsilon^2 P(X) + o(\varepsilon^2), \] (1.6b)
where the stream function \( \psi \) and the (modified) pressure \( p \) have been non-dimensionalised with respect to \( \varepsilon^2 U^x \) and \( \rho U^x^2 \) respectively. The conditions of zero normal velocity and zero stress at the (unknown) free surface \( y = 1 + \varepsilon^2 \eta(X) + o(\varepsilon^2) \) give, in turn,
\[ P(X) = \eta(X)/\text{Fr}, \] (1.7a)
\[ (P-A)(X) = \eta(X). \] (1.7b)
The first of these gives the expected hydrostatic variation of pressure. Next, in II, where \( \bar{Y} = y/\varepsilon = O(1) \), the solution which matches with I is given by
\[ \psi = \varepsilon \psi_0(\bar{Y}) + \varepsilon^2 A(X) \psi_0'(\bar{Y}) + o(\varepsilon^2) \] (1.8)
with (1.6b) again holding for the pressure. Here the undisturbed boundary-layer flow is given by \( \psi = \varepsilon \psi_0(\bar{Y}) \) where \( \psi_0(\bar{Y}) \to 1 + O(\exp) \) as \( \bar{Y} \to \infty \) and \( \psi_0 \sim \lambda \bar{Y} + \ldots \) as \( \bar{Y} \to 0; \lambda \) is the shear stress at the wall of the oncoming boundary layer. Finally in III, where \( Y = y/\varepsilon^2 \) is \( O(1) \), we set
\[ (u, \psi, p) = (\varepsilon U(X,Y), \varepsilon^3 \psi(X,Y), \varepsilon^2 P(X)) + \ldots \] (1.9)
(where we have anticipated \( p_y = 0 \) from the \( y \)-momentum equation).

With these expansions the Navier-Stokes equations reduce to the boundary-layer equations

\[
UU_x - U_x U_y = -P'(X) + U_{yy}, \quad U = \psi \tag{1.10a}
\]

in III. The boundary conditions are

\[
U = \psi = 0 \quad \text{at} \quad Y = \tilde{h}G(X), \quad \tag{1.10b}
\]

\[
U \to \lambda(Y + A(X)) \quad \text{as} \quad Y \to \infty, \quad \tag{1.10c}
\]

\[
(U, \psi, P, \lambda) \to (\lambda Y, \lambda Y^2, 0, 0) \quad \text{as} \quad X \to -\infty \tag{1.10d}
\]

for no slip at the wall, matching with II, and matching with the oncoming flow respectively. Also, from (1.7a,b) we have the interaction law

\[
P(X) = -\frac{A(X)}{(Fr - 1)}. \quad \tag{1.10e}
\]

The specification of the problem is completed by imposing suitable conditions far downstream (as \( X \to \infty \)) which depend on the geometry of the boundary. We shall discuss these boundary conditions at the end of the introduction.

The parameters \( \lambda \) and \( Fr \) may be scaled out of the lower deck problem by the transformation

\[
(X, Y, U, \psi, P, \lambda) \to (\lambda^{-1} \beta^2 X, \beta Y, \lambda \beta U, \lambda \beta^2 \psi, \lambda^2 \beta^2 P, \beta \lambda) \tag{1.11}
\]

where \( \beta = (\lambda^2 |Fr - 1|)^{-1} \). Finally, applying the Prandtl transposition theorem \( (U_p(X, Y_p) = U(X, Y), \quad \psi_p(X, Y_p) = \psi(X, Y_p) \) with \( Y_p = Y - \tilde{h}G(X) \) the equations become

\[
UU_x - U_x U_y = -P'(X) + U_{yy}, \quad U = \psi \tag{1.12a}
\]

\[
U = \psi = 0 \quad \text{at} \quad Y = 0, \tag{1.12b}
\]

\[
U \to Y + A(X) + hG(X) \quad \text{as} \quad Y \to \infty \tag{1.12c}
\]

\[
(U, \psi, P, \lambda) \to (Y, \beta^2 Y^2, 0, 0) \quad \text{as} \quad X \to -\infty \tag{1.12d}
\]

\[
P = -A \quad \text{for} \quad Fr > 1 \tag{1.12f}
\]

\[
P = +A \quad \text{for} \quad Fr < 1 \tag{1.12g}
\]
(dropping the subscript \( p \) for clarity) where \( h = \beta^{-1} \). Thus for each value of \( h \) there are just two distinct problems: *supercritical flow* (when \( Fr > 1 \)) and *subcritical flow* (when \( Fr < 1 \)). (A third case with \( Fr = 1 \) will not be discussed here.) These problems are addressed separately in the next two sections. They require numerical treatments in general, although analytical progress can be made in certain special cases e.g. when \( h \) itself is small.

The above flow structure was first derived by Gajjar & Smith (1983) although they considered free interactions; that is, non-trivial solutions when \( G(X) \neq 0 \). For eigensolutions starting in the form

\[
P \sim b e^{\kappa X} \quad \text{as} \quad X \to -\infty \quad \text{with} \quad \kappa = (-3A_{i}'(0))^3 \approx 0.4681
\]

are possible when \( Fr > 1 \) (whereas for \( Fr < 1 \) no such eigensolutions exist). Here \( b \) is unknown and depends on the ultimate downstream form. The above authors calculated the subsequent development of a disturbance starting with this form (with \( b > 0 \) so that the flow separates; see also Brown, Stewartson & Williams 1975) and showed theoretically that then, far downstream,

\[
P \sim P_1(X-X_s)^m \quad \text{as} \quad X - X_s \to \infty
\]

with \( m = \frac{2}{3} (-2 + \sqrt{7}) = 0.43050... \)

where \( X_s \) is the position of separation, a prediction supported by their numerical results. This free interaction is believed to describe the flow separation far ahead of an obstacle larger than the one given by (1.5), as mentioned later. One aim, then, of the work presented in \( \S 2 \) is to see if the free interaction does emerge far ahead of the obstacle (1.5) as its height \( h \) increases. Other features of the large-scale separated flow, such as reattachment and the ultimate free surface shape, are also of interest. To this end, we consider for the most part flow over the tanh step, defined by
\[ G(X) = \%\left( 1 + \tanh(X) \right) \quad (1.15) \]

for which separation is expected to occur only upstream of \( X = 0 \), with reattachment occurring on the forward face or the upper level of the step — see the sketch in figure 3. For this step the downstream boundary conditions are

\[(U, f, P, A) \rightarrow (Y, \%Y^2, h, -h) \text{ as } X \rightarrow \infty \quad (1.12e)\]

so that the flow returns to the oncoming uniform shear, apart from the lateral displacement due to the height of the step.

In §3 we discuss the subcritical case with particular reference to flow over an isolated hump (see equation (3.1)). The downstream boundary conditions appropriate to that problem are

\[(U, f, P, A) \rightarrow (Y, \%Y^2, 0, 0) \text{ as } X \rightarrow \infty \quad (1.12e')\]

Because of the lack of upstream influence, separation only takes place on the surface of the hump. Some analytical progress is possible in the limit \( h \rightarrow \infty \), and it suggests that, in that limit, separation takes place by means of a removable Goldstein singularity. Numerical solutions of separated flows are also given, and these tend to support the proposed (large \( h \)) structure of separation, although a complete description of the reattachment process downstream of the hump has not yet been found.
§2 Supercritical flow.

Linear theory when \( h \) is small.

As remarked earlier analytical progress can be made when the parameter \( h \ll 1 \), as the governing equations (1.12a–f) may then be linearised. At the very least the linear theory can serve as a useful check on the numerical schemes described later. In fact, it does much better than that, predicting some general flow features of the fully nonlinear problem for quite a wide range of values of \( h \).

Perturbing from the basic flow, we set

\[
(U, \xi, \psi, A) = (Y, \xi Y^2, 0, 0) + h(\tilde{U}, \tilde{\psi}, \tilde{A}) + O(h^2)
\]

as \( h \to 0 \). Substituting these expansions into the equations (1.12) we obtain a linear system at \( O(h) \), which may be solved for the Fourier transforms defined by

\[
\tilde{u}^*(\alpha, Y) = \int_{-\infty}^\infty u(X, Y) e^{-i\alpha X} \, dX \quad \text{etc.}
\]

We find, in fact, that

\[
\dot{\tilde{u}}^*(\alpha, Y) = \int_{-\infty}^\infty u(X, Y) e^{-i\alpha X} \, dX \quad \text{etc.}
\]

We find, in fact, that

\[
\hat{P}^*(\alpha) = -\hat{A}^*(\alpha) = \frac{Q^*(\alpha)}{1 - (i\alpha/\kappa)^{1/3}}
\]

where \( \kappa \) is given in (1.13) and \( Ai \) is Airy's function. Also the function \( (i\alpha)^{1/3} \) has a branch cut from \(+0i\) to \(+\infty\) in the complex \( \alpha \)-plane and its argument lies between \(-3\pi/2\) and \(\pi/2\). (The contour of the inverse transform integral then lies just below the \( \text{Real}(\alpha) \) axis.) The transforms may be inverted using convolutions; for example

\[
\hat{P}(X) = 3\kappa \int_0^\infty G(X-\xi) e^{i\xi} \, d\xi - \frac{\sqrt{3} \kappa}{2\pi} \int_0^\infty \int_0^\infty \frac{e^{-\xi} G(X-\xi)}{1-\kappa^{1/3}+s^{2/3}} \, dsd\xi
\]

but in general it is more straightforward to invert the transforms.
directly using a Fast Fourier Transform (FFT) algorithm.

As an example of flow over a forward-facing step, consider the disturbance produced by a simple abrupt rise in the wall, for which

\[ G(X) = \begin{cases} 
0 & \text{for } X < 0 \\
1 & \text{for } X > 0
\end{cases} \quad (2.5) \]

The leading-order pressure perturbation is then

\[ \tilde{p}(X) = \begin{cases} 
3e^{\kappa x} & \text{for } X < 0, \\
1 + \frac{\sqrt{3}}{2\pi} \int_{-\infty}^{0} \frac{e^{-\kappa x s} \, ds}{s^{2/3}(1-s^{1/3}+s^{2/3})} & \text{for } X > 0.
\end{cases} \quad (2.6) \]

(Notice the pressure variation is continuous at \( X = 0 \) even though the wall shape is not. The pressure gradient is irregular there, however. Its exact behaviour is of no concern to us because, for the most part, we will be considering the flow over smooth steps.) In scaled terms, then, we have the interesting result that an abrupt step of height \( h \) can produce a pressure rise, and hence a displacement of the free surface, of \( 3h \), three times the step height.

Next, we can find the asymptotic form of the pressure as \( X \to -\infty \). When \( (-X) \gg 1 \) the behaviour of the pressure is dominated by the pole in its transform at \( \alpha = -ix \) (as long as \( G(X) \) decays to zero sufficiently fast as \( X \to -\infty \)). Then

\[ \tilde{p}(X) \sim 3\kappa G^x(-ix)e^{\kappa x} \quad \text{as } X \to -\infty. \quad (2.7) \]

For comparison with later numerical results for the fully nonlinear problem we calculate the linearised solution for flow over the smooth tanh step given by (1.15). This function has the Fourier transform

\[ G^x(\alpha) = \frac{\pi}{2i\sinh(\kappa\alpha\pi)}. \quad (2.8) \]

Results for the pressure and wall shear \( \tau \), obtained by using the FFT, are given in figure 4. The up- and downstream asymptotes for the pressure are given by
The results from the FFT were (partially) checked by comparing the value that it returned for \( \tilde{p}(0) \) with the value obtained from the real integral representation

\[
\tilde{p}(0) = \frac{1}{2} + \frac{4}{\pi} \int_0^\infty \frac{s^{1/3} \, ds}{(s^{2/3} - \sqrt{3}s^{1/3} + 1) \sinh(\pi s)}
\]

This integral was evaluated using Simpson's rule giving \( \tilde{p}(0) = 2.2321\), in good agreement with the value of \( \tilde{p}(0) \) in figure 4. Furthermore, the asymptotes (2.9a,b) are reproduced well by the numerical inversion of (2.3a).

These linearised solutions reveal some important, and physically sensible, features of the flow. Firstly, there is a strong adverse pressure gradient, and a corresponding fall in the wall shear, ahead of the step. Secondly, the minimum wall shear occurs just ahead of the step, so that separation is most likely to take place there first as \( h \) increases. Thirdly, the decay to the downstream form (1.12e) is very slow. We note also that there is good quantitative agreement between these results and those below of the full nonlinear problem for \( h \) up to about 0.7.

**First Numerical Procedure for \( h = O(1) \).**

The basic problem (1.12a-f) was solved using an iterative finite difference procedure based on a method used by Davis (1984) for solving subsonic and supersonic interactive boundary-layer flows over humps and ramps. The idea is to introduce time dependence into the problem, through a fictitious time derivative, and to solve an
initial-value problem in the hope that its solution will approach the steady solution of (1.12a-f). Accordingly, we replace the interactive law (1.12f) by

$$A = -P + P_{xt}. \quad (2.10)$$

In order to capture the elliptic nature of the problem an alternating direction method is used to solve (1.12a-e) with (2.10). The numerical procedure takes the following steps:

1. set an initial distribution for P at time $t = 0$;
2. calculate A and P at the next time step by integrating the boundary layer equations, marching forward in the X-direction;
3. find new values of P by integrating the interaction law (2.10), marching backwards in X, and using an appropriate downstream boundary condition;
4. return to (2) until convergence has been achieved.

Before describing steps (2)-(4) in detail, we explain why (2.10) was chosen - it was arrived at after a number of trials. Consider the stability of the 'flow' in the absence of any forcing (that is, the flow with the fictitious law (2.10) rather than an actual flow). A small disturbance to the basic flow proportional to $\exp[i(\alpha x - \omega t)]$, with real wavenumber $\alpha$, has the dispersion relation

$$\Omega = \frac{1}{\alpha} \left[ 1 - (i\alpha/\kappa)^{1/3} \right]. \quad (2.11)$$

Then $\text{Im}(\Omega) < 0$ for all wavenumbers $\alpha$, so that the disturbance decays and the flow is stable to small disturbances, at least. Moreover, for $\alpha > 0$, $\text{Re}(\Omega) \geq 0$ when $\alpha \lesssim (2\pi/\sqrt{3})^{3/5}$. This means waves can travel both up- and downstream, yielding the desirable property of influence in both directions. An alternative to (2.10) is given in (2.19) later on; other simple alternatives that we tried yielded instability. We should
also mention that another view of the computational stability question is given later in this section.

Step (2). The integration was started at \( X = X_{-m} < 0 \) using the upstream asymptotic form

\[
\begin{align*}
U(X,Y) & \sim -3x^{1/3}b\exp(x) \int_0^Y \text{Ai}[x^{1/3}\xi] \, d\xi \quad \text{as } X \to -\infty \tag{2.12a} \\
P(X) & \sim b\exp(x) \tag{2.12b}
\end{align*}
\]

there. \( P(X_{-m}) \) (and hence \( b \)) is known from the previous time step. The equations are written in finite difference form using the Keller box scheme (Keller and Cebeci 1971). Introducing the new variable \( \tau = U_Y \) the equations can be written in terms of first derivatives only. These are then discretised using four-point central differencing at the current \((n^{th})\) time level. (The scheme can then easily accommodate non-uniform steps in \( X \) and \( Y \).) The interaction law (2.10) is discretised in the following way:

\[
\frac{A_i^n + A_{i-1}^n}{2} = -\frac{P_i^* + P_{i-1}^*}{2} + \frac{(P_i^n - P_{i-1}^n) - (P_i^* - P_{i-1}^*)}{(X_i - X_{i-1})(\Delta t/2)} \tag{2.13}
\]

at the \( X \)-station \( X_1 \), where the superscript \( n \) denotes the time level, and \( \Delta t \) is the time step between time levels \( n \) and \( n+1 \). The starred quantities are the (known) pressure values at the last intermediate time level \( n-\frac{1}{2} \) obtained from the last reverse sweep (step (3) below, or step (1) initially). With the pressure-displacement law written in this way, we are, loosely speaking, solving the boundary-layer equations with \( A^n \) balancing \( +P^n \) (rather than \( -P^n \)) during each forward sweep; as such the free interaction, per sweep, is prevented so that marching forward is stable. The discretised system is solved using a Newton iteration procedure. The Newton linearised system requires the inversion of a matrix with six non-zero diagonals and two non-zero
columns (for the pressure and displacement) although the final column can easily be removed 'by hand'. During the first sweep the initial guess for the flow variables at each X-station are taken from the known solution at the previous X-station. The entire flow-field is stored, however, and during subsequent forward sweeps the initial guess is taken from the previous time step. As a result only one iteration is required at each X-station, for although at first this produces only $O(\Delta x)$ accuracy (where $\Delta x$ is the grid width), $O(\Delta x^2)$ accuracy will be achieved after a number of sweeps: see also the grid-size checks later. This speeds up the computations considerably.

**Step (3).** This step is computationally much easier, as only the interaction law (2.10) is integrated. Here we write

$$\frac{A_1^n + A_{i+1}^n}{2} = -\frac{P_1^* + P_{i+1}^*}{2} + \frac{(P_{i+1}^* - P_1^*) - (P_{i+1}^n - P_1^n)}{(x_1 - x_{i+1})(\Delta t/2)}. \quad (2.14)$$

Starting from a suitable downstream boundary condition (see below) this is solved in the reversed (-X) direction for the starred variables, which give the pressure at the next intermediate time level $n+\frac{1}{2}$. Quantities with superscript $n$ are assumed to be known from the last forward sweep. We note that this formulation is explicit in time.

**Step (4).** The procedure is deemed to have converged when, after a forward sweep,

$$\text{Max} \left| \frac{P_1^n - P_1^*}{(\Delta t/2)h} \right| < 0.001 \quad (2.15)$$

The factor of $h$ in the denominator is included to ensure graphical accuracy when $h$ is small.

Rather than using a prohibitively large number of grid points, we
took uniform steps of $\Delta \xi$ in the stretched variable $\xi$, defined by

$$X = \xi - s_1 \tanh(s_2 \xi).$$

(2.16)

We took $s_1 = 4.212$ and $s_2 = 0.9/s_1$ (giving a minimum step length of $\Delta \xi/10$ in $X$ near $X = 0$, and a maximum step length of $\Delta \xi$ far from $X = 0$). Again, to avoid the need for an excessively long computational grid, we replaced the downstream boundary condition (1.12e) by an asymptotic expression. As $X \to \infty$ we have

$$P \sim h + P_0 X^{-1/3} + O(X^{-2/3})$$

(2.17)

where the constant $P_0$ is unknown (but $P_0 \sim 0.9511h$ as $h \to 0$ from (2.9b)). This condition was applied in the form

$$P'(X) \sim \frac{1}{3} \left[ \frac{h-P(X)}{X} \right]$$

(2.18)

which was central differenced between the last two $X$-stations (and evaluated at the (relevant) intermediate time level).

For the record, we should mention two more details of the calculations. Firstly, for all but the smallest value of $h$, an initial guess was made for the entire flow field (not just the pressure as in step (1)). This was found by scaling up the flow field for the next biggest value of $h$ according to the linear theory. Secondly, the so-called Flare approximation was used (Reyhner & Flugge-Lotz, 1968) in regions of reversed flow. This is a minor point, however, because the scheme failed at the onset of all but the weakest reversed flow.

The computational grid used in the calculations is defined by the following parameters: $\Delta \xi$ and $\Delta \eta$, the (uniform) step lengths in $\xi$ and $\eta$; $I$ and $J$, the number of grid points in the $X$- and $Y$-direction respectively; and $X_{-\infty}$, the upstream boundary. We also write $X_\ast$ and $Y_\ast$ for $X_I$ and $Y_J$. The centred-difference solutions presented below were obtained using $\Delta \xi=1.2$, $\Delta \eta=0.6$, $I=73$, $J=26$ and $X_{-\infty}=-40.19$ (so that $X_\ast=37.79$ and $Y_\ast=15$). The results were checked by repeating the
<table>
<thead>
<tr>
<th>Y</th>
<th>Δγ</th>
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<th>$P_{X=0.0}$</th>
<th>$P_{X=19.8}$</th>
<th>$P_{X=-2.39}$</th>
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<tr>
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<td>3.5303</td>
<td>5.0806</td>
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Table 1a: effect of changing $Y$. Here $J=26$, $I=81$, $X_{-w}=49.8$, $X_w=37.8$, $Δt=1.2$, $h=2.7$.  

<table>
<thead>
<tr>
<th>I</th>
<th>$X_{-w}$</th>
<th>$X_w$</th>
<th>$P_{X=-15.0}$</th>
<th>$P_{X=0.0}$</th>
<th>$P_{X=15.0}$</th>
<th>$P_{X=-3.36}$</th>
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<td>-40.2</td>
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<td>3.0055</td>
<td>4.8729</td>
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<tr>
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<td>-49.8</td>
<td>37.8</td>
<td>3.0052</td>
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</tr>
<tr>
<td>81</td>
<td>-40.2</td>
<td>49.8</td>
<td>3.0029</td>
<td>4.8723</td>
<td>3.8432</td>
<td>0.9801</td>
</tr>
</tbody>
</table>

Table 1b: effect of changing $X$-range. Here $Δt=1.2$, $Δγ=0.6$, $Y=15$, $h=2.7$.  

<table>
<thead>
<tr>
<th>Δγ</th>
<th>$J$</th>
<th>$Y$</th>
<th>$P_{X=-19.8}$</th>
<th>$P_{X=0.0}$</th>
<th>$P_{X=19.8}$</th>
<th>$P_{X=-2.39}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
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<td>2.2807</td>
<td>4.8741</td>
<td>3.7565</td>
<td>5.9149</td>
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<tr>
<td>-6</td>
<td>26</td>
<td>15.0</td>
<td>2.2752</td>
<td>4.8729</td>
<td>3.7569</td>
<td>6.1296</td>
</tr>
<tr>
<td>-4</td>
<td>39</td>
<td>15.2</td>
<td>2.2717</td>
<td>4.8723</td>
<td>3.7575</td>
<td>6.2873</td>
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</tbody>
</table>

Table 1c: effect of changing $Δγ$. Here $I=73$, $Δt=1.2$, $X_{-w}=-40.2$, $X_w=37.8$, $h=2.7$.  

<table>
<thead>
<tr>
<th>Δt</th>
<th>I</th>
<th>$X$</th>
<th>$P_{X=-20.6}$</th>
<th>$P_{X=0.0}$</th>
<th>$P_{X=20.6}$</th>
<th>$P_{X=-2.39}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
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<td>4.4088</td>
<td>3.3609</td>
<td>3.3445</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>73</td>
<td>1.5018</td>
<td>4.4109</td>
<td>3.3606</td>
<td>3.3460</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>109</td>
<td>1.5034</td>
<td>4.4124</td>
<td>3.3604</td>
<td>3.3443</td>
<td></td>
</tr>
</tbody>
</table>

Table 1d: effect of changing $Δt$. Here $Δγ=0.6$, $Y=15$, $J=26$, $X_{-w}=-39.8$, $X_w=38.2$, $h=2.4$.  

Table 1
calculations with different grid parameters; comparisons are given in tables la-d. The comparisons suggest that rather a large upstream range is required, despite the exponential decay of the perturbation flow there. On the other hand, \( X_\infty \) need not be too large, even though the disturbance decays much more slowly as \( X \to \infty \). Also, the apparently coarse grid is quite adequate, at least for graphical accuracy, especially in view of the convergence criterion used in step (4). The artificial time step \( \Delta t \) was set to 0.4 in all the calculations, this value giving the most rapid convergence to the steady state for the smallest disturbance (for which \( h = 0.1 \)).

When the method works it gives rapid convergence, typically in about 25 global iterations or less. Its major drawback is that it was unable to cope with any significant reversed flow, either with or without the Flare approximation. Careful calculations showed that separation first occurs when \( h = 2.78 \); solutions were obtained for \( h \) up to 2.88. The results for the pressure \( P(X) \) and the skin friction \( \tau_w(X) (= U_Y(X,0)) \) for various values of \( h \) are presented in figure 5. The results, although rather limited, do show that the position of separation is pushed upstream as \( h \) increases beyond 2.78, as we would expect if the free interaction of Gajjar & Smith is to emerge as \( h \to \infty \). Further comments on the flow properties, and the numerical method and its stability, are made in the last part of this section after we have described two more numerical approaches to this problem. These approaches were undertaken partly for purposes of comparison, and as checks on accuracy, partly for their intrinsic interest, and partly also to find out if solutions with more reversed flow could be generated.
Second Numerical Procedure for \( h = O(1) \).

Another way of introducing artificial time dependence into the problem (1.12a-f) through the pressure-displacement law is by taking

\[ A_t = P_x + A_x. \]  
(2.19)

Again, this was found after some trial and error. As with the previous law (2.10), the unforced 'flow' described by (1.12a-e) with (2.19) is linearly stable and allows both up- and downstream influence; the dispersion relation for small disturbances proportional to \( \exp[i(\alpha X - \omega t)] \) here is given by

\[ \omega = \alpha \left( i\alpha /\kappa \right)^{1/3} - 1 \]  
(2.20)

so that \( \text{Im}(\omega) < 0 \) and \( \text{Re}(\omega) > 0 \) when \( \alpha \gtrsim (3\sqrt{3}/8) \kappa \). This alternative law provides the basis for this and the next numerical procedure, both of which execute steps (1)-(4) above, in essence; the major changes arise from alternative treatments of (2.19) in steps (2) and (3). Here we simply adapt the method just described. Thus in step (2) we write

\[ \frac{A^n_i - A^*_{i-1} + A^{n-1}_{i-1} - A^{*}_{i-1}}{2(\Delta t/2)} = \frac{A^*_{i} - A^*_{i-1}}{X_i - X_{i-1}} + \frac{P^n_i - P^{n-1}_i}{X_i - X_{i-1}} \]  
(2.21)

in place of (2.13), where again the starred quantities are the known displacement values at the previous intermediate time level \((n-\%\)). In step (3) we write

\[ \frac{A^*_{i+1} - A^n_{i+1} + A^*_{i} - A^n_{i}}{2(\Delta t/2)} = \frac{A^*_{i+1} - A^*_{i}}{X_{i+1} - X_i} + \frac{P^n_{i+1} - P^n_{i}}{X_{i+1} - X_i} \]  
(2.22)

in place of (2.14), which is solved for the starred variables giving the displacement function at the next intermediate time level \((n+\%\)). The only other differences between this and the first scheme are, firstly, that in step (1) an initial guess must be made for the displacement function (rather than the pressure), and, secondly, that \( A \) replaces \( P \) in the convergence criterion in step (4). In regions of reversed flow
both the Flare approximation and upwind differencing were used, but again this method was unable to cope with significant reversed flow, unfortunately.

Using the same computational grid as before, we obtained solutions for $h$ up to 2.9. We set $A_t = 6.0$ throughout, a surprisingly large value (see below), this value giving the most rapid convergence to the steady state when $h = 0.1$. For all values of $h$, convergence was even more rapid than before; typically, only 10 to 15 global iterations were required. The results were virtually identical with those already presented.

**Third Numerical Procedure for $h = O(1)$.**

An alternative, and perhaps simpler, approach to solving (1.12a–e) with (2.19) is to fix the displacement function $A$ during the forward march through the boundary layer (step (2)). The resulting pressure distribution can be used to update $A$ for the next forward march. Carter (1979) used a very similar approach to solve incompressible external interactive boundary-layer flows. We note that the boundary-layer equations with prescribed displacement are free from singularities (Catherall & Mangler 1966) so that forward marching (in step (2)) is possible. The alternative of prescribing the pressure gradient is virtually certain to give the classical Goldstein (1948) singularity at the onset of separation. Thus in step (2) we simply set

$$A_i^n = A_i^*,$$

(2.23)

and solve the steady boundary-layer equations at time level $n$ (giving $P_i$). During step (3) we discretise (2.19) exactly as before (see (2.22)). This is slightly different from Carter’s approach which is based on
comparing the pressure distributions which emerge from step (2) and from the interaction law (1.12f). Carter's original method was found to yield divergence, whereas the present method, essentially comparing the pressure-gradient distributions $P_X$ instead, as done in (2.19), proved to be convergent. The whole procedure is deemed to have converged when, after a forward sweep,

$$\max \left| P^n_i + A^*_i \right| < 0.001h.$$  \hspace{1cm} (2.24)

As in the second method above, the optimum value of the artificial time step $A_t$ for the smallest disturbance was 6·0. However, convergence to the steady state was considerably slower than before: about 50 global iterations were needed for smaller step heights (with $h < 2$), while the larger steps required up to 80. On the other hand, this method was considerably more successful than the others in the presence of reversed flow. The computational mesh used with the previous methods proved to be adequate for $h$ up to about 3·3. Beyond that, the upstream influence was so great that the asymptotic form (2.12) was not attained even at $X = -40·19$. Thus for larger values of $h$ an extended mesh was used, with $X_{-\infty} = -70·19$ (and $I = 98$). Solutions were obtained for values of $h$ up to 5·3; the results for the pressure and wall shear distributions are presented in figures 6a,b. We note that the upper-surface shape is proportional to the pressure $P$ in view of (1.7a). For $h$ up to 2·7, these results are virtually identical with those found by the previous methods; when $h = 2·7$ the calculated pressures agree to within 0·1%, in fact.
Further Comments.

These numerical solutions do show some clear trends as the height parameter $h$ increases, which is the regime of most theoretical and practical interest and difficulty as it produces grossly separated flow. Firstly, the upstream influence and the point of separation $X_s$ are pushed increasingly far upstream, as we might expect on physical grounds. The approach to the asymptotic form (2.12) occurs surprisingly far upstream, in fact. Figure 8a shows the variation of $X_s$ (found by linear interpolation) with $h$ in log-log form. Secondly, the free interaction of Gajjar & Smith does indeed seem to be emerging far ahead of the step. The comparisons of the pressure and wall shear distributions when $h = 5.3$ with their free interaction solutions, given in figure 7, show that the free interaction solution continues right up to the step. Thirdly, the rapid rise of $\tau_w$ on the forward face of the step forces reattachment to occur quite abruptly there. The point of reattachment moves downstream a fraction, but it seems reasonable at present to conjecture that it would continue to occur on the forward face of the step, rather than on top of it, as $h$ increases further. Thus, the flow seems to be beginning to split into two separate regions, even at these modest values of $h$: separation is occurring sufficiently far upstream that it is self induced, whilst reattachment is forced to occur in the vicinity of the step. We note, however, that the length scale of the free interaction is vast; Gajjar & Smith show in their figure 2a that $\tau_w$ is still decreasing when $X-X_s = 300$, even though $\tau_w \to 0$ as $X-X_s \to \infty$ according to the asymptotes associated with (1.14). So we cannot expect separation and reattachment to be distinct flow features until $h$ is increased well beyond the current range. Fourth, the trends in $P$ and $X_s$ versus the step-height
parameter $h$ (shown in figure 8a), although not conclusive, may give some guidance to a large-$h$ analysis of the governing equations (1.12). In figure 8a we plot $P$ at $X = -2.390$, at which point the wall shear is close to its minimum, and as such is the point where the step begins to exert its influence. One cannot be too cautious when interpreting log-log plots, of course, but the points for the pressure in figure 8a do lie close to a straight line with gradient 0.66 (close to $2/3$), approximately. If indeed $P \sim P_0 h^{0.66}$ as $h \to \infty$, as this suggests, then

$$-X_8 \approx [(P_0/P_1)h^{0.66}]^{1/m} (\approx O(h^{1.53})) \quad \text{as} \quad h \to \infty$$

from (1.14), a prediction which would be in good agreement with the behaviour of $X_8$ also shown in figure 8a. The approach to the asymptotic form (1.14) as $h \to \infty$ is shown more explicitly in figure 8b.

With regard to the relative merits of the three numerical schemes used here, we see that the first two, based on Davis’s (1984) method, are very efficient when the flow is everywhere forward, but they have so far proven virtually useless otherwise i.e. when reversed flow is present, whilst the third scheme, based on an adaptation of Carter’s (1979) method, is rather slow but does at least converge to a solution in the presence of reversed flow. This suggests that a hybrid scheme, incorporating both methods, might meet with even more success. A number of other quite different methods are available for solving interactive boundary-layer problems. A shooting method – in which $b$ in (2.12) is altered iteratively until the downstream form (2.18) is attained at $X_8$ – was also attempted here, and met with little success in the present context. Pseudo-spectral methods and other artificial-or real-time dependent methods (e.g. based on the full unsteady boundary-layer equations) are also in current use: see, for example, Rizzetta, Burggraf & Jenson (1978). Again, however, all such methods to date fail to compute grossly separating flow with satisfactory
As a final point, we note that the stability of the three numerical methods, as applied to the linearised version of (1.12a-e) with (2.10) or (2.19), may be investigated using von Neumann's method (see Roache 1972, for example). We write

$$\hat{u} = u - \hat{u} \text{ etc.,}$$

where $u$ (and $\psi, p, a$) is the solution of the linearised system and $\hat{u}$ etc. is the solution of the discretised system. Suppose the grid points are at $X_r = X_{-1} + (r-1)\Delta_x$ with $r = 1, 2, ..., N$. At these points, and at every time level, we may decompose $\hat{u}$ into $N$ Fourier components: at $t = q\Delta_t$ we have

$$\hat{u}_{r,q} = \hat{u}(X_r, Y, q\Delta_t) = \sum_{n=-N/2}^{N/2} \hat{u}_n(Y)V_q\exp[i\alpha_n r\Delta_x] \text{ etc.,} \quad (2.26)$$

where $\alpha_n = n\pi/N\Delta_x$, and $V_q$ is the amplitude factor at the $q^{th}$ time level. Strictly speaking, we should also decompose $\hat{u}$ in the $Y$ direction, but then little progress can be made. Substituting these Fourier expansions into the discretised, homogeneous, linearised boundary-layer equations we obtain

$$\left\{ \begin{array}{l}
Y \frac{(E - 1)\hat{u}}{\Delta_x} - \frac{(E - 1)\hat{\psi}}{\Delta_x} = - \frac{(E - 1)\hat{p}}{\Delta_x} + \frac{(E + 1)\hat{u}_{yy}}{2} \\
\hat{u} = \hat{\psi}_Y, \quad \hat{u} = \hat{\psi} = 0 \text{ at } Y = 0, \quad \hat{u} \rightarrow \hat{a} \text{ as } Y \rightarrow -\infty
\end{array} \right. \quad (2.27a-d)$$

for each Fourier component (dropping the subscript $(n)$ for clarity), where $E = \exp(i\alpha\Delta_x)$. In the usual way (e.g. as in (2.2)-(2.7)) the solution for $\hat{u}$ can be written in terms of Airy's function. From the boundary conditions we obtain the relation

$$\theta\hat{p} = -\hat{a}, \quad \text{with } \theta = \left( \frac{2\ell S}{\kappa\Delta_x C} \right)^{1/3}, \quad (2.28)$$

where $C = \cos(\%\Delta_x)$ and $S = \sin(\%\Delta_x)$. Now, for the first numerical method, from step (2) we have, in moving from time level $q-\%$ to $q$,
from (2.13), where \( d = \Delta_x \Delta_t / 4 \). So the relative increase in the amplitude of the errors during step (2) is

\[
G_f = \frac{V_q}{V_{q-1/2}} = \frac{(E - 1) + d(E + 1)}{(E - 1) + d(E + 1)}.
\]

During step (3) we move from time level \( q \) to \( q+\tau \), so that

\[
(V_{q+1/2} - V_q)(E - 1) = d \left( \Delta V_{q+1/2} + \Delta V_q \right)(E + 1)
\]

from (2.14). The relative increase here is

\[
G_b = \frac{V_{q+1/2}}{V_q} = \frac{(E - 1) - d(E + 1)}{(E - 1) - d(E + 1)}.
\]

Overall, then, the relative increase in amplitude in each cycle is

\[
G = G_fG_b = \frac{iS + dC}{iS + dC} \cdot \frac{iS - dC}{iS - dC}.
\]

The method is stable as long as \(|G| < 1 \) for all the Fourier modes i.e. for \( 0 < \pi \Delta_x < \pi \). Here

\[
|G|^2 = 1 - \frac{2d(\alpha/\kappa)^{2/3} \sin(\alpha \Delta_x)}{3d^2(\alpha/\kappa)^2/3C^2 + (d(\alpha/\kappa)^{1/3}C + 2S)^2},
\]

showing that this method is unconditionally stable. Applying this analysis to the second method results in the same expression (2.29) for \( G \), but with \( d \) replaced by \( \Delta_x / \Delta_t \), so this method is also unconditionally stable. For the third method we have

\[
G_f = 1,
\]

\[
G_b = \frac{(E - 1) - d(E + 1)}{\theta (E - 1) - d(E + 1)}
\]

where \( d = \Delta_x / \Delta_t \) here. Thus the method is stable when \(|G_b|^2 < 1 \) for \( 0 < \alpha < \pi / \Delta_x \), which, after some rearrangement, reduces to

\[
\phi(1 - \phi) < \frac{2d}{\kappa \Delta_x}, \text{ for } 0 < \phi < 1,
\]

where \( \phi = \left( \frac{2 \tan(\pi \Delta_x)}{\kappa \Delta_x} \right)^{2/3} \).

So this method is stable when
\[ \Delta t < \frac{8}{\kappa} \approx 17.09 \] (2.32)
giving only conditional stability. However, the maximum time step in (2.32) is large, much larger than the time step chosen in the above calculations (which had \( \Delta t = 6.0 \)), and it is also noteworthy that the condition (2.32) is independent of the grid width \( \Delta x \) unlike many familiar criteria of conditional stability.

We should emphasize that the above stability analysis of the three numerical schemes is not exact, and that the stability criteria will change as the basic flow alters. In particular, this rather simple analysis is not appropriate when reversed flow is present. Nevertheless, it is, at the very least, useful as a guide in formulating unsteady-flow problems with artificial time variables of the type presented in this section.
§3 Subcritical flow.

When the oncoming flow is subcritical, that is, such that Fr < 1, the flow in the lower deck, governed by (1.12a–e',g), has quite different properties from the corresponding supercritical-flow theory studied earlier. For a start, the equations admit no upstream influence, so that if the obstacle (1.5) is such that G(X) = 0 for X < 0 then the oncoming uniform shear flow (1.12d) will not be significantly disturbed ahead of X = 0. The disturbance there is only determined by the higher-order terms in the expansions (1.9). The equations governing these terms are linear, and as such separation is not possible ahead of the obstacle, in marked contrast to supercritical flow. However, on physical grounds we would expect the flow over an isolated hump to separate when the height parameter h is large enough. This possibility is investigated numerically in the present section. The numerical solutions are found for flow over a wall shape given by

\[
G(X) = \begin{cases} 
16X^2(1 - X)^2 & \text{for } 0 \leq X < 1 \\
0 & \text{elsewhere.}
\end{cases}
\] (3.1)

Moreover, some analytical progress can be made in the limit h → ∞, which is the limit of greatest practical importance, of course, as it concerns the behaviour of grossly separated motion. This analysis is presented below, although some problems remain with describing the process of reattachment which occurs far downstream of the isolated hump. In an attempt to throw more light on the process of reattachment, we formulate below a physically reasonable two-parameter problem for which we do have a complete, self-consistent, description of separation and reattachment, and of which the present problem appears to be an interesting singular limit.
Numerical solutions for $h = O(1)$.

The numerical task of integrating (1.12a-e,g) is much more straightforward than the integration of the corresponding supercritical-flow equations, due mainly to the lack of upstream influence. The equations were discretised using Keller's box scheme, as before. The discrete equations were solved by marching forward in the $X$-direction from $X_-$ (just upstream of $X = 0$) using Newton iteration at each $X$-station. The tolerance parameter for the Newton iteration was set to $10^{-7}$. Usually four or five, but occasionally up to seven, iterations were required for convergence. However, this procedure became unstable in the presence of reversed flow, as expected. Therefore, during the first forward sweep, the Flare approximation was used (in which the term $UU_X$ in (1.12a) was set to zero whenever $U < 0$). The resulting approximate solution was refined by subsequent marching in the reversed $X$-direction followed by a further forward sweep, this time using the upwind-differencing formulation; that is, at the $i^{th}$ $X$-station $X_i$, the term $UU_X$ is centred at $X_i + \delta i / 2$ whenever $U < 0$ whilst all the other terms are centred at $X_i - \delta i$ as usual. This requires the computational mesh to be longer (in the $X$-direction) than the extent of the reversed flow region. During each such sweep, either forward or backward, it is only necessary to integrate the equations over the reversed flow region, in fact, rather than over the entire (pre-set) $X$ range. This pattern can be repeated until some convergence criterion is met, but we found that sufficiently accurate solutions were obtained after only four forward sweeps.

Due to the rapid variation of the flow variables over the hump
(given by (3.1)), and the much slower variation downstream of it, we took uniform steps of $\Delta \xi$ in the stretched $X$ co-ordinate $\xi$ defined by

$$X = \begin{cases} 
  s\xi & \text{for } 0 < X < 1 \\
  s\xi + \frac{1 - s^2}{4(C - 1)} \left( \xi - \frac{1}{s} \right)^2 & \text{for } 1 < X < C \\
  \xi - \frac{(1 - s)(sC + 1)}{s(1 + s)} & \text{for } C < X
\end{cases} \quad (3.2)$$

where $s$ and $C$ are constants. Note that $X = f(\xi)$ is monotonically increasing, and is continuous with a continuous first derivative. Also, $\max(\Delta X) = \Delta \xi$ (obtained when $X > C$) and $\min(\Delta X) = s\Delta \xi$ (obtained when $0 < X < 1$) if $s < 1$. We took the values $C = 5$ and $s = 0.02$ after some preliminary trials. The grid spacing in the $Y$-direction was uniform (and equal to $\Delta Y$).

With the above values of $C$ and $s$ there remain four parameters that specify the grid: $\Delta \xi$, $\Delta Y$, $Y_e$, and $X_\infty$. The numerical results for $h = 8$ were checked by altering these parameters (although the results are independent of $X_\infty$, in fact, as long as it is beyond the point of reattachment $X_{reatt}$). Comparisons of the results obtained using three different grid spacings are presented in figure 9; these grids have (a) $\Delta \xi = 0.5$, $\Delta Y = 0.4$, (b) $\Delta \xi = 0.375$, $\Delta Y = 0.3$, (c) $\Delta \xi = 0.25$, $\Delta Y = 0.2$, with $X_\infty$ and $Y_e$ fixed. It can be seen that the coarsest grid (a) gives an overestimate of the length of the reversed flow region; using linear interpolation the point of reattachment (where $r_e = 0$) is found to be at $X_{reatt} = 10.174$, 9.902, 9.662 for the grids (a), (b), (c) respectively. The numerical results for the pressure appear to converge quite well as the grid spacings are reduced. The wall shear is a little more sensitive, especially near reattachment, but at most $X$-stations it converges well also. The effect of varying $Y_e$ was tested using the grid spacings (a) with the edge values $Y_e = 20, 28, 36$. The resulting pressure distributions
agreed to at least four decimal places at each X-station, and the wall shear distributions to three. So these variations are much less significant than those due to the grid spacings.

The results for three values of the height parameter $h$ are presented in figure 10. We see that there is a rapid rise in the displacement function $(-A)$ over the forward face of the hump, and therefore a corresponding fall in the pressure. This favourable pressure gradient forces a strongly attached flow to develop there, as reflected in the rapid increase in $\tau_w$: see figure 10a. The pressure then reaches a minimum just beyond the peak of the hump; thereafter, the strong adverse pressure gradient quickly causes separation, which occurs in a regular fashion, of course, within this interactive flow structure. Beyond separation the pressure remains practically constant for some distance downstream. This implies that the flow bounded by the wall and the separation streamline is relatively slow. On a longer scale (figure 10b) the flow reattaches to the wall, and then, further downstream, returns to a uniform shear flow. The streamline plots in figure 10c show that the separation streamline and the wall bound a slowly recirculating eddy which gets longer (and wider) as the parameter $h$ increases. There is clearly one main eddy; others may exist but may only appear for larger values of $h$ or may only become apparent with much more detailed calculations. But we can already see that the curvature of the streamlines in the eddy is quite large close to reattachment (and near separation too), which indicates that reattachment may occur on a shorter length scale than the eddy length itself as $h$ increases.
The limit $h \rightarrow \infty$. 

When $h > 1$ suppose $P \sim \chi > 1$ where the order of magnitude $\chi$ of $P$ is to be determined. We expect the oncoming uniform-shear flow to change on the $O(1)$ length scale of the hump, initially at least i.e. up to separation. Then, from the boundary-layer equations (1.12a), we see that a viscous/inertia/pressure balance is confined to a thin layer adjacent to the wall in which $Y \sim \chi^{-1/4}$ and $U \sim \chi^{1/2}$. Outside this layer the governing equations are inviscid, and the solution here is therefore a direct continuation of the outer boundary condition (1.12c) i.e.

$$U = Y + P + hG(X)$$ (3.3)

(using (1.12g)) outside the viscous wall layer. Integrating this with respect to $Y$, and using (1.12a), we obtain

$$\frac{d}{dY} = \frac{1}{2}(Y + P + hG(X))^2 + P$$ (3.4)$$

Matching this solution to the wall layer fixes $\chi$, and also $P$ to leading order. Since $U \ll P$ in the wall layer, (3.3) implies that $\chi \sim h$ and

$$P = -hG(X) + ...$$ (3.5a)

yielding a simple (and classical) pressure-shape relationship. Further terms in (3.5a) can be found from the behaviour of the solution (3.4) near the wall, for $\frac{d}{dY}$ must drop to $O(h^{-1/4})$ when $Y \sim h^{-1/4}$. In terms of the inner variable $Z$, defined by $Y = h^{-1/4}Z$, (3.4) becomes

$$\frac{d}{dZ} = \frac{1}{2}h^{-1/2}Z^2 + h^{-1/4}Z(P + hG) + \frac{1}{2}(P + G)^2 + P$$ (3.6)

This implies the expansion

$$P = -hG + (2hG)^{1/2} - 1 + h^{-1/4}p_0(X) + ...$$ (3.5b)

where the function $p_0(X)$ is unknown at this stage. The first three terms of this expansion accurately predict the earlier numerical results for the pressure distribution ahead of separation. Then, from (3.3), $U$ expands in the asymptotic sequence $h^{1/2}$, 1, $h^{-1/4}$, ... when
Z = 0(1), which in turn gives the asymptotic sequence for \( t \) as \( h^{1/4}, \ h^{-1/4}, \ h^{-1/2} \), etc. Yet more terms in (3.5) can be calculated using only this asymptotic sequence for \( t \) and (3.6). We find after some working that

\[
P = -hG + (2hG)^{1/2} - 1 + h^{-1/4}P_0(X) + \frac{1}{2}(2hG)^{1/2} + h^{-2}P_1(X) \quad (3.5c)
\]

where, again, \( P_1(X) \) is an unknown function of \( X \) at this stage. So in the wall layer we deduce the following expansions:

\[
\begin{align*}
U &= (h^2u_0 + u_1 + h^4u_2 + h^2u_3 + h^{-2}u_4 + \ldots)(X,Z) \\
\psi &= (h^2\psi_0 + h^2\psi_1 + h^4\psi_2 + h^4\psi_3 + h^{-4}\psi_4 + \ldots)(X,Z)
\end{align*} \quad (3.7)
\]

using (3.3) and (1.12a) with (3.5c). Substituting these expansions into (1.12), we obtain the classical boundary-layer equations at the leading (zeroth) order:

\[
\begin{align*}
&u_0u_0x - \psi_0u_0z = G'(X) + u_0zz, \quad u_0 = \psi_0, \\
u_0 = \psi_0 = 0 \quad \text{at} \quad Z = 0, \\
u_0 \to (2G(X))^{1/2} \quad [\psi_0 \sim (2G)^{1/2}(Z + p_o)] \quad \text{as} \quad Z \to \infty.
\end{align*} \quad (3.8)
\]

The boundary conditions are for no slip on the hump, and matching with the outer flow (3.4). The solution of this system determines \( p_o(X) \). Subsequent terms in (3.7) can also, in principle, be determined in a systematic fashion.

However, in general the classical boundary-layer equations governing the zeroth-order problem break down as separation is approached at a finite value of \( X \), at \( X_s \) say, when the flow is forced by a sufficiently strong adverse pressure gradient. This would normally occur on the backward facing slope of an isolated hump such as (3.1), and this breakdown occurs in the form of the well-known Goldstein (1948) singularity. That raises the question of whether the singularity can be removed on a shorter length scale. If not, we are forced to conclude that the above structure, holding
upstream of $X_s$, is incorrect. Fortunately, it is possible to remove the Goldstein singularity in a physically sensible fashion in some flow situations, although these are relatively few in number. Smith & Daniels (1981) studied the large-\(h\) limit of a boundary-layer flow governed by (1.12a-e') with (1.12g) replaced by \(A = 0\). These equations, like ours, also admit no upstream influence, and a structure similar to the one above holds ahead of separation. They showed that the Goldstein singularity may be removed by a sequence of double structures holding near separation. Separation then takes place in a regular fashion, and a further singularity which arises just beyond separation can also be removed on a still shorter length scale. The same kind of structure holds for the present problem (see figure 11), as we now show, although the various length scales involved are different.

First, as \(X \to X_s-\), the solution of (3.8) approaches separation in the following way. The wall layer itself divides into two zones. In the outer Goldstein zone, in which \(Z = O(1)\), we have

\[
\psi_0 = \psi_{0s}(Z) + \frac{2\lambda_{00}}{\mu_0} (X_s - X) + \frac{2\lambda_{01}}{\mu_0} (X_s - X)^2 \psi_{0s}(Z) + \ldots
\]

from Goldstein (1948). Here \(\psi_{0s}(Z)\) is the zeroth-order stream function at separation, with the properties

\[
\psi_{0s} \sim \begin{cases} 
\frac{1}{8} \mu_0 Z^3 - \frac{\lambda_{00}}{60} Z^5 + \ldots & \text{as } Z \to 0 \\
(2G_s)^{1/2} (Z + p_{0s}) + o(1) & \text{as } Z \to - \infty,
\end{cases}
\]

and

\[
G_s = G(X_s), \quad G_s' = G'(X_s), \quad \mu_0 = - G_s' > 0.
\]

The constants \(\lambda_{00}, \lambda_{01}, p_{0s}\) (and all similarly named constants below) are unknown, and depend on the properties of the whole solution upstream of \(X_s\). This solution matches with the inner Goldstein zone,
in which \( \eta = (X - X)^{-1/4} \) is \( O(1) \), and

\[
\psi_0 = \frac{\mu_0}{\mu} (X_0 - X)^{\frac{3}{2}} \eta^2 + \lambda_{10}(X_0 - X) \eta^2 + (X_0 - X)^{\frac{3}{2}}(\lambda_{10} \eta^2 - \frac{\lambda_{10}^2 \eta^4}{60}) + \ldots \tag{3.12}
\]

From (3.8), (3.9) and (3.12) we may infer the irregular behaviour of the displacement function \( p_0(X) \) and the wall shear \( \tau_0(X) \) (\( \mu_0 \) at \( Z = 0 \)):

\[
p_0 = p_0 + \frac{2 \lambda_{10}}{\mu_0} (X_0 - X)^{\frac{1}{2}} + \ldots \tag{3.13}
\]

\[
\tau_0 = 2\lambda_{10}(X_0 - X)^{\frac{1}{2}} + 2\lambda_{10}(X_0 - X)^{\frac{7}{2}} + \ldots \text{ as } X \to X_0. \tag{3.14}
\]

The behaviour of the higher-order terms in (3.7) can also be calculated. Substitution of (3.7) into the governing equations and boundary conditions (1.12a-e, g) yields the following sets of equations at successive orders of \( \mu \):

\[
\mathcal{N}(\psi_1, u_1) = \frac{G(X)}{(2G(X))^{1/2}} \tag{3.15a}
\]

\[
\mathcal{N}(\psi_2, u_2) = 0 \tag{3.15b}
\]

\[
\mathcal{N}(\psi_3, u_3) = u_1 u_1 X - \psi_1 u_1 \tag{3.15c}
\]

\[
\mathcal{N}(\psi_4, u_4) = (u_1 u_2)_X - \psi_1 u_2 - \psi_2 u_1 \tag{3.15d}
\]

where \( \mathcal{N}(\psi, u) = u_{zz} - (u_0 u)_X + \psi_0 u_2 + \psi X u_0 \); \( u_1 = \psi_1 \) for \( i = 1, 2, \ldots \) \( \tag{3.15e} \)

\( u_1 = \psi_1 = 0 \) at \( Z = 0 \) for \( i = 1, 2, \ldots \) \( \tag{3.15f} \)

\( u_1 \to -1 \) [and \( \psi_1 \sim Z + (2G(X))^{\frac{1}{2}} p_1(X) \)] as \( Z \to -\infty \) \( \tag{3.15g} \)

\( u_2 \sim Z + p_0(X) \), \( u_3 \to \psi(2G(X))^{\frac{1}{2}} \), \( u_4 \to p_1 \) as \( Z \to -\infty \). \( \tag{3.15g} \)

The outer boundary conditions match the solution here with (3.3) and (3.4) using (3.5c). The behaviours of the solutions of the first-, second-, and third-order problems close to separation are similar to the behaviours of the zeroth-order problem above. We find that
\[ \psi_1 = \psi_{1s}(Z) + \frac{2\lambda_{10}}{\mu_0} (X_s - X)^{\frac{1}{2}} \left\{ \psi_{1s}' + \frac{\lambda_{10}}{\lambda_{00} + \eta^2} \psi_{0s}' \right\} + \frac{2\lambda_{01}}{\mu_0} (X_s - X)^{\frac{3}{2}} \left\{ \psi_{1s}' + \frac{\lambda_{10}}{\lambda_{00} + \eta^2} \psi_{0s}' \right\} + \ldots, \]  

(3.16a)

\[ \psi_1 = -\frac{1}{6}\mu_1 (X_s - X)^{\frac{3}{2}} \eta^2 + \lambda_{10}(X_s - X)\eta^2 + (X_s - X)^{\frac{5}{2}} \left\{ \lambda_{11}\eta^2 - \frac{\lambda_{00}\lambda_{10}}{30} \eta^5 \right\} + \ldots, \]  

(3.16b)

as \( X \rightarrow X_s^- \) in the outer and inner Goldstein zones respectively. Here

\[ \mu_1 = -\frac{G'\eta}{(2G_s)^{1/2}} > 0 \]  

(3.17)

and \( \psi_{1s}(Z) \) is the (first-order) stream function at separation, having the following properties

\[ \psi_{1s} \sim \begin{cases} -\frac{\mu_1}{6} Z^3 - \frac{\lambda_{00}\lambda_{10}}{30} Z^5 + \ldots & \text{as } Z \rightarrow 0 \\ -Z + (2G)^{\frac{1}{2}} p_{1s} & \text{as } Z \rightarrow 0. \end{cases} \]  

(3.18)

Again, we can compute the behaviour of \( p_1(X) \) and \( \tau_1(X) \) (the first-order wall shear) near separation from these solutions. We find

\[ p_1 = p_{1s} + \frac{2\lambda_{10}}{\mu_0} (X_s - X)^{\frac{1}{2}} + \ldots, \]  

(3.19)

\[ \tau_1 = 2\lambda_{10}(X_s - X)^{\frac{1}{2}} + 2\lambda_{11}(X_s - X)^{\frac{3}{2}} + \ldots. \]  

(3.20)

The second- and third-order solutions also follow in a reasonably straightforward manner:

\[ \psi_2 = \psi_{2s}(Z) + \frac{2\lambda_{10}}{\mu_0} (X_s - X)^{\frac{1}{2}} \left\{ \psi_{2s}' + \frac{\lambda_{20}}{\lambda_{00}} \psi_{0s}' \right\} + \frac{2\lambda_{01}}{\mu_0} (X_s - X)^{\frac{3}{2}} \left\{ \psi_{2s}' + \frac{\lambda_{20}}{\lambda_{00}} \psi_{0s}' \right\} + \ldots, \]  

(3.21a)

\[ \psi_3 = \psi_{3s}(Z) + \frac{2\lambda_{10}}{\mu_0} (X_s - X)^{\frac{1}{2}} \left\{ \psi_{3s}' + \frac{\lambda_{10}}{\lambda_{00}} \psi_{0s}' \right\} + \frac{2\lambda_{01}}{\mu_0} (X_s - X)^{\frac{3}{2}} \left\{ \psi_{3s}' + \frac{\lambda_{10}}{\lambda_{00}} \psi_{0s}' + \frac{\mu_1}{\mu_0} \psi_{1s} \right\} + \ldots \]  

(3.22a)

in the upper Goldstein zone, and

\[ \psi_2 = \lambda_{20}(X_s - X)\eta^2 + (X_s - X)^{\frac{3}{2}} \left\{ \lambda_{21}\eta^2 - \frac{\lambda_{00}\lambda_{20}}{30} \eta^5 \right\} + \ldots \]  

(3.21b)

\[ \psi_3 = \lambda_{30}(X_s - X)\eta^2 + \ldots \]  

(3.22b)
in the inner zone. For the record, we note the following behaviour of the second- and third-order stream functions at separation:

\[ \psi_2 \sim -\frac{\lambda_2\lambda_{20}}{30} Z^2, \quad \psi_3 \sim -\frac{(\lambda_1\lambda_{20} + \lambda_{20}^2)}{30} Z^3 \; \text{as} \; Z \to 0. \quad (3.23) \]

which reflect not only the fact that the wall shear falls to zero at these orders as \( X \to X_s^- \), but also the smallness (or absence) of forcing in (3.15b,c) in the inner Goldstein zone.

So far, the forcing terms in (3.15) have been so small that the first four terms, with suffices zero to three above, of the expansions (3.7) remain well ordered as separation is approached. As such, they cannot indicate the new length scale on which the Goldstein singularity may be removed. Turning to the equations governing the fourth order, however, we see that the pressure gradient driving the flow in (3.15d) is singular, and of order \( (X_s - X)^{-1/2} \) from (3.19).

This suggests that \( \psi_4 \) is of order \( (X_s - X)^{1/4} \) in the inner Goldstein zone, which is larger than the dominant terms above (in (3.16b), (3.21b), (3.22b)). However, the resulting ordinary differential equation for the leading order term of \( \psi_4 \) has no solution satisfying the matching conditions. The same problem arises in Smith & Daniels' study. The solution, as these authors note, is to include three eigensolutions which are more singular than the one above; in fact, we require

\[ \psi_4 = \ln(X_s - X)g_{0L}(\eta) + g_0(\eta) + (X_s - X)^{\frac{1}{4}}\ln(X_s - X)g_{1L}(\eta) \]

\[ + (X_s - X)^{\frac{1}{4}}g_1(\eta) + \ldots \]

in the lower Goldstein zone. Substitution of this expansion into (3.15d), together with (3.12), (3.13), (3.16b), (3.21b), yields the solutions

\[ g_{0L}(\eta) = -\frac{1}{2} B_{0L} \eta^2, \quad g_0(\eta) = \frac{1}{2} B_0 \eta^2, \quad g_{1L}(\eta) = \frac{1}{2} B_{1L} \eta^2 \quad (3.25) \]
for the three eigenfunctions, where the constants $B_{0L}$, $B_0$, $B_{1L}$ are unknown at present. The equation governing $g_1(\eta)$ is then

$$
\begin{align*}
\psi'''' + \frac{1}{3} \mu_0 \eta^3 \psi'' + \frac{1}{4} \mu_0 \eta^2 \psi' - \frac{1}{2} \mu_0 \eta \psi_1 &= \lambda_{00} (B_{0L} \eta^2 - \mu^{-1}) \\
g_1(0) = g_1'(0) &= 0, \quad g_1(\eta) = o(\exp) \text{ as } \eta \to -
\end{align*}
$$

A solution of (3.26) satisfying these boundary conditions only exists if a certain solvability condition is met, and this condition fixes $B_{0L}$ uniquely (see Smith & Daniels' equations (2.14a-c)). We find that

$$
B_{0L} = \frac{\Gamma(\frac{1}{4})}{(2\mu_0)^{1/2} \Gamma(\frac{1}{4})}
$$

So the leading term in (3.24) is indeed necessary. The constants $B_0$, $B_{1L}$, $B_1$ can also be determined at higher order. Then in the upper Goldstein zone we obtain

$$
\begin{align*}
\psi_s = (X_s - X)^{-1} & \mu_0 \psi_0(Z) \left[ - B_{0L} \ln(X_s - X) + B_0 \\
&+ B_{1L}(X_s - X)^{1/2} \ln(X_s - X) + B_1(X_s - X)^{1/2} + \ldots \right]
\end{align*}
$$

Thus the expansion (3.7) breaks down as $X \to X_s^-$, and a new expansion is required when the fourth-order term is comparable in magnitude with the zeroth-order term. This occurs when $X_s^- - X$ is $O(h^{-3/4} \ln(h))$. Accordingly, we set

$$
X = X_s + h^{-3/4} \ln(h) \hat{X}.
$$

On this shorter length scale, defined by $\hat{X} = 0(1)$, the flow in the wall layer again divides into two (see figure 11): region (i), in which $Z = 0(1)$, where the flow properties are mainly inviscid; and region (ii), a viscous sub-layer, in which $Z = O(h^{-3/16} \ln(h)^{1/4})$. It turns out that the Goldstein singularity is not removed on this shorter length scale, but is merely postponed, occurring slightly further downstream of $X_s$. This also occurs in Smith & Daniels' study.

The new expansions in regions (i) and (ii) are obtained from (3.7)
using the behaviours of the zeroth- to fourth-order solutions given above as (3.29) comes into play. In region (i) the expansion for \( t \) is found to be

\[
t = h \psi_{0_{_G}}(Z) + h \psi_{1_{_G}}(Z)
  + h \frac{3}{4} L \frac{1}{4} \psi_{0_{_G}}(Z) \left\{ \hat{\alpha}_1(\hat{X})L + \hat{\alpha}_1(\hat{X})\ln(L) + \hat{\alpha}_2(\hat{X}) \right\} + h \frac{1}{4} \psi_{3_{_G}}(Z)
  + h \frac{1}{4} L \frac{1}{4} \psi_{0_{_G}}(Z) \left\{ \hat{\alpha}_3(\hat{X})L + \hat{\alpha}_3(\hat{X})\ln(L) + \hat{\alpha}_4(\hat{X}) \right\} + h \frac{1}{4} \psi_{3_{_G}}(Z)
  + h \frac{1}{4} L \frac{1}{4} \left\{ \hat{\alpha}_3(\hat{X})\psi_{0_{_G}}(Z) + \hat{\alpha}_1(\hat{X})\psi_{1_{_G}}(Z) \right\} + \ldots
\]  

(3.30)

where \( L = \ln(h) \). The expansion for \( p \) can be determined by matching this solution with the outer-flow solution (3.4), giving

\[
P = - h G_s + (2h G_s)^{1/2} - 1 - h^{3/4} L \frac{1}{4} \hat{X} + h^{1/4} p_{_{_G}} + \frac{1}{2} (2h G_s)^{-1/2}
  + h \frac{3}{4} L \left[ \frac{X G_s}{(2G_s)^{1/4}} \right] + h \frac{3}{4} p_{_{_G}}
  + h \frac{3}{4} L \frac{1}{4} \left\{ \hat{\alpha}_1(\hat{X})L + \hat{\alpha}_1(\hat{X})\ln(L) + \hat{\alpha}_2(\hat{X}) \right\} + \ldots
\]  

(3.31)

The unknown functions \( \hat{\alpha}_1(\hat{X}) \) etc. in (3.30) are determined by the solvability conditions that arise in region (ii). For in (ii) we find

\[
\hat{t} = h \frac{1}{4} L \frac{3}{4} \left( \frac{1}{2} \mu_0 z^5 \right)
  + h^{-1} \left( \frac{1}{2} \mu_0 z^2 \right) \left\{ \hat{\alpha}_1(\hat{X})L + \hat{\alpha}_1(\hat{X})\ln(L) + \hat{\alpha}_2(\hat{X}) \right\}
  - h \frac{1}{4} L \frac{3}{4} \left( \frac{1}{2} \mu_1 z^3 \right)
  + h \frac{1}{4} L \frac{3}{4} \left\{ \hat{\psi}_3(\hat{X}, z) + \hat{\psi}_3(\hat{X}, z)\ln(L) + \hat{\psi}_4(\hat{X}, z) \right\}
  + \ldots
\]  

(3.32)

from (3.7) together with the above solutions in the lower Goldstein zone. Here \( Z = h^{-5/16} L^{1/4} z \) with \( z = O(1) \). Substitution of (3.32) into (1.12) gives the following equation for \( \hat{\psi}_3 : \)

\[
\hat{\psi}_{zzz} = \frac{1}{2} \mu_0 z^2 \hat{\psi}_3 \hat{X}_z \mu_0 z \hat{\psi}_3 \hat{X} + \frac{1}{2} \mu_0 z^3 \hat{\alpha}_1(\hat{X}) \hat{\alpha}_1(\hat{X}) .
\]  

(3.33a)

The boundary conditions (from (1.12) and the join with (3.30)) are

\[
\hat{\psi}_3 = \hat{\psi}_3 = 0 \quad \text{at } z = 0 ,
\]

\[
\hat{\psi}_3 = \frac{1}{2} \mu_0 \hat{\alpha}_3(\hat{X}) z^2 \quad \text{as } z \to \infty .
\]  

(3.33b)
These equations serve to determine $\hat{\alpha}_1(\hat{x})$, for a solution only exists if

$$\frac{1}{2}\mu_0^2\hat{\alpha}_1(\hat{x})\hat{\alpha}'_1(\hat{x}) = -\lambda_0^2 \quad \text{i.e.} \quad \mu_0^2\hat{\alpha}_1^2(\hat{x}) = -4\lambda_0^2\hat{x} + C_1 \quad (3.34)$$

(see Smith & Daniels) in which case the outer boundary condition in (3.33b) gives $\hat{\psi}$, throughout region (ii). The constant $C_1$ in (3.34) can be determined by the match as $\hat{x} \rightarrow -\infty$ with the oncoming Goldstein form as $X \rightarrow X_\infty^-$. From (3.7), (3.12), (3.24) and (3.25) we find specifically

$$\frac{1}{2}\mu_0\alpha_1(\hat{x}) \sim \lambda_0^2|\hat{x}|^{1/2} + \frac{5B_0\lambda_0}{4\lambda_0^2}|\hat{x}|^{-1/2} + \ldots \quad \text{as} \quad x \rightarrow -\infty \quad (3.35)$$

which is compatible with (3.34) only if

$$C_1 = 5B_0\lambda_0 \quad . \quad (3.36)$$

Thus we have

$$\hat{\alpha}_1(\hat{x}) = \frac{2\lambda_0}{\mu_0} \left[ -\hat{x} + \frac{5B_0\lambda_0}{4\lambda_0^2} \right]^{1/2} \quad . \quad (3.37)$$

Other unknown functions in (3.30) can also be calculated by working to higher orders in region (ii). To leading order, the wall shear is given by

$$\tau(\hat{x}) = \frac{1}{h^2}\frac{1}{\mu_0}\hat{\alpha}_1(\hat{x}) \quad . \quad (3.38)$$

We see, therefore, that the Goldstein singularity is merely reproduced at

$$\hat{x} = \hat{x}_0 = \frac{5B_0\lambda_0}{4\lambda_0^2} \quad . \quad (3.39)$$

Examination of further terms in (3.30) reveals the next relevant shorter length scale in the neighbourhood of $\hat{x}_0$. Alternatively, since the flow on the $X = O(1)$ length scale is essentially identical to the oncoming Goldstein form, we may deduce the new length scale from further examination of the singular behaviour of (3.7) as $X \rightarrow X_\infty^-$. Either way, we find that the new length scale is given by $\hat{x} = O(1)$
where

\[ X = x_0 + \frac{5B_0 k}{4\lambda_{oo}} h^\frac{3}{2} \ln(h) + h^\frac{5}{4} x. \]  

(3.40)

On this new length scale, the flow again divides into two: a region
(a), in which \( Z \) is \( O(1) \) again; and a viscous sub-layer, region (b), in
which \( Z \) is \( O(h^{-5/16}) \). In region (a) we find

\[ \bar{t} = h^\frac{5}{4} \bar{\psi}_{oo}(Z) + h^\frac{5}{4} \bar{\psi}_{11}(Z) + h^\frac{5}{16} \bar{\psi}_{01}(\bar{X}) \psi_{oo}(Z) + ... \]  

(3.41)

from (3.30), (3.40), and (1.12), whilst the pressure is given by

\[
P = - hG_s + (2hG_s)^{\frac{1}{2}} - 1 - h^\frac{1}{2} \frac{\gamma}{L} \bar{X}_0 + h^\frac{1}{2} (p_{oo} - \bar{G}_s) + \frac{1}{2} (2hG_s)^{-\frac{1}{2}}
\]

\[ + h^\frac{1}{2} L \left[ \frac{\bar{X}_0 G_s}{(2G_s)^{1/2}} \right] + h^\frac{1}{2} \left[ p_{1s} + \frac{\bar{G}_s}{(2G_s)^{1/2}} \right] + h^\frac{1}{16} \bar{\psi}_{11}(\bar{X})
\]

+ ...  

(3.42)

in view of (3.31) and the join with (3.4) as (3.40) comes into play.

Again, the unknown function \( \bar{\alpha}_1(\bar{X}) \) is determined by solvability
conditions that arise in the viscous region. In region (b), setting

\[ z = h^{5/16} Z, \]

we find

\[
\bar{t} = h^\frac{11}{16} (\frac{1}{4} \mu_0 \bar{z}^3) - h^\frac{1}{16} \bar{\alpha}_1(\bar{X}) (\frac{1}{4} \mu_0 \bar{z}^3) - h^\frac{11}{16} (\frac{1}{4} \mu_1 \bar{z}^3) + h^\frac{11}{16} \bar{\psi}_2(\bar{X}, \bar{z})
\]

+ ...  

(3.43)

from (3.32), (3.40) and (1.12). The differential equation and boundary
conditions governing \( \bar{\psi}_2 \) are

\[
\bar{\psi}_{zzz} - \frac{1}{2} \mu_0 \bar{z}^2 \bar{\psi}_{z} + \mu_0 \bar{z} \bar{\psi}_{x} = \bar{\alpha}_1 \bar{X}_0 + \frac{1}{2} \bar{z}^2 \mu_0 \bar{\psi}_{11}(\bar{X}) \bar{\psi}_{11}(\bar{X}) \]  

(3.44a)

\[
\bar{\psi}_2 = 0 \text{ at } \bar{z} = 0, \quad \bar{\psi}_2 = o(\exp) \text{ as } \bar{z} \to 0. \]  

(3.44b)

The solvability condition for the existence of a solution of this
system is

\[
C_2 - \mu_0 \bar{\psi}_{11}(\bar{X}) \bar{\alpha}_{11}(\bar{X}) = \frac{\Gamma(\frac{3}{4}) (2\mu_0)}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\bar{X}} \frac{\bar{\psi}_{11}''(\xi)}{\bar{\psi}_{11}(\bar{X} - \xi)^{1/2}} d\xi \]  

(3.45)
(see Smith & Daniels) where the constant $C_2$ is found by matching (3.42) as $\tilde{x} \to -\infty$ with (3.31) as $\tilde{x} \to \tilde{x}_o^-$. This match gives

$$\tilde{x}_1 \sim \frac{2\lambda_2}{\mu_0} \left| x \right|^{\frac{1}{2}} \text{ as } \tilde{x} \to -\infty \quad (3.46)$$

giving

$$C_2 = -\frac{2\lambda_2}{\mu_0} \quad (3.47)$$

The central problem (3.45) with (3.47) was also derived by Smith & Daniels (to within positive constants which can be factored out anyway). Their numerical investigation of (3.45) reveals two important features. Firstly, separation takes place in a regular fashion, at $\tilde{x} = \tilde{x}_s$ say, where $\tilde{x}_1(\tilde{x}_s) = 0$, and, secondly, the solution of (3.45) itself breaks down, at $\tilde{x} = \tilde{x}_o$ say, downstream of separation. The exact value of $\tilde{x}_o$ can only be determined numerically, but we may deduce from a balance of terms in (3.45) that

$$\tilde{x}_1(\tilde{x}) \sim -\frac{4B_{ol}}{\mu_0} (\tilde{x}_o - \tilde{x})^{-\frac{1}{2}} \text{ as } \tilde{x} \to \tilde{x}_o^- \quad (3.48)$$

implying an increasingly fast reversed flow as $\tilde{x} \to \tilde{x}_o^-$, since $B_{ol} > 0$. This singular behaviour of the displacement function (or equivalently of the pressure or wall shear distributions) in the neighbourhood of $\tilde{x}_o$ implies a breakdown of the expansions (3.41), (3.42), (3.43) there. These expansions, together with (3.48), single out a new shorter length scale of $O(h^{-5/3})$ for study. Therefore we set

$$\tilde{x} = \tilde{x}_s + \hat{x}_o h^{-\frac{3}{10}} \ln(h) + \tilde{x}_oh^{-\frac{1}{2}} + \tilde{x}_h h^{-\frac{1}{3}} \quad (3.49)$$

As before, on this new length scale the flow divides into two regions: region (I), a mainly inviscid outer layer in which $\zeta = O(1)$; and region (II), a viscous, nonlinear sub-layer. In region (I) we obtain the solution

$$\tau = h^{\frac{1}{3}} \zeta_0 \zeta(Z) + h^{\frac{1}{3}} a(x) \zeta_0'(Z) + ... \quad (3.50)$$
of (1.12), in view of (3.41), (3.48) with (3.49). The pressure expansion, implied by (3.50) with (3.48) and (3.49), is now

\[ P = -hG_{a} + (2hG_{a})^{\frac{1}{2}} - 1 - h^{\frac{1}{4}}L_{G_{a}}(\hat{\chi})_{0} + h^{\frac{1}{4}}(p_{0} - \hat{\chi}_{0}G_{a})^{\prime} \]

\[ + \frac{1}{2}(2hG_{a})^{\frac{1}{2}} + h^{\frac{3}{2}}\hat{\psi}(\hat{\chi}) + \ldots \]  

(3.51)

The join with the outer solution (3.6) now gives the relation

\[ \hat{p}(\hat{\chi}) = \hat{\alpha}(\hat{\chi}) - G_{a}^{\prime} \hat{\chi} \]  

(3.52)

between the unknown pressure and displacement distributions. In region (II) we have

\[ \hat{\psi} = h^{-1}\hat{\psi}(\hat{\chi}, \hat{z}) + \ldots \]  

(3.53)

from (3.43), (3.48) and (3.49), where \( \hat{z} = h^{3/12}z \) is O(1). Substitution of (3.53) into (1.12a) gives the boundary layer equations

\[ \hat{u} \hat{u}_{\hat{\chi}} - \hat{\psi}\hat{u}_{\hat{z}} = -\hat{p}^{\prime}(\hat{\chi}) + \hat{u}_{\hat{z}} \hat{z} \hat{u} = \hat{\psi}_{\hat{z}} \]  

(3.54a)

The boundary conditions follow from (1.12b), the join with the solution (3.50), and the join with region (b) upstream, giving

\[ \hat{u} = \hat{\psi} = 0 \text{ at } \hat{z} = 0 \]  

(3.54b)

\[ \hat{\psi} \sim \frac{1}{\hat{\alpha}} \mu_{0}(\hat{z} + \hat{\alpha})^{3} \text{ as } \hat{z} \rightarrow \infty \]  

(3.54c)

\[ \hat{\psi} \rightarrow \frac{1}{\hat{\alpha}} \mu_{0}\hat{z}^{3} \text{ as } \hat{\chi} \rightarrow -\infty \]  

(3.54d)

The central problem (3.54a-d) with (3.52) describes the complete, regular breakaway of the whole sub-layer from the wall. The ultimate form of the solution far downstream, proposed by Smith & Daniels and supported by their numerical calculations, has

\[ \hat{\alpha}(\hat{\chi}) - G_{a}^{\prime} \hat{\chi} - \hat{p*} \]  

\[ \hat{p} \rightarrow \hat{p*} \text{ as } \hat{\chi} \rightarrow \infty \]  

(3.55)

Also, the flow far downstream is concentrated in a detached shear layer, expanding like \( \hat{X}^{1/4} \), and surrounding the dividing streamline \( \hat{\xi} = 0 \text{ at } \hat{z} = -G_{a}^{\prime} \hat{\chi} \). Thus the emerging shear layer lies parallel to the
undisturbed wall. Above the shear layer, the flow solution is a direct continuation of the outer boundary condition (3.54c). Below it, the flow is a relatively slow, almost uniform, reversed motion. Finally, a thin viscous layer of reversed flow is provoked between the reversed uniform stream and the wall.

So the question of regular separation in the limit \( h \rightarrow \) seems to be settled by the above flow structure. We turn our attention now to the subsequent development of the flow beyond separation. The above flow structure on the \( O(h^{-5/3}) \) length scale just studied continues until the shear layer, of thickness \( O(h^{-2/3} \bar{x}^{1/4}) \) as \( \bar{x} \rightarrow \bar{x}_s \), expands to 'fill' the entire wall layer, of thickness \( O(h^{-1/4}) \). This occurs when \( \bar{x} = O(h^{5/3}) \), that is, when \( X = O(1) \) from (3.49), which is a physically sensible length scale to emerge downstream. Thus, returning to the \( O(1) \) length scale of the hump, we have then that the flow consists of a shear layer of thickness \( O(h^{-1/4}) \) separating a uniform shear flow above and a reversed motion below. One eminent possibility is that the reversed flow is very much slower than the flow in the shear layer, being driven, perhaps, merely by the requirement of viscous entrainment of fluid into the shear layer. If this supposition is correct, it implies that the streamwise flow velocity \( U \) is of order \( h^{-3/4} \) in the reversed flow region, which has width \( O(h) \). Then the pressure gradient can only be \( O(h^{-3/2}) \) at most, implying from the match with (3.51) that

\[
P = -hG_a + (2hG_a)^{1/2} - h^{-\frac{1}{4}}LG_a^\infty Y_0 + \ldots \quad (3.56)
\]

on the \( X-X_s = O(1) \) length scale. The constancy of the pressure beyond separation to leading order is to some extent borne out by the numerical results given earlier (see figure 10). Above the shear layer the solution (3.4) again holds. Then the position of the dividing
streamline, on which \( f = 0 \), is given by \( Y = Y_d(X) + O(h^{-1/4}) \), where

\[
Y_d(X) = h(G_s - G(X)) + h^{1/2} \ln(h)G_s'X_0
\]

(3.57)

in view of (3.4), (3.56) and the requirement that \( f = O(h^{1/4}) \) in the shear layer (from the join with regions (I), (II) upstream). Therefore in the shear layer we set \( Y = Y_d(X) + h^{-1/4} \tilde{y} \) and

\[
(U, f) = (h^{1/2}U, h^{1/4}f)(X, \tilde{y}) + \ldots
\]

(3.58)

Substitution of these expansions into (1.12a) gives the boundary layer equations

\[
\tilde{U}_x - \tilde{f}_x \tilde{U}_y = \tilde{U}_{yy}, \quad \tilde{U} = \tilde{f}_y
\]

(3.59a)

subject to the boundary conditions

\[
\tilde{U} \to (2G_s)^{1/2} \quad \text{as } \tilde{y} \to +
\]

(3.59b)

\[
U \to 0 \quad \text{as } \tilde{y} \to -
\]

(3.59c)

from (3.4) with (3.56), and the requirement that the flow below the shear layer is relatively slow. Starting conditions, as \( X \to X_s^+ \), can also be found from the match with regions (I) and (II) upstream. Thus the shear layer is of the familiar Chapman form (or, more precisely, it tends to that form in the limit \( X \to + \) when the influence of the incoming flow profile is negligible) driven by a constant external velocity (3.59b) above, with zero velocity (3.59c) below. Far downstream, then, as \( X \to - \), the shear-layer width expands as \( X^{1/2} \).

The Chapman form breaks down when the velocity gradient, of order \( h^{3/4}X^{-1/2} \) as \( X \to - \), is comparable with the external shear of (3.3). That occurs when \( X = O(h^{3/2}) \).

The same behaviour is found to hold if we allow for much larger velocities in the reversed flow region, incidentally, as long as \( U < h^{1/2} \) there. For example, if
\[ P = -hG_s + h^{\frac{1}{2}}P_1(X) + \ldots \quad (3.56') \]

in the current stage (so that \( \tilde{P}(X) \to (2G_s)^{1/2} \) as \( X \to X_s^+ \)) then the dividing streamline is such that

\[ Y_d(X) = h(G_s - G(X)) + ((2G_s)^{\frac{1}{2}} - \tilde{P}_1(X))h^{\frac{1}{2}} + \ldots \quad (3.57') \]

Then the expansions (3.58) lead to the same problem (3.59) in the shear layer. The indeterminacy of the exact position to \( O(h^{1/2}) \) of the dividing streamline in (3.57') presents no problems here, as long as \( \tilde{P}_1(X) \) remains finite in \( X_s < X < \), for when the Chapman form breaks down, that is, when \( X \sim h^{3/2} \) downstream, the shear-layer thickness itself is also \( O(h^{1/2}) \).

On the longer \( O(h^{3/2}) \) length scale we again suppose that the pressure is constant to leading order:

\[ P = -hG_s + h^{\frac{1}{2}}P(\bar{x}) + \ldots \quad (3.60) \]

where \( X = h^{3/2}\bar{x} \). In the shear layer we set

\[ Y = hG_s + h^{\frac{1}{2}}(\bar{y} - \tilde{P}(\bar{x})) ; \quad (\bar{U}, \bar{v}) = (h^{\frac{1}{2}}\bar{U}, h^{\frac{1}{2}}\bar{v})(\bar{x}, \bar{y}) + \ldots \quad (3.61) \]

where the scalings are implied by the oncoming Chapman form. Then \( \bar{U}, \bar{v} \) obey the boundary-layer equations (3.59a) again, with

\[ \bar{U} - \bar{v} \to 0 \text{ as } \bar{y} \to \ast \quad (3.62a) \]
\[ \bar{U} \to 0 \text{ as } \bar{y} \to -\ast \quad (3.62b) \]

from (3.3) with (3.60), (3.61), and also to merge with the slower flow below. A small-\( \bar{x} \) expansion shows that it is possible to match the solution here with the previous stage (as \( X \to \ast \)). The shear layer expands like \( \bar{x}^{1/3} \) as \( \bar{x} \to \ast \), and begins to fill the entire region \( Y > 0 \) when \( h^{1/2}\bar{x}^{1/3} \sim h \) i.e. \( \bar{x} \sim h^{3/2} \) or \( X \sim h^3 \). This assumes that \( \tilde{P}(\bar{x}) \) remains finite within the current range, of course. Thus in the final stage of the motion we set
(U, U, P, A, X, Y) = (hU, h2W, hP, hA, h2X, hY) + ... (3.63)

as implied by the shear layer flow above when \( \hat{x} \to \infty \). The governing equations (1.12) then give

\[
\hat{U} \hat{U}_{\hat{x}} - \hat{y} \hat{U}_{\hat{y}} = \hat{U}_{\hat{y}} \hat{y} , \quad \hat{U} = \hat{y} 
\]

subject to the conditions

\[
\hat{U} - \hat{y} \to \hat{p}(x) \quad \text{as} \quad \hat{y} \to \infty \\
\hat{U} = \hat{y} = 0 \quad \text{at} \quad \hat{y} = 0
\]

(3.64a) (3.64b) (3.64c)

The starting conditions are

\[
\hat{U} \to \hat{y} - \hat{G}_s \quad \text{for} \quad \hat{y} > \hat{G}_s \quad \text{as} \quad \hat{x} \to 0^+ \\
\hat{U} \to 0 \quad \text{for} \quad \hat{y} < \hat{G}_s \quad \text{as} \quad \hat{x} \to 0^+ \\
\hat{p}(0) = - \hat{G}_s
\]

(3.64d) (3.64e) (3.64f)

The above description of the flow is therefore complete provided that a solution of (3.64a-f) can be found such that

\[
(\hat{U}, \hat{y}, \hat{p}) \to (\hat{y}, \frac{1}{2} \hat{y}^2, 0) \quad \text{as} \quad \hat{x} \to \infty
\]

(3.64g)

so that reattachment and an eventual return of the flow to its original upstream form are achieved. A numerical integration would be required to verify that such a solution of (3.64) exists. This has not been attempted as it seems virtually certain that these equations are unable to describe reattachment anyway, owing to the absence of an adverse pressure gradient in (3.64a). Moreover, a small-\( \hat{x} \) expansion of the solution has not been found.
So the proposal here of a viscous eddy filling the entire lower deck seems to lead to an inconsistency. The same proposal, however, does lead to a self-consistent account of separation and reattachment in the flow studied by Smith & Daniels, which we recall has

\[ A = 0 \]

in place of (1.12g). The final stage of the motion in their case has \( U, \tau, P, X, Y \) of orders \( h, h^2, h^2, h^3, h \) respectively, so that a nonlinear/viscous/pressure balance governs reattachment and the ultimate return to a uniform shear flow. These scalings suggest that the same account is valid for a wider class of flows, with (1.12g) replaced by

\[ P = \gamma h A \quad (3.65) \]

with \( \gamma \) an \( O(1) \) factor. The equations (1.12a-e') with (3.65) do arise in a physically sensible way in the present context, incidentally, for they describe subcritical liquid-layer flow over a hump of length \( Lh^{-\gamma} \lambda^{-5}(1-Fr)^{-3} \) and height \( \varepsilon^2 \gamma^{-1} \lambda^{-2}(1-Fr)^{-1} \) in terms of \( \gamma \) and the flow parameters defined in §1. In terms of the two-parameter system (1.12a-e'), (3.65) our main interest is in the double limit \( h \to \infty, \gamma \to 0 \), with \( \gamma h = 1 \). To throw more light on that problem, we give first the following brief description of the flow in the limit \( h \to \infty \) with \( \gamma \) fixed and of order unity.

Ahead of separation, a classical boundary layer of thickness \( O(h^{-1/2}) \) is again provoked between the majority of the lower deck, wherein the solutions are

\[ U = Y + A + hG \quad (3.66a) \]

\[ \tau = \frac{1}{2}(Y + A + hG)^2 + P , \quad (3.66b) \]

and the wall at \( Y = 0 \). In the classical boundary layer \( U, \tau \) are of order \( h, h^{1/2} \) respectively. From (3.66b) and (3.65) we may then infer that
\[ A = hA_0(X) + h^{\frac{1}{2}}A_1(X) + \ldots \] (3.67)

where
\[ A_0(X) = -\gamma - G(X) + \left(\gamma^2 + 2\gamma G(X)\right)^{\frac{1}{2}}. \] (3.68)

The pressure gradient driving the flow in the classical boundary layer is then \( \gamma^{-1}A_0'(X) \), and in general the Goldstein singularity is again encountered at \( X = X_s \) beyond the top of the hump; here \( X_s \) depends on \( \gamma \), of course, as well as on the hump shape \( G(X) \). The Goldstein singularity is again removable by the same sequence of double structures as detailed above, although the length scales involved are now those of Smith & Daniels. Beyond separation the detached shear layer of thickness \( O(h^{-1/2}) \) (again of the Chapman form; see (3.59)) lies at a constant distance \( hG_s \) from the undisturbed wall, assuming constancy of pressure to leading order here; specifically
\[ P = h^2\gamma^{-1}A_0(X_s) + o(h^2) \quad \text{in} \quad 0 < X - X_s = O(1). \] (3.69)

Above the shear layer the flow is given by the uniform shear
\[ U = Y + hA_0(X_s) + hG(X) \] (3.70)

Below the shear layer the flow is relatively slow \( (U = o(h)) \) due to the uniform pressure in (3.69). The shear layer expands, and begins to fill the whole lower deck when \( X \sim h^3 \), at which stage the scalings
\[ (U, \hat{\gamma}, P, A, X, Y) = (h\hat{U}, h^2\hat{\gamma}, h^2\hat{P}, h\hat{A}, h^3\hat{x}, h\hat{y}) + \ldots \] (3.71)

are implied by the oncoming flow. With (3.71) the controlling equations (1.12) become
\[ \hat{U}\hat{U}_{xx} - \hat{\gamma}_x\hat{U}_y = -\hat{P}'(\hat{x}) + \hat{U}_{y}\hat{y}, \quad \hat{U} = \hat{\gamma} \] (3.72a)
\[ \hat{U} = \hat{\gamma} = 0 \quad \text{at} \quad \hat{y} = 0 \] (3.72b)
\[ \hat{U} \sim \hat{\gamma} + \hat{A}(\hat{x}) \quad \text{as} \quad \hat{y} \to \pm \] (3.72c)
\[ \hat{P} = \gamma\hat{A} \] (3.72d)
In addition, the match with the upstream flow provides the following starting conditions

\[
\begin{align*}
\hat{U} &\rightarrow \hat{y} + \hat{A}(0) \quad \text{for } \hat{y} > G_s \quad \text{as } \hat{x} \rightarrow 0^+ \\
\hat{U} &\rightarrow 0 \quad \text{for } 0 < \hat{y} < G_s \quad \text{as } \hat{x} \rightarrow 0^+ \\
\hat{A}(0) &= -\gamma - G_s + \left(\gamma^2 + 2\gamma G_s\right)^{1/2}
\end{align*}
\] (3.72e)

Finally, the recovery of the uniform shear flow far downstream gives the boundary condition

\[
(\hat{U}, \hat{\gamma}, \hat{P}, \hat{A}) \rightarrow (\hat{y}, \frac{1}{2}\hat{y}^2, 0, 0) \quad \text{as } \hat{x} \rightarrow \infty .
\] (3.72f)

Thus (3.72) is a closed problem to determine the point of reattachment \(x_{\text{reatt}}\) and the return to the original undisturbed uniform shear flow. The solution depends on just the one parameter \(\hat{\gamma} = \gamma/G_s\) since we see that \(\hat{U}/G_s, \hat{\gamma}/G_s, \hat{P}/G_s, \hat{A}/G_s\) depend only on \(\hat{\gamma}/G_s, \gamma/G_s\) and \(\hat{\gamma}\). We expect, incidentally, that \(G_s\), which itself depends on \(\gamma\), will remain of order unity for all values of \(\gamma\) since separation almost certainly takes place just beyond the maximum of the hump in all but a few exceptional cases.

The solution of (3.72) provides an account of reattachment which is consistent with the proposed form of separation upstream. For reattachment here takes place under the action of viscosity. In addition, the pressure gradient driving the flow is unknown and interacts nonlinearly with the boundary-layer displacement. The alternative of a mainly inviscid eddy bounded by the wall and a (relatively thin) viscous shear layer leads to a contradiction. For the eddy would then be of the Prandtl-Batchelor type, with closed streamlines and constant vorticity (Batchelor 1956). Then a simple algebraic equation follows for the eddy shape, which in turn implies that the eddy has constant width, contradicting the assumption that
The problem (3.72) needs a numerical treatment in general. Smith & Daniels' computations correspond to the limiting case $\gamma \to \infty$; their results predict reattachment at $x = 0.076G_s^2$. Our main interest is in the other extreme $\gamma \to 0$, of course. The necessary numerical solutions for a range of values of $\gamma$ would be of great interest, partly because they may throw more light on the process of reattachment in our original problem (1.12a-e',g), and partly because the enlarged problem (1.12a-e') with (3.65) makes some physical sense in its own right, describing flows over humps of the same height as those defined by (1.5) but with shorter length scales. However, we cannot say with any certainty that solutions of (3.72) exist for all (positive) values of $\gamma$, although Smith & Daniels' solution strongly suggests that solutions do exist for some (sufficiently large) values of $\gamma$.

Some numerical solutions of the complete original flow problem (1.12a-e') with (3.65) for a range of values of $\gamma$ with $h$ fixed (and reasonably large), which were obtained using the numerical scheme described earlier in this section, are given in figure 12. These give some insight into the behaviour of the large-$h$ solutions as $\gamma$ varies from $O(1)$ values to $O(h^{-1})$ values. Upstream of separation the leading order prediction of the pressure (from (3.65) with (3.67), (3.68)) is in reasonable agreement with the calculations. The three cases shown in figure 12 are for $h = 8$. Separation takes place at $X = X_s = 0.615$, $0.625$, $0.645$ for $\gamma = 1$, $1/2$, $1/8$ respectively, so that $G_s = 0.897$, $0.879$, $0.839$ in turn. The most significant change, however, is the substantial decrease in the length of the eddy as $\gamma$ decreases; rather surprisingly, reattachment appears to occur much more abruptly when $\gamma$ is small.
Captions for figures.

Figure 1. Definition sketch for the undeveloped flow in a liquid layer over a small two-dimensional hump, showing the coordinate system, the undisturbed liquid depth $t^*$ and the uniform fluid velocity $U^*$ outside the thin boundary layer.

Figure 2. Schematic diagram of the three regions I-III of flow over a hump of (non-dimensional) height $H$ and length $L$, at a location where the boundary layer is of thickness $\varepsilon$. Region III is of thickness $\delta_w = O(H) < \varepsilon$.

Figure 3. Possible form of a flow with large-scale separation ahead of a step of finite dimensions in a supercritical stream, with the free interaction of Gajjar & Smith (1983) occurring asymptotically far ahead of the step.

Figure 4. Linearized solutions for the pressure (* surface displacement) and the wall shear distributions for supercritical flow over the tanh step (1.15). Also shown (dotted lines) are the up- and down-stream asymptotes (2.9a,b) of the pressure distribution.

Figure 5. Solutions of (1.12a-f) (with (1.15)), for the various values of the step-height parameter $h$ shown, generated by the first numerical method. Regular separation is seen to take place just upstream of the hump when $h = 2.78$. (a) pressure (or surface displacement) (b) wall shear (c) wall shape (1.15).
Figure 6. Solution of (1.12a-f) generated by the third numerical method: (a) pressure, (b) wall shear. Here solutions with substantial regions of reversed flow are obtained. Note the rapid reattachment on the forward face of the step, and the long upstream response.

Figure 7. Comparison of the present solutions (with h = 5.3) with the free-interaction solutions of Gajjar & Smith (1983). (a) Pressure: — present, o Gajjar & Smith, (b) wall shear: — present, x Gajjar & Smith.

Figure 8. (a) Variation of position of separation $X_s$ and pressure $P$ (at $X = -2.39$) with step height $h$. (b) Variation of $P$ with $X_s$: x numerical, — — — predicted asymptote (1.14).

Figure 9. Effect of grid refinement on the numerical solution of (1.12a-e',g) with $h = 8$ at various streamwise locations: (i) pressure, (ii) wall shear. The symbols $\Delta_x, \Delta_y, \circ$ correspond to grids (a) $\Delta_x = 0.5,\Delta_y = 0.4$; (b) $\Delta_x = 0.375,\Delta_y = 0.3$; (c) $\Delta_x = 0.25, \Delta_y = 0.2$.

Figure 10. Separated-flow solutions of (1.12a-e',g) with (3.1) (sub-critical flow over a hump) for $h = 4, 8, 12$. (a) Pressure and wall shear distributions in the vicinity of the hump, showing rapid rise on forward face, then rapid fall, separation, and approach to seemingly uniform conditions (for large $h$) downstream; (b) on a longer scale, showing reattachment and increasing dimensions of reversed flow region; (c) streamline plots for $h = 8, 12$. 
Figure 11. Schematic diagram showing the approach to separation from the hump surface, and the various length scales and flow regions, in the limit $h \rightarrow \infty$.

Figure 12. Solutions of the two-parameter problem (1.12a-e'), (3.65) for $h = 8$ and $\gamma = 1, \frac{1}{2}, \frac{1}{8}$ as shown. Pressure and wall shear distributions are shown (a) over the hump, and (b) on a longer scale downstream of the hump. (c) Streamline plots showing the substantial decrease in the length of the eddy as $\gamma$ decreases to $1/h$. 
\[ \frac{p(x)}{h} \]

\[ \frac{(\pi(x) - 1)}{h} \]

**Figure 4.**
Figure 5
Figure 6
Figure 8.
\begin{figure}
\centering
\begin{tabular}{ccc}
\hline
$P(\lambda)$ & $X$ & $T(\lambda)$ \\
\hline
3.29 & .70 & 8.80 \\
4.75 & 6.0 & 1.30 \\
4.20 & 50 & 1.00 \\
4.20 & 100 & 1.00 \\
4.20 & 1.10 & 3.00 \\
4.10 & 2.50 & 7.16 \\
2.50 & .80 & 10.16 \\
.80 & 10.10 & .20 \\
.40 & 13.16 & .60 \\
.21 & 19.16 & .84 \\
\hline
\end{tabular}
\caption{(i)\hspace{0.1cm} (ii)\hspace{0.1cm} Figure 9}
\end{figure}
Figure 10(a)
\textbf{FIGURE 10 (b)}
Figure 12(a)
Figure 12 (b)
CHAPTER THREE

The stationary hydraulic jump: comparison between experiments and theory.
CHAPTER THREE

§1 Introduction.

Craik, Latham, Fawkes & Gribbon (1981) describe some interesting experiments on a quasi-stationary circular hydraulic jump, of the type commonly encountered in, for example, the initial filling of a kitchen sink with water from a tap or in numerous other such filling-up processes of smaller or larger scale. Under certain circumstances, depending on the local Reynolds number and other conditions, even though the jump itself may be very pronounced it can remain in practice predominantly laminar, steady and two-dimensional (when viewed locally), with only relatively minor fluctuations and three-dimensionality present; in other situations three-dimensionality and unsteadiness are overwhelming features. Craik et al (1981) report on some experimental properties of both cases. Our main concern here is in the former experimental case and in making a quantitative comparison with the recent theory of Gajjar & Smith (1983) for steady hydraulic jumps in supercritical liquid-layer flow. The theory is based on the ideas of viscous-inviscid free interaction and upstream influence, leading to regular separation, and it regards the "jump" as a continuous local phenomenon. Earlier, and rather different, theoretical or numerical studies are given by Watson (1964), Bouhadef (1978) and others [see also Tani (1949) and references in Lighthill (1978), Gajjar & Smith], with the jump being treated broadly as an inviscid discontinuous phenomenon, unlike in Gajjar & Smith. Other experimental studies are referred to in the Craik et al and Gajjar & Smith papers.

In the following, §2 summarizes the Gajjar & Smith theory, a main
result of which is the prediction (2.4) below for the surface shape (= pressure) just downstream of the effective start of the jump. This prediction relies on both viscous and inviscid effects, incidentally, and it would not emerge from an inviscid approach. Then §3 presents comparisons between the theoretical predictions and the Craik et al experimental findings. The quantitative agreement is found to be quite close, especially in view of the largeness of the Froude number and other extreme conditions holding in the experiments and the assumption of high Reynolds number in the theory. Additional comparisons of a qualitative nature are also described and overall the agreement seems reasonably favourable. Further comments are given in §4.
The hydraulic-jump theory.

Here we summarize, normalize and re-interpret the Gajjar & Smith viscous-inviscid approach.

The theory assumes inter alia steady, laminar, planar motion locally, in the vicinity of the jump, with a large local Reynolds number \( Re \) and an oncoming liquid-layer flow [depth \( \ell_d \)] which is predominantly uniform with a relatively thin viscous sublayer [of depth \( \varepsilon \ell_d \)], relatively far from the source of liquid. All this is on a fixed horizontal flat surface, given by \( y_d = 0 \), in cartesian coordinates \( x_d, y_d \). The subscript \( d \) where used stands for dimensional quantities, while \( Re = u_d(0)\ell_d/\nu_d \) where \( \nu_d \) is the kinematic viscosity of the liquid and \( u_d(0) \) is the oncoming flow speed at the top free surface, \( \rho_d \) is the liquid density and \( x_d = x_d(0) \) denotes the stream-wise location of the jump. Close to the surface the liquid velocity components \( u_d, v_d \), in the \( x_d, y_d \) directions respectively, and the pressure \( p_d \) take the form

\[
\begin{align*}
  u_d &= u_d(0)\varepsilon(F - 1)^{-1}\lambda^{-1}U(X,Y) \\
  v_d &= u_d(0)\varepsilon^{-2}Re^{-1}(F - 1)\lambda^2V(X,Y) \\
  p_d &= p_d(0) + \rho_d u_d(0)^2\varepsilon^2(F - 1)^{-2}\lambda^{-2}P(X)
\end{align*}
\]

with

\[
x_d = x_d(0) + \varepsilon^5Re(F - 1)^{-3}\lambda^{-5}X \quad \text{and} \quad y_d = \varepsilon^5(F - 1)^{-1}\lambda^{-2}Y,
\]

where \( p_d(0) \) is the atmospheric pressure, \( F = u_d^2/\gamma\ell_d \) is the Froude number \( (F > 1) \) and the oncoming nondimensional skin friction \( \tau_w(\varepsilon u_d(0)/\gamma u_d/\gamma y_d) \) at \( y_d = 0 \) = \( \lambda\varepsilon^{-1} \) is large, of order \( \varepsilon^{-1} \).

Then, within the sub-boundary-layer where \( Y \) is \( O(1) \), to leading order \( U, V, P \) satisfy the parameter-free problem consisting of the
scaled boundary-layer equations in terms of $X$ and $Y$, subject to the
no-slip condition at $Y = 0$ and the sublayer displacement condition
$U \sim Y + A$ as $Y \to -\infty$, where the pressure-displacement relation
between $P$, $A$ is $P = -A$, due to the gravity force ($g$). The corres-
ponding shape of the upper surface is given by

$$\frac{Y_d}{\ell_d} = 1 + \varepsilon^2 F(F - 1)^{-2} \lambda^{-2} P(X)$$

(2.2)
in nondimensional form.

The nonlinear parameter-free problem summarized above yields
Gajjar & Smith's viscous-inviscid description of the development of a
hydraulic jump. It allows a free interaction to begin upstream, as an
exponential departure from the oncoming motion, and computations
and analysis then show that downstream, for $X$ large and positive,
the pressure increases in the form

$$P(X) \sim P_1 X^m \text{ (as } X \to \infty \text{)}$$

(2.3)

where the constants $P_1$ and $m$ are given in (2.4b) below. Hence the
nondimensional deviation of the upper surface there is predicted to
have the (convex) form:

$$\frac{v_d - \ell_d}{\ell_d} = \frac{V}{U} m^{-2} F(F - 1)^{-2} P_1 \left[ \frac{x_d - x_d(0)}{\text{Re} \ell_d} \right]^m$$

(2.4a)

with

$$P_1 = 0.94796..., \quad m = 2(\sqrt{7} - 2)/3 = 0.43050...$$

(2.4b)

This prediction and others are compared with experiments below.
§3. Comparisons with experiments.

Craik et al. experimentally established the quasi-stationary hydraulic jump, in an approximately radial flow on a flat horizontal table, by making an axisymmetric water jet (of radius $a$) impinge vertically at the centre and then spread out on the table. The conditions far from the centre were either controlled or allowed to vary freely, this influencing the position $(x_d^{(0)})$ of the jump produced.

For a given volume flux $Q$ in the jet, conservation of mass and momentum implies the relations $2\pi x_d t_d u_d^{(0)} = Q = \pi a^2 u_d^{(0)}$ to leading order provided that, as assumed in the theory, the upper-surface pressure remains at $p_d^{(0)}$ and the water-layer velocity $u_d$ remains predominantly at $u_d^{(0)}$, between the centre $x_d = 0$ and the jump $x_d = x_d^{(0)}$. Hence the equations

$$\ell_d = \frac{a^2}{2x_d^{(0)}} \quad (3.1)$$

$$F \left(= \frac{u_d^2}{\ell_d} \right) = \frac{Q^2}{4\pi^2 x_d^{(0)} \ell_d^3} \quad (3.2)$$

$$Re \left(= \frac{u_d \ell_d}{\nu_d} \right) = \frac{Q}{2\pi x_d^{(0)} \nu_d} \quad (3.3)$$

express $\ell_d$, $F$, $Re$ in terms of the quantities $Q$, $a$, $x_d^{(0)}$ measured by Craik et al. To apply (2.4) we need also the skin-friction parameter $\tau_w$ as $x_d \rightarrow x_d^{(0)}$, which follows from the classical boundary-layer properties assumed to hold upstream of the jump as

$$\tau_w = \sqrt{3} Re^{\frac{1}{2}} \left(\frac{x_d^{(0)}}{\ell_d}\right)^{-\frac{1}{2}} \left[\hat{\lambda} = 0.332\ldots\right] \quad (3.4)$$

where $\hat{\lambda}$ is the traditional planar Blasius skin-friction value and the extra factor $\sqrt{3}$ is due to the axisymmetry of the oncoming flow (see Watson 1964). The results (3.1)–(3.4) now allow the prediction (2.4) for
the upper-surface shape, downstream of the jump, to be worked out for given experimental values of \( Q, a, x_d(0) \). For water flow (2.4) then has the form

\[
\frac{y_d - t_d}{t_d} = (0.0352) \frac{Q^{15m/2-3}(x_d(0))^{2-5m/2}}{a^{15m/4-4}} (x_d - x_d(0))^{15m/2}
\]

[c.g.s.] (3.5)

in c.g.s. units, we note, with \( \nu_d \) taken as 0.01cm\(^2\)/sec and \( g \) as 981cm/sec\(^2\). In some of Craik et al.'s measurements for water flow the value of \( a \) is not given, however [e.g. in their figure 6], and in such cases we estimate \( t_d \) from their graphs and convert the prediction (2.4) to the form

\[
\frac{t_d^{15m/2-2}(x_d(0))^{10m-4}}{Q^{15m/2-3}} \left( \frac{y_d - t_d}{t_d} \right) = (0.0149) \left[ \frac{x_d - x_d(0)}{t_d} \right]^{15m/2}
\]

[c.g.s.] (3.6)

instead, using only (3.2)-(3.4).

To start comparing theory and experiment quantitatively we address first a single representative case, \( Q = 18\text{cm}^3/\text{sec.}, x_d(0) = 3.2\text{cm.}, t_d = 0.015\text{cm.}, \) read from figure 6 of Craik et al. Here (3.2)-(3.4) yield the values \( F = 242.1, Re = 89.53 \) and \( \tau_w = 0.3725 \) in turn. So (2.4) predicts an increase \( y_d - t_d = 0.0292\text{cm.} \) in the free-surface height at a representative position 0.25cm. downstream of the jump (i.e. at \( x_d = 3.45\text{cm}; \) the same result follows from (3.6). This is fairly close to the measured result in Craik et al.'s figure 6. Applying the above theme to other \( x_d \) values and to the other two experimental cases available, we obtain the direct comparisons shown in our figure 1. Conversely, (3.6) predicts a universal form holding, as \( Q, t_d, x_d(0) \) vary, and this is tested in figure 2. All three experimental curves there (for \( Q = 11, 18, 29 \)) collapse quite near the
predicted universal form just beyond the jumps. In both figures, the quantitative agreement near the jumps is reasonably encouraging.

Several other comparisons of a more qualitative kind can also be made. First, the local Reynolds number Re used here is typically about 100 in the quantitative comparisons described above where steady laminar flow is supposed and hence it lies below the approximate critical value of 147 for instability noted by Craik et al [their $R_j$ is equal to our Re]. Second, Craik et al (their pp. 360, 361 and figure 10) observe that the eddy length falls as the size of the downstream disturbance is increased. This can be explained theoretically to some degree by examination of the scalings in §§2,3. It is a familiar feature in viscous–inviscid interactive flows [see also §4] that increasing the disturbance size forces upstream separation to move further upstream, thus decreasing $x_d^{(o)}$. So then, for fixed Froude number $F$, the typical length scale of interaction, and therefore probably of the eddy, varies like $Re\tau_w^{-3} \approx x_d^{(o)} Q^{-3/2}$ (from (3.1) –(3.4)) and hence decreases, as observed experimentally, for fixed flow rate $Q$. The predicted decrease of length like $(x_d^{(o)})^4$, for given $Q$, is not inconsistent with the experiments, and neither is the combined dependence on $x_d^{(o)} Q^{-3/2}$ for varying $x_d^{(o)}$, $Q$ in Craik et al's figures 9 and 10, although there the disturbance conditions downstream are not always controlled. Likewise, (3.1)–(3.4) would imply an increasing of $\tau_w$, a decreasing of $\varepsilon$ and hence a relative rise in the top-surface displacement in (2.4), as the downstream disturbance is increased, and this again is not out of line with the observed trends. Third, the observed small rise and then dip of the water depth just prior to the main jump would seem to be accountable by surface-tension effects, as indeed Craik et al suggest. More specifically, surface tension can add a term $= d^2A/dX^2$ to the
pressure-displacement interaction law, giving \( P = -A + \gamma d^2 A/dX^2 \)

where \( \gamma \) is a scaled surface-tension parameter (see chapter 6). The free interaction may then be capable of producing a rise upstream, followed by a fall and then a rise of the pressure, and hence the liquid depth, culminating in the downstream behaviour (2.3) which is dominated by gravity effects. Fourth, and as an alternative, we consider the three representative experimental runs (a) \( Q = 11.6, a = 0.155, x_d^{(o)} = 2.3 \), (b) \( Q = 19.8, a = 0.18, x_d^{(o)} = 3.2 \), (c) \( Q = 26.0, a = 0.215, x_d^{(o)} = 4.0 \), in Craik et al's table 1, where again we use c.g.s. units throughout. These runs are analogous to those shown in our figure 1. For run (b), for instance, (3.1)-(3.4) predict the values \( t_d = 0.005065 \text{cm}, F = 7630.5, \text{Re} = 98.48, \tau_w = 0.227 \), in turn, and hence (2.4) implies a relative increase \((y_d-t_d)/t_d\) of 13.82 at a distance \( x_d-x_d^{(o)} = 1 \text{cm.} \) beyond the jump; (3.5) yields the same value, as a check. The corresponding predictions for runs (a), (c) are relative increases of 13.01, 11.68 at 1cm. beyond the jumps. The closeness of these relative increases, and perhaps even their ordering in sizes, i.e. (b), (a), (c), are not inconsistent with the analogous cases in Craik et al's figure 6, and with the closeness of the downstream depths shown in their table 1. Notice also that the relative estimates for (a) - (c) are larger than the relative increases in figure 1 but this is compensated by the smaller values of \( t_d \) estimated here. Fifth, Craik et al's experimental description, in their pp. 359, 360, of the motion near separation and around the subsequent eddy all appears to tie in with the theoretical description, in which the free interaction causes the majority of the fast-moving incoming liquid layer to be converted into an out-going, detached, shear layer riding over slower flow nearer the solid surface.
§4. Further Comments.

The quantitative and qualitative comparisons in §3 between theory and experiment seem reasonably affirmative and encouraging on the whole. Of course, a number of factors need to be borne in mind, especially the extreme values of certain parameters and the experimental difficulties of measurement. The theoretical predictions are a little sensitive to the value of the liquid depth $t_d$ and skin friction $\tau_w$ just prior to the jump, for instance, since from (3.2) the Froude number $F = t_d^{-3}$ for given $Q$, $x_d^{(0)}$ [or $F = a^{-6}$ from (3.1)], but $t_d$, $\tau_w$ are difficult to measure accurately in practice, as is the jet radius $a$. Although the values of $t_d$, $\tau_w$ could be deduced instead from simple extra theoretical notions, such as in (3.1), these in turn introduce extra approximation. Further, the Froude number $F$ can be very large ($\approx 10^2 - 10^4$) and the thickness parameter $\varepsilon$ is about 2 in practice, whereas the theory takes $F$ to be an $O(1)$ parameter and $\varepsilon$ is assumed to be small. The sometimes excessive Froude number in fact is a most significant numerical factor in (2.4) as regards the experimental comparisons, because $F$ in (2.4a) is raised to the power $3m-1 (\approx 0.29150...)$ effectively [since $F-1$ is approximately equal to $F$] which is comparable with the powers $5m-2 (\approx 0.15250...)$, $-m (\approx -0.43050...)$ of $\tau_w$, $Re$, respectively, but these last two parameters are not so excessive. Nevertheless, with allowance for the relative smallness of the Reynolds number $Re$ as well as the extreme parameter values above, it would appear that all the main features of stationary hydraulic jumps observed experimentally by Craik et al do lean fairly strongly towards the Gajjar & Smith theory.

Certain related theoretical matters should also be considered here. The first is that the complete shape of the free surface beyond
the jump starting from the form (2.4) is dependent on the flow conditions downstream, as is the jump location, but the theoretical dependence of both is unknown as yet. It requires more understanding of the complex problem of large-scale separated flow. The computations of liquid-layer flow past a small forward-facing step presented in chapter 2 are aimed at that end. Although the computations prove difficult we note that they do confirm the increase of upstream influence and upstream separation with increasing step height, and the emergence of the free-interaction result (2.4), in line with experimental findings. Next, it seems evident that a large-Froude-number analysis should be pursued, possibly comparing $F$ with various powers of $Re$. This may also provide some theoretical insight into the different categories of free-surface shape, depending on the value of $F$, suggested empirically by Ishigai, Nakanishi, Mizuno & Imamura (1977). Another very useful development, again implied by the experimental comparisons above, and one that we would emphasize, is to consider the longer-scale stage where $x_d^{(0)}$ is $O(\epsilon_d Re)$, so that $\epsilon$ is $O(1)$ and the motion of the whole layer is controlled by the interactive boundary-layer equations. Finally, it would seem worthwhile for the theory to be extended also to the study of unsteady effects, and in particular the instabilities which are found experimentally to originate in the separated flow downstream, as well as to the study of non-stationary hydraulic jumps or bores. Throughout, the significance and sensitive nature of viscous-inviscid interaction in the presence of a solid surface would need to be incorporated; these certainly seem the crucial aspects of the above hydraulic-jump experiments, according to §§2,3.
Captions.

Figure 1. Comparison between theory and experiments, for various flow rates $Q$ (cm$^3$/sec) as shown, for the stationary hydraulic jump. ———: theory (from equation (3.6)); ———: experiments (from Craik et al., figure 6).

Figure 2. Alternative comparison based on the universal form (3.6), re-expressed as $\dot{y} = \dot{x}^m$ with $\dot{y} = (y_d - t_d)/\alpha t_d$, $\dot{x} = (x_d - x_d(o))/t_d$, where

$$\alpha = \frac{0.0149Q^{1.5m/2-3}}{t_d^{1.5m/2-3}(x_d(o))^{10m-4}}$$

———: theory. Experimental results:

$x$, $Q = 11$cm$^3$/sec ($x_d(o) = 2.7$, $t_d = 0.02$cm),

$\circ$, $Q = 18$cm$^3$/sec ($x_d(o) = 3.2$, $t_d = 0.015$cm),

$\Box$, $Q = 29$cm$^3$/sec ($x_d(o) = 4.35$, $t_d = 0.02$cm),

the values of $x_d(o)$, $t_d$ being estimated from figure 6 of Craik et al.
Figure 1.

Figure 2.
CHAPTER FOUR

Large-scale separation in a fully-developed stream.
Large-scale separation in a fully-developed stream.

We continue our investigation of steady two-dimensional laminar liquid-layer flows over obstacles with a discussion of the effects of including vorticity in the mainstream. There may, in some circumstances, be a gradual adjustment from an undeveloped flow over a flat horizontal surface, like those studied in chapter 2, to a fully-developed flow when the boundary layer eventually grows to fill the entire liquid layer. In the high-Reynolds-number limit this adjustment takes place on a vast $O(R)$ length scale; here the Reynolds number $R = U^* \ell^*/\nu$ where $U^*, \ell^*$ are the velocity at the free surface and the depth of the fluid at a station where the flow is fully developed, and $\nu$ is the fluid's kinematic viscosity. Although the adjustment is clearly a result of diffusion of vorticity from the wall, a complete theoretical description of the process has so far not been found, by the way; indeed, it may be that the boundary-layer equations, which control the motion across the entire depth of the fluid on the long $O(R)$ length scale, have no steady solution for free-surface flows over a horizontal wall.

Leaving these theoretical difficulties aside, we consider disturbances, in the vicinity of $x = 0$, say, where the flow is fully or partly developed (see figure 1), that provoke a response on a length scale which is shorter than $O(R)$, the development length of the boundary layer. The undisturbed flow is given in non-dimensional form by

$$u = U_\delta(y), \quad \psi = \psi_\delta(y), \quad p = \sigma(1 - y) \quad \text{in} \quad 0 < y < 1 \quad (1)$$

to leading order, where the inverse Froude number $\sigma = g \ell^*/U^*^2 \quad (= Fr^{-1})$, the external pressure is taken to be zero, and
\[ UB(Y) = t_b(Y), UB(0) = t_b(0) = 0, \]
\[ UB(1) = 1, UB(1) = 0, UB(0) = \lambda. \]

On a length scale \( L < R \) viscous effects are confined to a thin layer of thickness \( O(L^{1/3}R^{-1/3}) \) adjacent to the wall, in view of the behaviour of the basic velocity profile close to the wall. Outside the wall layer the flow is governed mainly by inviscid mechanics. Now suppose first that gravity effects are negligible so that \( \sigma \) is small. In such circumstances the flow is analogous to that in a two-dimensional channel. The free surface of the liquid layer corresponds to that streamline in the channel flow on which the vorticity is zero; recall that the vorticity is conserved to leading order on streamlines that lie outside the viscous wall layer. (The correspondence between these two flows is not quite exact, incidentally, since we have the additional constraint of constant pressure along the free surface in the liquid-layer flow). Smith's study of asymmetrically constricted channel flows (Smith 1976a) is therefore relevant here. He shows that there is a significant response on an \( O(R^{1/7}) \) length scale, which, although long, is still much shorter than the development length of the boundary layer. For then the curvature of the streamlines in the mainstream of the flow, which is caused by the displacement effect of the wall layer, is just sufficient to cause a variation of the pressure across the liquid layer of the same order of magnitude as the pressure in the wall layer. So there is an interaction between the inviscid mainstream, in which \( y = O(1) \), and the viscous wall layer, in which \( y = O(R^{-2/7}) \), on that length scale. The expansions implied in the two distinct regions of the flow indicate that gravity first begins to exert its influence when \( \sigma \) rises to \( O(R^{-2/7}) \). Specifically, we find in the mainstream
\[ u = U_b(y) + R^{-2}u_1(X,y) + \ldots \]  
(3a)
\[ \psi = \psi_b(y) + R^{-2}\psi_1(X,y) + \ldots \]  
(3b)
\[ p = R^{-2}\sigma(1-y) + R^{-4}p_1(X,y) + \ldots \]  
(3c)

from Smith (1976a) (using also (1)), where \( \sigma = \varnothing R^{-2/7} \). Substituting these expansions into the equation of continuity and the x-momentum equation, and solving, we obtain

\[ \psi_1 = A(X)U_b(y) \]  
(4)

The y-momentum equation gives

\[ p_{1y} = U_b(y)\psi_{1xx} \]  
(5)

which on integration and use of (4) yields

\[ p_1(X,y) = P(X) + A''(X)\int_0^Y U_b(s) \, ds \]  
(6)

The pressure and displacement functions \( P(X), A(X) \) are unknown at this stage, but are related to the (unknown) displacement of the free surface at \( y = 1 + R^{-2/7}\eta(X) + \ldots \), where the dynamic and kinematic boundary conditions give, in turn,

\[ \eta(X) = - A(X) \]  
(7a)
\[ P(X) + A''(X)\int_0^1 U_b(s) \, ds = \varnothing\eta(X) \]  
(7b)

In the wall layer, wherein \( Y \approx R^{2/7}y = O(1) \), the expansions are

\[ u = R^{-2}U(X,Y) + \ldots \]  
(8a)
\[ \psi = R^{-4}\psi(X,Y) + \ldots \]  
(8b)
\[ p = \sigma R^{-2} + R^{-4}( - \sigma Y + P(X) ) + \ldots \]  
(8c)

(where the expansion for \( p \) follows from the y-momentum equation and matching with (6)). Then the x-momentum equation and continuity yield the boundary-layer equations at leading order:

\[ U_Ux - \frac{Ux}{Uy} = - P'(X) + U_{YY} \]  
\( U = \xi_Y \) .  
(9a)

The join with the mainstream and the oncoming flow upstream
requires

\[ U \sim \lambda(Y + A(X)) \quad \text{as} \quad Y \to \infty \quad \text{(9b)} \]

\[ (U, \dot{U}, P, A) \to (\lambda Y, \frac{1}{2} \lambda Y^2, 0, 0) \quad \text{as} \quad X \to -\infty \quad \text{(9c)} \]

while the no-slip condition at the wall gives

\[ U = \dot{U} = 0 \quad \text{at} \quad Y = 0. \quad \text{(9d)} \]

Finally, eliminating \( \eta(X) \) between (7a) and (7b) gives the pressure/displacement relation

\[ P(X) = -\tilde{\sigma}A(X) - A''(X) \int_0^1 U_0(s) \, ds. \quad \text{(9e)} \]

Equations (9a–e) serve to determine the leading-order solution in the wall layer (at least when appropriate downstream conditions are given). One solution is evidently a continuation of the uniform shear flow (9c) throughout the current stage. However, other (non-trivial) solutions, starting in the form

\[ P(X) \sim b e^{\varepsilon X} \quad \text{as} \quad X \to -\infty, \quad \text{(10)} \]

are also possible. The exact value of \( b \) in (10) depends on the entire solution (and in particular on the ultimate downstream form as \( X \to \infty \)); \( \varepsilon \) on the other hand is readily found to be the unique real solution of

\[ \frac{\gamma}{\varepsilon^{1/3}} = \tilde{\sigma} + q \varepsilon^2 \quad \text{(11)} \]

where \( \gamma = -3A\imath'(0) (> 0) \) and

\[ q = \int_0^1 U_0(s) \, ds. \]

The rescaling \( (X, Y, U, \dot{U}, P, A) \to (\tilde{\sigma}^2 X, \tilde{\sigma} Y, \tilde{\sigma} U, \tilde{\sigma}^2 \dot{U}, \tilde{\sigma}^3 P, \tilde{\sigma} A) \) of (9) suggests that, in the limit \( \tilde{\sigma} \to 0 \) the flow in the wall layer is still governed by (9a–d) but with (9e) replaced by

\[ P = -A \quad \text{(12)} \]
in terms of the rescaled variables, to leading order in $\tilde{\sigma}^{-1}$. Thus the curvature of the streamlines has only a minor influence when $\sigma$ is increased beyond the present $O(R^{-2/7})$ stage. When $\sigma$ rises to order unity, so that $\tilde{\sigma} = O(R^{3/7})$ formally, the above scalings suggest that the flow again responds on the $O(R)$ [$= O(R^{1/7}\tilde{\sigma}^3)$] length scale, and that there are no subscales in the $y$-direction.

Let us now suppose that $\sigma = O(1)$ and that the flow encounters a small hump in the vicinity of $x = 0$. For definiteness, we suppose that the hump length $L$ is of order unity, comparable with the depth of the fluid. Such a hump is expected to force a response locally on the same $O(1)$ length scale. The only other important streamwise length scale is $O(R)$, as explained above. The interaction on the long $O(R)$ length scale produces only a relatively small linear response however (as in Smith, Brighton, Jackson & Hunt's (1981) study of short humps in external flows) so that, inter alia, separation is not a possibility there. A complete analysis of the leading-order flow perturbations should take into account the flow features on the $O(R)$ length scale, therefore, but the major nonlinear response occurs only on the $O(1)$ length scale of the hump. For humps of relative length $O(1)$, then, the first critical height scale that produces a nonlinear interaction is $O(R^{-1/3})$. Thus, we begin by confining our attention to humps of the form

$$y = R^{-1/3}hG(x)$$

(13)

where $h$ is an $O(1)$ height parameter of the hump. The flow on the $O(1)$ length scale essentially divides into two regions: the mainstream, region (I), in which $y = O(1)$, and the wall layer, region (II), in which $Y = O(1)$ where $y = R^{-1/3}Y$. [A viscous layer is also required adjacent to the free surface, but it has no effect to our order of
working.] The basic flow (1) implies that the expansions

\[
(u, \psi) = (R^{-\frac{1}{3}} U, R^{-\frac{2}{3}} \psi)(x, y) + \ldots \tag{14}
\]

\[
p = \sigma - \sigma R^{-\frac{1}{3}} y + R^{-\frac{2}{3}} p(x) + \ldots
\]

hold in region (II). From the x-momentum and continuity equations we obtain the boundary-layer equations

\[
UU_x - \psi_x U_y = -P'(x) + U_{yy} , \quad U = \psi_y \tag{15a}
\]

subject to the conditions

\[
U = \psi = 0 \text{ at } Y = hG(x) \tag{15b}
\]

\[
(U, \psi, P) \to (\lambda Y, \frac{1}{2} \lambda Y^2, 0) \text{ as } x \to -\infty \tag{15c}
\]

In the mainstream we may then expect the expansions to proceed in powers of \(R^{-1/3}\):

\[
u = U_0(y) + R^{-\frac{1}{3}} u_0(x, y) + R^{-\frac{2}{3}} u_1(x, y) + \ldots \tag{16a}
\]

\[
\psi = \psi_0(y) + R^{-\frac{1}{3}} \psi_0(x, y) + R^{-\frac{2}{3}} \psi_1(x, y) + \ldots \tag{16b}
\]

\[
p = \sigma(1-y) + R^{-\frac{2}{3}} p_1(x, y) + \ldots \tag{16c}
\]

in order to match with (II), with the free surface given by

\[
y = 1 + R^{-\frac{1}{3}} \eta_0(x) + R^{-\frac{2}{3}} \eta_1(x) + \ldots \tag{16d}
\]

It turns out, however, that \(u_0 = \psi_0 = \eta_0 = 0\), mainly because of the condition \(p = \text{constant}\) at the free surface. The match with region (II) then gives

\[
U - \lambda Y \to 0 \text{ as } Y \to -\infty \tag{15d}
\]

The matching condition (15d) reflects the lack of displacement that the wall layer produces in the outer flow; a hump of height \(O(R^{-1/3})\) causes a displacement of the free surface of \(O(R^{-2/3})\) only. This lack of displacement effect is also apparent in other such boundary-layer flows when the length of the hump is less than the interaction length.
scale; see Smith, Brighton, Jackson and Hunt (1981) for short humps in external flows. Again, (15a–d) is also relevant to the flow in a symmetrically constricted channel (Smith 1976a) where the lack of displacement derives from the necessary symmetry of the flow in the core of the channel. In the present case, a significant displacement effect only occurs when \( \sigma \) falls to \( O(R^{-1/3}) \), formally, but before that stage is reached the longer \( O(R^{1/7}) \) length scale, mentioned above, reasserts its influence on the dominant flow features.

Equations (15a–d) form a closed system [given appropriate downstream conditions] and solutions have been obtained for \( h < 1 \) (when linearisation is possible) and \( h = O(1) \) for a variety of hump shapes (Smith 1976a). An important feature of the flow is the lack of upstream influence, to leading order, so that if \( G(x) = 0 \) for \( x < 0 \) then

\[
U = \lambda y, \quad \psi = \frac{1}{2} \lambda y^2, \quad P = 0 \quad \text{for} \quad x < 0. \quad (17)
\]

In region (I) the (non-trivial) leading-order perturbations in (16) satisfy

\[
U_0 u_1 x - U_0' \psi_{1x} = - p_{1x} \quad (18a)
\]

\[
U_0 \psi_{1xx} = p_{1y} \quad (18b)
\]

\[
u_1 = \psi_{1y} \quad (18c)
\]

from the \( x \)- and \( y \)-momentum equations, and the continuity equation respectively. Elimination of \( p_1 \) yields

\[
\psi_{1xx} + \psi_{1yy} = \frac{U_0''(y)}{U_0(y)} \psi_1. \quad (19a)
\]

The conditions at the free surface require \( \psi_1(x, 1) + \eta_1(x) = 0 \), \( p_1(x, 1) = \sigma \eta_1(x) \), which combine to give

\[
\psi_{1y} = \sigma \psi_1, \quad \text{at} \quad y = 1 \quad (19b)
\]
using (10), while the match with region (II) gives

\[ \psi_1 = \lambda^{-1} P(x) \text{ at } y = 0. \]  

(19c)

In addition, we impose the up- and downstream conditions

\[ \psi_1 \to 0 \text{ as } x \to -\infty, \]  

(19d)

\[ \psi_1 \to 0(\exp) \text{ as } x \to \infty. \]  

(19e)

Once \( P(x) \) has been determined from (15), solutions can be found for \( \psi_1 \). The system (19a-e) is different from that describing the flow in a symmetrically constricted channel (Smith 1976a) because of the free-surface condition (19b). In effect the channel flow is a special case of the present work, with \( \sigma = \phi \). Tillett (1968) derived the same system (with \( \sigma = \phi \) again) for a jet of fluid emerging from a channel; in that case \( P(x) = 0 \) also.

Upstream of the hump the solution of (19a-e) can be written

\[ \psi_1(x, y) = \sum_{n=1}^{\infty} e^{\alpha_n x} \pi_n f_n(y) \]  

(20)

where the constants \( \pi_n \) depend on the flow downstream, and \( \alpha_n \) are the eigenvalues (with \( 0 < \alpha_1 < \alpha_2 < \ldots \)) and \( f_n \) the corresponding eigenfunctions of the system

\[ f'' + \left( \alpha^2 - \frac{U_b''}{U_B} \right)f = 0, \]

\[ f(0) = 0, \quad f'(0) = 1, \quad f'(1) = \sigma f(1). \]  

(21)

The eigenvalues of (21) depend on the oncoming velocity profile \( U_b(y) \) as well as on \( \sigma \), of course. The dependence of the first few eigenvalues on \( \sigma \) when the undisturbed flow is half-Poiseuille, with \( U_b(y) = 2y - y^2 \), is graphed in figure 2. Tillett (1968) calculates the first five eigenvalues in the limit case where \( \sigma = \phi \), finding \( \alpha_n = 2.587, 5.969, 9.196, 12.384, 15.556 \) for \( n = 1, \ldots, 5 \). In the neighbourhood of the origin of figure 2, we find from (21) that
\[
\sigma = -\frac{8\alpha^2}{15} + O(\alpha^4);
\]

for an arbitrary shear flow (21) gives
\[
\sigma = -\alpha^2 q + O(\alpha^4). \quad (22)
\]

However, only the top half of figure 2 (\(\sigma > 0\)) is relevant to liquid-layer flows. The behaviour (22) indicates that \(\alpha_1\) (the smallest positive eigenvalue) remains order unity as \(\sigma \to 0^+\).

The analysis of the flow in regions (I) and (II) in the limit \(h \to +\infty\) gives certain vital clues to the form of the flow over humps larger than those given by (13). First, Smith & Daniels (1981) show that \(P \sim h^2\) over the hump when \(h \gg 1\); see also Chapter 2. They also give a self-consistent account of separation and reattachment based on Kirchhoff free-streamline theory. Reattachment then takes place on a long \(O(h^3)\) length scale. The pressure remains \(O(h^2)\) throughout, and forces \(\psi_1\) in the mainstream to become \(O(h^2)\) also, via (19c). This in turn forces flow perturbations of \(O(h^2R^{-2/3})\) in the streamwise velocity in the wall layer ahead of the hump. The response in region (II) then becomes nonlinear when \(h^2R^{-2/3} \sim R^{-1/3}\) i.e. when \(h\) rises to \(O(R^{1/6})\) formally. Thus moderate humps of height \(R^{-1/6}h_M\) (with \(h_M = O(1))\) provoke a significant response and separation ahead of the hump. As the hump height is increased further, \(h_M \gg 1\), the point of separation \(x = x_{\text{sep}}\) is pushed increasingly far upstream; it can be shown, in fact, that
\[
x_{\text{sep}} = -\frac{2}{\alpha_1} \ln(h_M) + O(1) \quad \text{as} \quad h_M \to +\infty. \quad (23)
\]

The arguments leading to (23) follow Smith's (1978) analysis of the analogous flow in a symmetrically constricted channel. Eventually, when the hump height is order unity, comparable with its length and with the fluid’s depth, that is, when \(h_M\) rises to \(O(R^{1/6})\) formally,
separation takes place in the manner of a free interaction asymptotically far upstream at a distance $O(\ln(R))$ ahead of the hump, from (23), as well as on the hump itself. We now go on to describe the grossly-separated flow over a hump, again using Kirchhoff free-streamline theory on the grounds of self-consistency. We observe that the nonlinear adjustment of the flow ahead of the hump takes place on a length scale which is much shorter than the $O(R)$ length scale mentioned earlier, so that even for grossly-separated flows we need not consider the adjustment on that longer scale.
Large-scale separated motion.

The proposed asymptotic structure of the grossly separated motion is shown schematically in figure 3. The up- and downstream free streamlines $C_0C_1$ and $C_2C_3$, which bound eddying motions, are taken to be at constant modified pressure, $p+\sigma y$, according to the Kirchhoff model. Within the eddies there is then a relatively slow recirculatory motion, being driven only by the necessary entrainment of fluid into the thin shear layers surrounding $C_0C_1$, $C_2C_3$. The separation of $C_2C_3$ from the surface of the hump takes place in the manner of a triple-deck interaction (Smith 1977, Sychev 1972) on a short $O(R^{-3/8})$ length scale surrounding $C_2$. The separation process forces a small $O(R^{-1/16})$ correction to the mainly inviscid flow in the mainstream. The same kind of asymptotic structure has been proposed by Smith (1979a) for the high-Reynolds-number flow in a severely, symmetrically constricted channel; our analysis below therefore follows his.

Thus, in the mainstream, which is that part of the flow bounded by the free surface, $C_0C_1$, $C_2C_3$, and the hump between $C_1$ and $C_2$, where $x$ and $y$ are $O(1)$, we have

$$\psi, u, p = (\psi_0, u_0, p_0) + R^{-1/16}(\psi_1, u_1, p_1) + \ldots \quad (24)$$

Then $\psi_0$, $u_0$ and $p_0$ satisfy the inviscid Euler equations, which on integration yield the vorticity equation

$$\psi_{0xx} + \psi_{0yy} = -\omega(\psi) \quad (25a)$$

where the function $\omega$ is determined by the oncoming flow:

$$\omega(\vartheta) = -\vartheta_{yy} \quad (25b)$$

The boundary conditions for (25a) are

$$\psi_0 \to \vartheta_{yy}(y) \quad \text{as} \quad x \to -\infty, \quad (25c)$$

$$\psi_0 = 0 \quad \text{on} \quad C_0C_1, \quad C_2C_3 \quad \text{and} \quad \text{the body surface}, \quad (25d)$$
\( \psi_0 = \psi_0(1) \) on the free surface, \( \psi_0 = 0 \) on the free surface, \( p_0 + \sigma y = \sigma \) \([u_0 = 0]\) on \( C_0C_1 \), \( p_0 + \sigma y = \sigma + p_{oo} \) \([u_0 = u_a]\) on \( C_2C_3 \).

Here \( p_{oo}, u_a \) are constants, and the statements in square brackets follow from Bernoulli's theorem. The shapes of the free streamlines and the free surface are also unknown, as are the positions where \( C_0C_1 \) and \( C_2C_3 \) meet the solid surface; we write

\[
C_0C_1 : y = F_1(x) \pm \ldots , \ x < x_1 + \ldots , \\
C_2C_3 : y = F_2(x) \pm \ldots , \ x > x_2 + \ldots , \\
\text{free surface} : y = 1 + \eta(x) \pm \ldots ,
\]

where the corrections in each case are \( O(R^{-1/16}) \). In addition to the boundary conditions (25c-h) we require separation at \( x = x_2 \) to be smooth. Finally, the specification of the problem is complete when the ultimate downstream form is also given. As \( x \to \pm \) we expect a parallel flow to emerge, with

\[
F_2(x) \to c , \quad \eta(x) \to \Delta
\]

for some unknown constants \( c \) and \( \Delta \); also

\[
u_o \to \begin{cases} 
U_0(y) \quad \text{for} \quad c < y < 1 + \Delta \\
0 \quad \text{for} \quad 0 < y < c 
\end{cases}
\]

\[
p_o \to \sigma(1 + \Delta - y)
\]

when \( x \to \pm \). Here, (27c) reflects the lack of any significant flow between \( C_2C_3 \) and the wall when \( x = O(1) \). If we take \( \psi \) to be the independent variable, rather than \( y \), then Bernoulli's theorem, together with the proposed downstream form (27a-d), gives the simple result

\[ U_0^a(\psi) = U_0^b(\psi) + 2\sigma\Delta \ . \]
Evaluating this expression on the streamline \( \psi = 0 \) yields the following relationship between the slip velocity \( u_s \) along \( C_2C_3 \) and the ultimate displacement of the free surface:

\[
\Delta = -\frac{u_s^2}{2\sigma} ,
\]

(28b)

predicting a fall in the level of the free surface \( (\Delta < 0) \) for separated flows \( (u_s > 0) \). From conservation of mass, we deduce

\[
c = 1 - \frac{u_s^2}{2\sigma} - \int_0^Q \frac{d\psi}{(u_s^2 + U_0^2(\psi))^{1/2}}
\]

(28c)

using (28a) and (28b), where \( Q = I_6(1) \) is the flow rate. Alternatively (28c) may be written

\[
c = 1 - \frac{u_s^2}{2\sigma} - \int_0^1 \frac{U_0(y)dy}{(u_s^2 + U_0^2(y))^{1/2}} .
\]

(28c')

From (25h), (27d) and (28b) we obtain

\[
p_{oo} = \sigma \Delta = -\frac{1}{4} u_s^2 \quad ( < 0 \text{ for } u_s < 0 ) .
\]

(28d)

In addition, the drag \( C_0 \) on the hump can be found in terms of the parameter \( u_s \) by considering an integral momentum balance; we obtain

\[
C_0 = \frac{\sigma}{2} \left[ 1 - (1+\Delta)^2 \right] + \int_0^Q (U_0(\psi) - U_0(\psi)) \ d\psi
\]

\[
= q + \frac{u_s}{2} \left[ 1 - \frac{u_s^2}{4\sigma} \right] + \int_0^1 U_0(y)(u_s^2 + U_0^2(y))^{1/2}dy .
\]

(29)

It is a simple matter to show, incidentally, that

\[
\frac{dC_0}{du_s} = u_sc , \quad C_0(u_s= 0) = 0 ,
\]

(30)

from which we may deduce that \( C_0 > 0 \) as long as the motion is separated \( (c > 0) \).

So the ultimate downstream form (on the \( x = O(1) \) length scale) is determined once the parameter \( u_s \) is known. The requirement that separation be smooth at \( x_2 \) fixes \( u_s \), probably uniquely.
In general, the basic problem (25) (with (27), (28)) requires a numerical solution, which is by no means a straightforward task in view of the unknown positions of the boundaries and the mixed boundary conditions. However, analytical progress can be made in certain cases, when, for example, the hump is relatively long. For suppose the hump is given by

\[ y = F(X) \], with \[ x = LX \] and \[ L > 1 \].

Then, on the length scale of the hump, \( X = O(1) \), the governing equation (25a) suggests the expansion

\[ \psi_0 = \hat{\psi}_0(X, y) + \sum_{r=1}^\infty L^{-2r} \hat{\psi}_r(X, y) \],

of which the first term satisfies

\[ \hat{\psi}_{yy} = -\omega(\hat{\psi}_0) \].

The flow is therefore quasi-parallel to leading order. As such, we may use the results of (28) above, identifying \( c \) with \( F(X) \) at stations where the flow is attached; also \( u_n, \Delta \) now depend on \( X \). Using (28c'), we deduce the relation

\[ F(X) = 1 - \frac{u_n^2(X)}{2\sigma} - \int_0^1 \frac{U_B(y)}{(u_n^2(X) + U_B^2(y))^{1/2}} dy \].

between the hump shape and the leading-order slip velocity \( u_n(X) \) (which drives the classical boundary-layer flow between \( C_1 \) and \( C_2 \)).

We may draw some important conclusions by considering the graph of \( F \) vs. \( u_n \): see figure 4 for the representative case when the oncoming flow is half-Poiseuillean. Other basic velocity profiles (satisfying (2)) give similar features. For example, it can be shown from (34) that
moreover, $F$ (as a function of $u_s$) has a single maximum, $\tilde{F}$ say, in $u_s > 0$, and $\tilde{F} < 1$ for all (finite) values of $\sigma$ and all velocity profiles (provided only that $U_\delta(y) > 0$ for $0 < y < 1$). We can immediately deduce that the governing equations have no solution, in the long hump limit, if the maximum height of the hump $F_{\text{max}} > 1$. More precisely, for given upstream conditions, $\sigma$ and $U_\delta(y)$ are fixed, so that $\tilde{F}$ is fixed, which in turn imposes a maximum obstacle height for which solutions of the steady problem (25) (with (31)) exist. The implication of this result is that a hump with $F_{\text{max}} > \tilde{F}$ "blocks" the flow, causing an adjustment (in an unsteady fashion) of the upstream conditions, which alters $\sigma$ and $U_\delta(y)$ and hence also $\tilde{F}$. When eventually $F_{\text{max}} < \tilde{F}$ a steady solution of the above form may (possibly) emerge.

Assuming that $F_{\text{max}} < \tilde{F}$, so that a solution can be found, figure 4 shows that $u_s$ increases over the forward face of the hump, reaches a maximum at $X = X_{\text{max}}$ (where $F = F_{\text{max}}$), and thereafter decreases, provided the flow remains attached. However, smooth separation can only take place if there is no finite adverse pressure gradient. Therefore, separation must occur at $X = X_{\text{max}}$. Beyond separation $u_s$ is uniform (see (25h)), so that

$$y = F_2(X) = F_{\text{max}} + O(L^{-2})$$

(36)

gives the shape of the free streamline $C_2C_3$ as a straight line, to leading order; then $c = F_{\text{max}}$ also.

We observe that there is no significant upstream influence on the long $O(L)$ length scale of the hump. Instead, the major upstream effects are confined to $O(1)$ distances ahead of the start of the hump,
which is at \( X = 0 \), say. The details of the flow in the neighbourhood of \( X = 0 \) are strongly dependent on the initial shape of the hump. For definiteness, we take \( F(X) = 0 \) for \( X < 0 \), and \( F(X) \sim KX \) for \( X \rightarrow 0^+ \) with \( K \) the initial \( O(1) \) slope of the hump. On the \( x \)-scale the hump has the shape \( F = L^{-1} Kx + o(L^{-1}) \) in \( x > 0 \) so that \( |F| < 1 \) and linearisation of (25) is possible. Thus we set

\[
\psi_0 = \tilde{f}_B(y) + L^{-2} \tilde{\psi}(x,y) + \ldots \tag{37a}
\]

\[
u_0 = U_B(y) + L^{-2} \tilde{u}(x,y) + \ldots \tag{37b}
\]

where the correction to the basic flow is \( O(L^{-2}) \) because \( \psi_0 = 0 \) on the body surface, and \( \tilde{f}_B(y) \sim \frac{\lambda}{2} y^2 \) as \( y \rightarrow 0^+ \). The position of \( C_0C_1 \) is determined by the condition that \( u_0 = 0 \) along it; therefore

\[
F_1 = L^{-2} \lambda^{-1} \tilde{u}(x,0) + \ldots \tag{38}
\]

from (37b). So the upstream eddy is relatively thin. Reattachment occurs when \( C_0C_1 \) intersects the hump surface, which suggests that \( x_1 = O(L^{-1}) \) in view of (38). Therefore \( C_0C_1 \) does not appear in \( x = O(1) \) when \( x > 0 \).

Substituting (37) into (25) we obtain

\[
\tilde{\psi}_{xx} + \tilde{\psi}_{yy} = \frac{U_B''(y)}{U_B(y)} \tilde{\psi} \tag{39a}
\]

\[
\tilde{\psi} \rightarrow 0 \text{ as } x \rightarrow -\infty \tag{39b}
\]

\[
\tilde{\psi}(x,0) = 0 \quad (x < 0) \; , \; = -\frac{1}{2} \lambda K^2 x^2 \quad (x > 0) \tag{39c}
\]

\[
\tilde{\psi}_y(x,1) = \sigma \tilde{\psi}(x,1) \tag{39d}
\]

cf. (19a–e). The solution of (39) which matches (in the limit \( x \rightarrow -\infty \)) with the longer scale flow downstream (as \( X \rightarrow 0^+ \)) is

\[
\psi = \left\{ \begin{array}{ll}
\sum_{n=1}^{\infty} \pi_n e^{\alpha_n x} f_n(y) & (x < 0) \\
-\sum_{n=1}^{\infty} \pi_n e^{\alpha_n x} f_n(y) + \lambda^2 K^2 \left[ g_1(y) + \frac{1}{2} x^2 g_2(y) \right] & (x > 0)
\end{array} \right. \tag{40a}
\]

\[
(40b)
\]
where
\[ g_2(y) = U_b(y) \left\{ \sigma^{-1} - \int_y^1 U_b^{-2}(s) \, ds \right\} , \]
\[ g_1(y) = -U_b'(y) \left\{ \frac{1}{\sigma} \int_0^1 U_b(t) g_2(t) \, dt - \int_y^1 U_b^{-2}(s) \, ds \int_0^s U_b(t) g_2(t) \, dt \right\} \]
and
\[ \eta_n = \frac{\lambda^2 K^2}{2} \frac{\int_0^1 g_1(t) f_n(t) \, dt}{\int_0^1 f_n^2(t) \, dt} = -\frac{\lambda K^2}{2a_n^2} \left\{ \int_0^1 f_n^2(t) \, dt \right\}^{-1}. \] (40c)

From this solution we may compute the slip velocity along the hump in \( x = O(1) \):
\[ u_s(x) = \lambda (KL^{-1}x) - \lambda_2 (KL^{-1}x)^2 \ln(KL^{-1}x) + O(L^{-2}) \] (41)
when \( U_b(y) \sim \lambda y + \lambda_2 y^2 \) as \( y \to 0^+ \). We may also determine the position of reattachment, which, to leading order, is the point where \( C_0 C_1 \) (given by \( y = L^{-2} \lambda^{-1} \tilde{u}(x,0) \)) meets the hump (\( y = L^{-1} K x \)):
\[ x_1 = -\frac{1}{(\lambda KL)^{-1}} \sum_{1} \eta_n , \] (42)
which is positive (as it should be for physical sense) since each \( \eta_n \) is negative (see (40c)).

The above is an apparently complete description of the flow in the vicinity of a long hump, apart from the details of separation at \( C_2 \) and reattachment at \( C_1 \). The separation of the upstream free streamline at \( C_0 \) and the reattachment of the downstream free streamline take place on longer \( x \)-scales. It can be verified that the upstream separation takes place at an \( O(\ln(R)) \) distance ahead of the hump (and is given by (23) with \( h_m = R^{1/6} \)) using the solution (40a), which is appropriate in the limit \( x \to -\infty \) for any hump shape, not just those with length \( L > 1 \); the details of the separation at \( C_0 \) are the same as those given in Smith's (1979a) study. The reattachment of \( C_2 C_3 \) takes place on a longer \( O(R) \) length scale, when the viscous shear layer surrounding the dividing free streamline expands and
eventually 'fills' the entire liquid layer. The reattachment process therefore takes place under the action of viscosity, thereby preventing a strong backward jet of fluid into the reversed-flow region. The existence of such a jet would contradict the assumption that the eddy flow is relatively slow. The proposed flow structure, based on the free-streamline theory, seems to be self-consistent therefore.

Some qualitative comparisons may be made between the present theory and the experiments of Huppert & Britter (1982). In their experiments the Reynolds number was large, of the order of $10^3-10^4$, and moreover the long-hump theory above seems to be appropriate for the topography chosen in the experiments (apart from the initial hump shape). The conclusion that downstream separation is associated with a drop in the level of the free surface (from (28) ff.) is borne out by their finding; see their figures 3a,b. Furthermore, separation, when it takes place, occurs close to the crest of the hump, in line with the present theory (from just before (36)). On the other hand, upstream separation was not observed, which is not entirely surprising since, firstly, the initial shape of the topography was not wedge-like, but rose more gradually, and secondly, the predicted $O(\ln(R))$ distance of the point of separation ahead of the hump is not very large anyway for the range of Reynolds numbers taken in the experiments.
Captions for figures.

Figure 1. Schematic diagram of a fully-developed liquid-layer flow over a horizontal surface, indicating the high-Reynolds-number structure in the presence of a disturbance of characteristic streamwise length L.

Figure 2. Dependence on $\sigma$ of first four eigenvalues $\alpha_i$ ($i = 1, \ldots, 4$) of (21) when $U_\delta(y) = 2y - y^2$. Only the region $\sigma > 0$ is relevant to liquid-layer flows. Points marked $\Diamond$ correspond to $\sigma = \infty$ (Tillet 1968). Note that upstream influence decreases slightly (i.e. $\alpha_1$ increases) as $\sigma$ decreases (i.e. as the Froude number increases).

Figure 3. The main features of the grossly-separated flow over a hump. The free streamlines $C_0C_1$, $C_2C_3$ separate the fast oncoming stream from relatively slow eddying motions. The upstream separation of $C_0C_1$ and the downstream reattachment of $C_2C_3$ take place at distances $O(\ln(R))$, $O(R)$ from the hump, respectively.

Figure 4. The slip velocity $u_\delta$ along the hump surface as a function of the hump shape $F$, for various values of $\sigma$, from (34) (using $U_\delta(y) = 2y - y^2$).
FIGURE 2
FIGURE 4
PART II

Unsteady liquid-layer flows.
CHAPTER FIVE

A viscous instability of a nonlinear surface wave on a flow with shear.
CHAPTER FIVE

§1 Introduction.

Many studies have been made of waves propagating on the surface of a layer of water, and as a result numerous equations and systems of equations have been developed to describe them. The bulk of the theory to date has been concerned with inviscid fluid motion, and moreover many practical situations seem to be modelled quite successfully by linear inviscid theory. Nonlinear effects can become important when the wavelength of a small disturbance is sufficiently large compared with the depth of the fluid. Viscous effects have received comparatively little attention, on the other hand, no doubt partly because in practice high-Reynolds-number liquid-layer flows tend to be turbulent, on the whole, so that an elaborate viscous-flow theory based on laminar flow may be of little relevance, and partly because of the widely held view that when viscous effects are included in a laminar theory they often tend to be dissipative and are therefore unlikely to lead to particularly interesting (i.e. unstable) behaviour. However, we are able to show below that, on the contrary, viscous effects can destabilise a realistic flow of partly-developed or fully-developed form, in some circumstances at least.

In this chapter, in an attempt to gain some insight into the possible instabilities of some of the steady flows studied earlier (in Part I), we discuss the stability of a parallel shear flow in a layer of viscous fluid over a flat horizontal surface. To make progress analytically we restrict attention to a disturbance in the form of a weakly-modulated travelling wave which has amplitude $\delta$, say (with
primary wavelength $O(\delta^{-1/2})$ and phase velocity $O(1)$ (comparable with the mean flow), and which grows or decays on long temporal and spatial scales of relative order $\delta^3$. The above scalings are the familiar ones of classical inviscid KdV theory governing the development of nonlinear dispersive waves in shallow water. Our aim below is to introduce viscous effects in such a way that they are comparable with the effects of nonlinearity and dispersion, as a first step in a viscous-flow approach. We propose that the high-Reynolds-number flow can be analysed in terms of a two-layered structure, initially at least, in which the majority of the flow is governed by inviscid dynamics, while viscous effects are confined to a thin layer adjacent to the wall: see figure 1. The (thinner) viscous layer needed at the free surface can be neglected to our order of working. Now, the vertical displacement of the streamlines in the main stream is $O(\delta)$, comparable with the amplitude of the disturbance, so that we expect the thickness of the viscous layer to be $O(\delta)$ also, to reconcile the disturbance with the no-slip condition at the wall. Then a balance between the pressure gradient ($p_x \sim \delta/\delta^{-1/2}$), unsteadiness ($u_t \sim \delta/\delta^{-1/2}$) and viscous effects ($R^{-1}u_{yy} \sim R^{-1}\delta/\delta^2$) in the wall layer gives

$$\delta = R^{-2/5}$$

The viscous region is then a classical Stokes layer. We find the viscous effects to be very significant in certain cases.

We adopt a multiple-scales approach in the analysis of the unsteady flow, setting

$$\frac{\partial}{\partial x} = R^{-1/2} \frac{\partial}{\partial x} + R^{-2} \frac{\partial}{\partial x} + \ldots \quad (1.2a)$$

$$\frac{\partial}{\partial t} = -R^{-1} \frac{\partial}{\partial x} \frac{\partial}{\partial t} + R^{-2} \frac{\partial}{\partial \tau} + \ldots \quad (1.2b)$$
in the Navier–Stokes equations, as implied by the previous paragraph. Here $c$ is the (real) wavespeed of the disturbance.

In §2 we analyse the stability of a simple model of a flow with shear, using a 'straight-line' velocity distribution. The main conclusions we draw are that, firstly, growing modes exist, the instability being a result of viscous effects, and secondly, a finite-time breakdown of the equation governing the evolution of the disturbance (a viscosity-modified KdV equation) is possible. The repercussions of the finite-time breakdown are discussed at the end of the section. The stability of more realistic flows are discussed in §3; the major complicating feature is the existence of a strongly nonlinear critical layer within the layer for some modes. However, it is shown that the presence of the critical layer does not alter the evolution equation of the disturbance, and, as such, the conclusions drawn from §2 concerning the simpler straight-line profile still hold for more general profiles.
§ 2 A Simple Prototype.

We begin by analysing the stability properties of a simple shearing motion given by the 'straight-line' profile

\[ U_b(y) = \begin{cases} \lambda^{-1} & \text{for } \lambda^{-1} < y < 1 \\ \lambda y & \text{for } 0 < y < \lambda^{-1} \end{cases} \]  \hspace{1cm} (2.1)

This is perhaps a slightly crude model of an actual shear flow, but its stability properties are relatively simple to derive and understand, and they do give some insight into those of smooth velocity profiles studied in the next section.

The Rayleigh zone (in which \( y = O(1) \)) divides into two parts, \( I^+ \) and \( I^- \), above and below the material line, \( y = \zeta(x, \xi, \tau) \), across which the vorticity is discontinuous: see figure 1. In region \( I^+ \) the solution expands in the form

\[ u = 1 + R^{\frac{2}{5}} u_1^+ + R^{\frac{4}{5}} u_2^+ + \ldots \] \hspace{1cm} (2.2a)

\[ \psi = y - \frac{1}{2\lambda} + R^{\frac{2}{5}} \psi_1^+ + R^{\frac{4}{5}} \psi_2^+ + \ldots \] \hspace{1cm} (2.2b)

\[ p = \sigma(1 - y) + R^{\frac{2}{5}} p_1^+ + R^{\frac{4}{5}} p_2^+ + \ldots \] \hspace{1cm} (2.2c)

and the free surface and material line bounding region \( I^+ \) are given by

\[ y = 1 + R^{\frac{2}{5}} \eta_1 + R^{\frac{4}{5}} \eta_2 + \ldots \] \hspace{1cm} (2.3)

\[ y = \lambda^{-1} + R^{\frac{2}{5}} \xi_1 + R^{\frac{4}{5}} \xi_2 + \ldots \]

respectively. In (2.2) \( u_1^+ \) etc. are functions of \( X, \chi, \tau \) and \( y \), and the inverse Froude number \( \sigma = \frac{g\ell^x}{U^x} (= Fr^{-1}) \) is defined relative to the undisturbed depth \( \ell^x \) and free-surface velocity \( U^x \) of the fluid. Similarly, in region \( I^- \) we have

\[ u = \lambda y + R^{\frac{2}{5}} \bar{u}_1 + R^{\frac{4}{5}} \bar{u}_2 + \ldots \] \hspace{1cm} (2.4a)
\[ \psi = \frac{\lambda}{2} y^2 + R \frac{2}{5} \psi^2 + R \frac{4}{5} \psi^4 + \ldots \quad (2.4b) \]
\[ p = \sigma (1 - y) + R \frac{2}{5} p^2 + R \frac{4}{5} p^4 + \ldots \quad (2.4c) \]

Substitution of these expansions, together with (1.2), into the x-momentum and continuity equations gives

\[ \mathbf{L}^\pm(u^\pm_1, \psi^\pm_1, p^\pm_1) = \mathbf{M}^\pm_1, \quad u^\pm_1 = \psi^\pm_1, \quad \text{for } i = 1, 2 \quad (2.5a, b) \]

and the y-momentum equation gives

\[ p^\pm_{1y} = 0 \]
\[ p^\pm_{2y} = (1 - c)\psi^\pm_{1xx}, \quad p^\pm_{2y} = (\lambda y - c)\psi^\pm_{1xx} \quad (2.5c) \]

where the operators appearing in (2.5) are defined by

\[ \mathbf{L}^+(u, \psi, p) = (1 - c)u_x + p_x \]
\[ \mathbf{L}^-(u, \psi, p) = (\lambda y - c)u_x - \lambda \psi_x + p_x \]
\[ \mathbf{M}^+_1 = 0, \quad \mathbf{M}^-_1 = -u^+_x - u^+_x + u^+_x u^+_x + \psi^+_x u^+_y - p^+_x \]
\[ \mathbf{M}^-_2 = -u^-_x - \lambda u^-_x - u^-_x u^-_x + \psi^-_x u^-_y + \lambda \psi^-_x - p^-_x \]

The relevant boundary conditions are as follows:

(i) free surface conditions: at \( y = 1 \)

\[ p^+_1 = \sigma \eta_1 \]
\[ p^+_2 = \sigma \eta_2 - \eta_1 p^+_1 y \]
\[ \psi^+_1 x + (1 - c) \eta_1 x = 0 \]
\[ \psi^+_2 x + (1 - c) \eta_2 x = -\eta_1 u^+_1 x - \eta_1 x u^+_1 - \eta_1 x - \eta_1 - \psi^+_1 x \]

which derive from the dynamic and kinematic conditions respectively;

(ii) jump conditions between regions \( I^+ \) and \( I^- \): at \( y = \lambda^{-1} \)

\[ p^+_1 = p^-_1 \]
\[ p^+_2 + \zeta_1 p^+_1 y = p^-_2 + \zeta_1 p^-_1 y \]
\[ \psi^+_1 x + (1 - c) \zeta_1 x = 0 = \psi^-_1 x + (1 - c) \zeta_1 x \]
\[ \psi^+_2 x + (1 - c) \zeta_2 x + \zeta_1 x + u^+_1 \zeta_1 x + \zeta_1 x u^+_1 x + \zeta_1 x + \psi^+_1 x = 0 \]
\[ \psi^-_2 x + (1 - c) \zeta_2 x + \zeta_1 x + u^-_1 \zeta_1 x + \zeta_1 x u^-_1 x + \zeta_1 x + \psi^-_1 x + \lambda \zeta_1 \zeta_1 x = 0 ; \]
(iii) at the wall, \( y = 0, \)

\[
\psi_1 = 0 , \quad \psi_2 = \psi^* \]

in anticipation of the result that the efflux of fluid from the wall layer gives an \( O(1) \) value for \( \psi^* \) (see (2.10b) below).

At first order, the solution of (2.5a-c) (with \( i=1 \)) is readily found to be

\[
\begin{align*}
\rho^+_1 &= \sigma \eta_1 , \\
\eta^+_1 &= - \frac{\sigma \eta_1}{(1-c)} , \\
\psi^+_1 &= - \eta_1 \left[ \frac{\sigma (y-1)}{(1-c)} + (1-c) \right] \\
\rho^-_1 &= \sigma \eta_1 , \\
\eta^-_1 &= \frac{\sigma \eta_1}{c} , \\
\psi^-_1 &= \frac{\sigma \eta_1 y}{c}
\end{align*}
\]  

(2.6)

where we have used the conditions given in (i) and (iii) above, and also the dynamic condition in (ii). Then the kinematic condition of (ii) gives the following equation for the phase velocity:

\[
\frac{1}{\sigma} = \frac{c - \lambda^{-1}}{c (1-c)^2} .
\]  

(2.7)

The right-hand side of (2.7) is sketched in figure 2. Since \( \lambda > 1 \), there are always three real roots \( c_-, c_0, c_+ \), satisfying (9) (and no complex ones, of course) with \( c_- < 0, \ 0 < c_0 < 1, \ c_+ > 1 \).

It is interesting to note that there is always one neutral wave which propagates upstream against the flow no matter how strong the flow is. (A strong flow has \( \sigma < 1 \)). This is contrary to the classical inviscid result which states that, in a supercritical \( \sigma < 1 \) uniform flow, disturbances of all wavelengths (of which the longest are the fastest) are swept downstream. The two long-wave velocities given by the classical inviscid theory are reproduced by (2.7) in the limit \( \lambda > 1 \), as they should be on physical grounds, since in that limit the basic flow is uniform across almost the entire layer:
as \( \lambda \to 0 \) when \( \sigma < 1 \). In (2.8) the leading terms of \( c_+ \) and \( c_o \) are the classical results. If \( \sigma > 1 \) (subcritical flow) the roots of (2.7) (in the limit \( \lambda \to 0 \)) are again given by (2.8) but with \( c_o \) and \( c_- \) interchanged. The appearance of the third mode could have been anticipated from an unsteady analysis of the interactive boundary-layer flows studied in chapter 2. Also, (2.8) is equivalent to Burns' formula (Burns 1953)

\[
\frac{1}{\sigma} = \int_0^1 \frac{dy}{(U_0(y) - c)^2}
\]

with the profile (2.1), which he derived for neutral waves propagating in an arbitrary shear flow, although, as he states, his derivation is only valid (and the above integral only convergent) on the assumption that \( c < 0,1 \); see §3 however. As such he did not find the root \( c_o \).

An equation determining the arbitrary function \( \eta_1 \) in the above solution is found from a compatibility condition that arises in the second-order system. From (2.5c), the second-order pressure perturbations in the two regions \( I^+ \) and \( I^- \) are found to be

\[
\rho_2^+ = \sigma \eta_2 - \eta_{xx} \left\{ \frac{1}{2} \sigma (y-i)^2 + (1-c)^2 (y-i) \right\}
\]

\[
\rho_2^- = \sigma \eta_2 - \lambda^2 \eta_{xx} \left\{ \frac{1}{2} \sigma (\lambda-i)^2 - \lambda (\lambda-i)(1-c)^2 \right\}
\]

\[
+ \frac{\sigma \eta_{xx}}{c \lambda^2} \left\{ \frac{1}{3} \left( \lambda^3 y^3 - 1 \right) - \frac{1}{2} c \left( \lambda^2 y^2 - 1 \right) \right\}
\]

\[\text{(2.9a)}\]
where the dynamic conditions of (i) and (ii) have been used. Then, integrating (2.5a,b) with \( i = 2 \) gives

\[
\psi_2^+ = \frac{\eta_{xxx}}{1-c} \left\{ \frac{1}{6} \sigma (y-1)^3 + \frac{1}{2} (1-c)^2 (y-1)^2 \right\} \\
- \frac{\sigma (y-1)}{1-c} \left\{ \eta_{xx} + \frac{\eta_{x^2} + c \eta_{xx}}{1-c} \right\} \\
- \frac{\sigma \eta_{x^2} \eta_{xx}}{1-c} \right\} \\
\psi_2^- = \frac{\sigma y}{c^2} \left\{ \eta_{xx} + c \eta_{x^2} + \frac{\sigma}{c} \eta_{xx} - \frac{\lambda^2 c^2}{b \lambda^2} \gamma_{xxx} + \frac{\sigma}{c} \rho_{2x}^-(y=0) \right\} \\
- \frac{\sigma y}{b \lambda^2} (\lambda y-c)(\lambda y+c) \eta_{xxx} - \frac{(\lambda y-c)}{c} \psi_x^* \\
\right\} \\
(2.9b)
\]

where we have used the conditions in (i) and (iii). The two solutions (2.9b) are only compatible with the kinematic condition of (ii) if

\[
K_1 (\eta_{xx} + c \eta_{x^2}) + K_2 \eta_{xx} + K_3 \eta_{xxx} = \frac{(1-c)}{c} \psi_x^* \\
(2.10a)
\]

where the coefficients of (2.10a) are given by the following expressions:

\[
K_1 = 1 + \frac{\sigma (\lambda^2 - 1)}{\lambda (1-c)^2} + \frac{\sigma}{\lambda c^2} \\
K_2 = - \frac{\sigma^2}{\lambda c^2 (1-c)^2} \left\{ 3 \lambda c^3 - 6 c^2 + 4 c - 3 \right\} \\
K_3 = \frac{\sigma}{3 \lambda^2 c^4 (1-c)^2} \left\{ 2 - c - 3 \lambda + 3 \lambda c - \lambda^2 c^2 \right\}
\]

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the only result worthy of note here is that $K_i$ is positive for all values of the parameters $\lambda$ and $\sigma$.

In the viscous layer, region II, in which $y = R^{-2/5}Y$ with $Y = O(1)$, the flow quantities expand as

$$
\begin{align*}
\psi &= R^{-\nu_y} \varphi_i + \ldots, \\
p &= -\sigma R^{-\nu_y} Y + R^{-\nu_y} P_i + \ldots,
\end{align*}
$$

\tag{2.11}

in view of the behaviour of the solution in $I^-$ as $y \to 0$. Substitution of (2.11) into the Navier-Stokes and continuity equations results in

$$
-c U_{ix} = -P_{ix} + U_{iyy}, \quad P_{iy} = 0, \quad U_i = \Phi_{iy}.
$$

\tag{2.12a,b,c}

The join with $I^-$ then gives $P_1 = \sigma \eta_1$ and

$$
\Phi_i \sim \frac{\sigma \eta_1}{c} Y + \psi^* \quad \text{as} \quad Y \to \infty
$$

\tag{2.12d}

while the no-slip condition at the wall requires

$$
U_i = \Phi_i = 0 \quad \text{at} \quad Y = 0.
$$

\tag{2.12e}

The solution of (2.12a-e) which decays as $X \to -\infty$ can be found by taking a Fourier transform with respect to $X$, which gives

$$
(\hat{\psi}^*(\chi, \chi, \tau)) = \frac{\sigma}{c} \left( -\frac{i \chi}{m} \right) \hat{\eta}_1(\chi, \chi, \tau)
$$

\tag{2.13}

where $m^2 = -\sigma c$, $\text{Real}(m) > 0$; here $\hat{\cdot}$ denotes the Fourier transform, and $\alpha$ is the transform variable. Inverting (2.13), we find the efflux of fluid from the viscous wall layer:

$$
\psi^*(\chi, \chi, \tau) = \left\{ \begin{array}{ll}
-\frac{\sigma}{\pi \nu_y c \nu_2} \int_X^\infty \frac{d\chi}{\nu_2} \frac{d\xi}{(\xi - \chi)^{\nu_2}}, & c > 0 \\
-\frac{\sigma}{\pi \nu_2 |c| \nu_2} \int_X^\infty \frac{d\chi}{\nu_2} \frac{d\xi}{(\chi - \xi)^{\nu_2}}, & c < 0 \end{array} \right.
$$

\tag{2.10b}
Thus (2.10a,b) together give a nonlinear, integro-differential equation for the free-surface displacement, which is in the form of a KdV equation with viscous modification. (An inviscid fluid gives $\gamma^* = 0$.) A similar equation is derived by Kakutani & Matsuuchi (1975) for the evolution of waves on a fluid layer at rest; an extension of their analysis to continuously stratified fluids is given by Koop & Butler (1981). Cowley (1983) derives the equation for elastic jumps on fluid-filled tubes. The novel feature here is the existence of growing modes, as we now show.

By linearising (2.10a,b) we may consider the initial growth or decay of the three possible marginal waves on the slow $O(R^{4/3})$ time scale. Suppose a disturbance of (real, positive) wavenumber $\alpha$ is given by $\exp[i\alpha(X-\gamma t)]$ with $\varepsilon < 1$. The (complex) wavespeed is given by $c + R^{-2/3}\gamma$ where $c$ satisfies (2.7) (and is real) and $\gamma$ satisfies the dispersion relation

$$K_1 \gamma = -K_3 \alpha^2 + \frac{(1 - c)\sigma}{m c^2}$$

(2.14)

with $m^2 = -i\alpha c$, Real($m$) $> 0$ again. Thus the growth rates due to viscosity on the slow time scale, for the three marginal modes, are given by

$$\alpha \operatorname{Im}(\gamma) = \begin{cases} \frac{\sigma (1 - c)}{c^2 K_1} \left(\frac{\alpha}{2c}\right)^{\frac{1}{2}} & \text{for } c = c_0, c_+ \\ -\frac{\sigma (1 - c)}{c^2 K_1} \left(\frac{\alpha}{2|c|}\right)^{\frac{1}{2}} & \text{for } c = c_- \end{cases}$$

(2.15)

So a small sinusoidal disturbance propagating with speed $c_0$ grows exponentially fast (initially, at least) on the slow time scale. The other two modes decay. In the limit $\lambda > 1$, the decay rate of the
upstream-travelling wave \((c_-)\) is

\[
\lambda^{\frac{1}{2}} \left( \frac{\alpha (1-c)}{2 \sigma} \right)^{\frac{1}{2}}
\]

\((2.16)\)

when \(\sigma < 1\) (supercritical flow) which is very fast (albeit on the slow time scale); alternatively, the length scale on which this mode decreases in amplitude is very short. As such, these waves are less likely to be observed in practice. More important is the growth of the 'middle mode' \(c_o\). In the case of subcritical flow the existence of this growing mode could have been anticipated from the unsteady version of the boundary-layer problem studied in chapter 2 (section 3); the instability can therefore be associated with the growth of Tollmien-Schlichting waves. In the limit \(\lambda \gg 1\) again, the initial growth rate of this mode for supercritical [subcritical] flow is \(O(1)\) \([O(\lambda^3/2)]\).

Our main interest is in the combined effects of nonlinearity and viscosity when the disturbance size grows to \(O(1)\) (or larger).

Following Kakutani & Matsuuchi (1975) we can show that there are no (non-trivial) steady solutions of \((2.10a,b)\) satisfying the physically sensible conditions

\[
\eta_1 \to 0, \quad \eta_{1x} \to 0 \quad \text{as} \quad \chi \to \pm \infty.
\]

\((2.17)\)

To simplify matters we shall absorb the \(\chi\)-derivative into the \(\tau\)-derivative by a Galilean transformation (i.e. set \(\partial_\chi = 0\) in \((2.10a)\)) and confine attention to \(c > 0\); an exactly similar analysis holds when \(c < 0\). In addition, we note that \(1K_3^{-1/3}K_2\eta_1\) depends only on \(1K_3^{-1/3}X\), \(K_1^{-1}\tau\), and the parameter

\[
\mu = \frac{\sigma (1-c)}{\tau^{\frac{1}{2}} c^{\frac{3}{2}} / K_3^{\frac{1}{2}}}
\]

\((2.18)\)
(as well as on the signs of $K_2$ and $K_3$). By a suitable normalisation, then, (2.10a,b) can be written

$$
\eta_{t \tau} - \eta_x \eta_{xx} - \eta_{xxx} = -\int_{x}^{\infty} \frac{\eta_{11}(x, \tau)}{(x - \lambda)^{1/2}} \, d\xi \tag{2.19}
$$

where we have chosen, without significant loss of generality, the coefficient of the nonlinear term $\text{sign}(K_2) = -1$, and that of the dispersive term $\text{sign}(K_3) = -1$ also. Equation (2.19), as it stands, does hold when $\lambda > 1$, for example. Then we show in Appendix A that

$$
\frac{d}{d\tau} \int_{-\infty}^{\infty} \eta_i^2(x, \tau) \, dx = \sqrt{2\pi} \mu \int_{c}^{\infty} \alpha^{\frac{1}{2}} \left| \hat{\eta}_i(\alpha, \tau) \right|^2 \, d\alpha. \tag{2.20}
$$

The integral on the right-hand side of (2.20) is positive definite, so that the energy of the disturbance increases without bound for $0 < c < 1$, and decreases to zero for $c > 1$ (and also for $c < 0$). The energy growth found for the middle mode contrasts with the findings of Kakutani & Matsuuchi concerning viscous effects.

A physically plausible disturbance which satisfies (2.17) and moves with phase velocity $c_0$ cannot, therefore, develop into a steady solution in either a fixed or a moving frame of reference. On the face of it, two distinct types of behaviour seem possible for the growing mode: the energy created by the instability may be carried away from the initial location of the disturbance by a wave-like motion; or the disturbance may remain localised, forming, for example, a sharp peak in the displacement. The second of these suggests the possibility of a finite-time breakdown of the proposed flow structure. The existence of a finite-time breakdown can be shown analytically when viscous effects (which are gauged by the magnitude of the parameter $\mu$) are relatively weak ($\mu < 1$). The central problem (2.19) was derived
assuming \( \mu = O(1) \) effectively, but since (2.19) is independent of the Reynolds number the following multiple-scales approach is valid in the limit \( \mu \to 0 \). First, we note that when \( \mu = 0 \) one solution of (2.19) is the familiar solitary wave moving with speed \( \tilde{c} \), say, in the negative-X direction. Then, for \( 0 < \mu < 1 \), we set

\[
\frac{\partial}{\partial \tau} \quad \longrightarrow \quad \tilde{c}(T) \frac{\partial}{\partial X} + \mu \frac{\partial}{\partial T}.
\]  

(2.21)

The solution of (2.19) may be expanded as

\[
\eta_i(X, \tau) = \tilde{\eta}_i(X, T) + \mu \tilde{\eta}_i(X, T) + \ldots.
\]

(2.22)

Substitution of (2.22) into (2.19) yields, at first order,

\[
- \tilde{c} \tilde{\eta}_i'' + \tilde{\eta}_i \tilde{\eta}_i'' + \tilde{\eta}_i'' = 0
\]

(2.23)

which has the solitary wave solution

\[
\tilde{\eta}_i = 3 \tilde{c}(T) \text{sech}^2 \left[ \frac{X \sqrt{c(T)}}{2} \right].
\]

(2.24)

In general, an initial disturbance will not take the form of a solitary wave, of course. However, an arbitrary initial disturbance satisfying \( \int_{-\infty}^{\infty} \eta_1(X, 0) dX < \infty \) will evolve under (2.23) into a number of almost independent solitary waves, the exact number depending on the value of this integral (Segur 1973). The analysis below then applies equally to each soliton.

At second order we obtain

\[
\begin{align*}
- \tilde{c} \tilde{\eta}_i'' & - \tilde{\eta}_i \tilde{\eta}_i' + \tilde{\eta}_i \tilde{\eta}_i'' + \tilde{\eta}_i'' = \int_{-\infty}^{\infty} \frac{\tilde{\eta}_i}{(1 - X)^{1/2}} dX \\quad \text{at} \quad X = \pm \infty.
\end{align*}
\]

(2.25)

A solution of (2.25) exists only if a certain solvability condition is
satisfied. This is found by multiplying (2.25) by $\tilde{\eta}_o$ and integrating over $X$ from $-\infty$ to $\infty$:

$$\frac{1}{2} \frac{d}{dT} \int_{-\infty}^{\infty} \tilde{\eta}^2_o (X, T) dX = -\int_{-\infty}^{\infty} \tilde{\eta}_o (X, T) \int_{-\infty}^{\infty} \frac{\tilde{\eta}_o (\xi, T)}{(\xi - X)^{\nu_2}} d\xi dX$$

(2.26)

i.e.

$$\tilde{\xi} \frac{d \tilde{\xi}}{dT} = 4 \int \tilde{\xi}^{3/2}$$

(2.27)

where the constant

$$I = 2^{1/2} \int_{-\infty}^{\infty} \text{sech}^2 \nu \int_{\nu}^{\infty} \frac{\text{sech}^2 \nu \tanh \nu}{(\nu - \nu')^{3/2}} d\nu d\nu'$$

is positive. Imposing the initial condition $\tilde{\xi}(0) = 1$, say, we find that

$$\tilde{\xi}(T) = \frac{1}{(1 - IT)^{4}}$$

(2.28)

Thus the solution breaks down at the finite time $T = T_o = I^{-1}$ (or at $\tau = \mu I^{-1}$ in terms of the original time variable). As $T \to T_o-$ the above solution gives

$$\eta_i = O(T_o - T)^{-4} \quad \text{when} \quad \tilde{X} = O(1)$$

(2.29a)

where the coordinate $\tilde{X}$ is defined by

$$X = -\left\{ \frac{1 - IT_o (T_o - T)}{\mu I^5 (T_o - T)^4} \right\} + \left( T_o - T \right)^2 \tilde{X}$$

(2.29b)

The first term in (2.29b) simply gives the location of the centre of the solitary wave relative to its position at $\tau = 0$.

The above analysis for $\mu < 1$ suggests that a finite-time breakdown may also occur in a similar fashion when $\mu = O(1)$, but on the original time scale, i.e. as $\tau \to \tau_o-$, say, with $\tau_o = O(1)$. A preliminary
investigation, based partly on the small-μ solution above, suggests the following possibility of the nonlinear terminal form when μ = O(1). We expand

$$\eta_1 = \tilde{\tau}^{-4} \tilde{\eta}_1(\tilde{\chi}) + \tilde{\tau} \tilde{\eta}_1(\tilde{\chi}) + \ldots \quad \text{as} \quad \tilde{\tau} \to O^+ \quad (2.30)$$

where $\tilde{\tau} = \tau_0 - \tau$, and

$$\tilde{\chi} = \frac{\chi + (\tau_0 - \tau)^3 A}{(\tau_0 - \tau)^2}; \quad (2.31)$$

the similarity variable $\tilde{\chi}$ therefore requires the disturbance (a) to move increasingly rapidly upstream (since we take $A$ to be positive: see (2.36) below) relative to the present frame of reference, at least, which itself is moving downstream (relatively fast) relative to the wall, and (b) to be confined to a thinning region of streamwise extent of order $(\tau_0 - \tau)^2 R^{1/5}$. Substitution of (2.30) into (2.19) gives, at O$(\tilde{\tau}^{-10})$,

$$-3A \tilde{\eta}_1' + \tilde{\eta}_1 \tilde{\eta}_1' + \tilde{\eta}_1^{iii} = 0 \quad (2.32)$$

where primes denote differentiation with respect to $\tilde{\chi}$. Equation (2.32) has the solitary wave solution

$$\tilde{\eta}_1(\chi) = 9A \text{sech}^{\frac{3}{2}}\left(\frac{\tilde{\chi}}{\sqrt{3} A}\right) \quad (1.33)$$

in which, without loss of generality, we suppose the centre of the disturbance to be at $\tilde{\chi} = 0$. At O$(\tilde{\tau}^{\frac{5}{3}})$ we obtain the following equation for $\tilde{\eta}_1$:

$$-3A \tilde{\eta}_1' + (\tilde{\eta}_1 \tilde{\eta}_1')' + \tilde{\eta}_1^{iii} = 4 \tilde{\chi} \tilde{\eta}_1' - 2 \tilde{\chi} \tilde{\eta}_1' - \int_\tilde{\chi}^{+\infty} \tilde{\eta}_1'(\xi) d\xi \quad (2.34)$$

We require $\tilde{\eta}_1(\pm\infty) = \tilde{\eta}_1'(\pm\infty) = 0$ so that the expansion (2.30) remains
valid. As such (2.34) gives rise to the solvability condition

\[ 3 \int_{-\infty}^{\infty} \tilde{\eta}_0(\tilde{z}) d\tilde{z} = -\mu \int_{-\infty}^{\infty} \tilde{\eta}_0'(\tilde{z}) \left( \frac{\tilde{\eta}_1(\zeta)}{(\zeta - \tilde{z})^{1/2}} \right) d\zeta d\tilde{z}. \tag{2.35} \]

On substitution of the solution (2.33) the solvability condition reduces to

\[(3\Lambda)^{1/2} = \mu^{-1} I^{-1} \quad \text{i.e.} \quad \Lambda = \frac{1}{3(\mu I)^{1/2}} \quad \text{for} \quad \mu > 0 \tag{2.36} \]

which gives a positive value for \( \Lambda \), as assumed. Notice that if \( \mu \) is negative (corresponding to the two modes with \( c = c_- \), \( c_+ \)) then the first of (2.36) cannot be satisfied for \( \Lambda \) real, so in that case the terminal form (2.30) cannot be attained. We should point out that, for \( \tau < \tau_0 \), \( \Lambda \) is a function of \( \tau_0 - \tau \), in fact, and not a constant, but it remains \( O(1) \) as \( \tau \to 0^+ \) and takes the value given by (2.36) at \( \tau = 0 \). Further terms in the expansion of \( \Lambda \) near the breakdown time can be determined, in principle, by solvability conditions that arise at higher orders.

We observe that the terminal form (2.30) is, to leading order, controlled by inviscid dynamics, which give rise to the solitary wave solution. In the original frame of reference fixed in the wall, the solitary wave moves downstream with speed \( c_0 \) to leading order, but begins to slow down due to the increasing effects of viscosity (on the slow \( O(R^{3/5}) \) time scale) as \( \tau \to \tau_0^- \). The amplitude and streamwise extent of the soliton, and indeed the entire breakdown phenomenon, are fixed nevertheless by the relatively small viscous effect, so that viscosity remains important throughout.

The rapid growth of the disturbance continues until a new balance comes into play. The breakdown structure suggests that the
relative amplitude $O(R^{-2/3} \tilde{\tau}^{-4})$ and streamwise extent $O(R^{1/5} \tilde{\tau}^{2})$ of the disturbance both become $O(1)$, comparable with the fluid depth, when $\tilde{\tau} = O(R^{-1/10})$ formally. At that stage the full (steady) Euler equations control the flow in the majority of the layer, with unsteadiness (operating on an $O(R^{1/2})$ time scale) confined to a thin viscous wall layer. Viscosity is again important (as in (2.32)-(2.36)) in determining the exact amplitude of the inviscid solution. At this stage, however, the unsteady nonlinear flow in the viscous boundary layer may itself break down within a finite time, creating bursts of vorticity into the inviscid zone (see also Smith & Burggraf 1985), and thereby perhaps destroying the entire laminar structure of the flow.

Finally, we should emphasise that the breakdown structure (2.30) is only one possibility; other terminal solutions may be possible (although none has been found). Indeed, unsteady solutions valid for all time might conceivably occur, with the increasing energy of the perturbation being propagated away into the far-field in the form of free-surface waves. The small-$\mu$ analysis above suggests that a finite-time breakdown is almost certain to occur, however, although numerical solutions of the full problem (2.19) are needed to settle the issue decisively. The numerical task of integrating (2.19) is by no means straightforward, incidentally, because of the rapid growth on a short length scale of the disturbance (assuming the form (2.30) does emerge) as well as its increasingly rapid propagation upstream.
§3 Smooth basic flow.

The stability properties of more realistic fully- or partly-developed shearing motions, with smooth (i.e. twice-differentiable) velocity distributions \( U_\beta (y) \) throughout the liquid layer, are similar to those of the prototype 'straight-line' profile studied in the previous section, although some complicating features arise in the analysis. We consider velocity distributions with

\[
U_\beta (0) = 0 , \quad U_\beta (1) = 1 , \quad U_\beta ^{\prime} (1) = 0 , \quad U_\beta ^{\prime\prime} (y) < 0 . \tag{3.1}
\]

Then, if a neutral disturbance with phase velocity \( c \in (0,1) \) exists, there will be a critical layer at \( y = y_c \) where \( U_\beta (y_c) = c \). There is no need for a critical layer when the basic flow is given by the straight-line profile, since the solution is regular throughout the Rayleigh zone, mainly because \( U_\beta ^{\prime\prime} (y) = 0 \) everywhere - we leave aside the exceptional case where \( y_c \) and the junction of the straight lines coincide. For smooth velocity profiles we suppose a critical layer is present. Then, for the fundamental disturbance (of long wavelength \( O(R^{1/5}) \) and small amplitude \( O(R^{-2/5}) \)) under consideration, the critical layer is of the strongly nonlinear kind, with thickness \( O(R^{-1/5}) \), as implied by the behaviour of the solution in the Rayleigh zone close to the critical layer (see for example Bodonyi, Smith & Gajjar, 1983), rather than of the nonlinear-viscous kind (Haberman, 1972) or the classical linear viscous kind (Reid, 1965). The asymptotic structure of the flow is depicted in figure 3. The Rayleigh zone is in two parts, regions \( I^+ \) and \( I^- \), separated by the nonlinear critical layer, region CL. The viscous Stokes layer adjacent to the wall, region II, is again of thickness \( O(R^{-2/5}) \).

In the following, we analyse the flow in \( I^\pm \), where the expansions
are
\[ u = U_0(y) + \varepsilon^1 u_1 + \ldots + \varepsilon^4 u_4 + \ldots \]  
\[ \psi = \Psi_0(y) + \varepsilon^2 \psi_1 + \ldots + \varepsilon^4 \psi_4 + \ldots \]  
\[ p = \sigma(1-y) + \varepsilon^2 p_1 + \varepsilon^4 p_4 + \ldots \]
in which we have set \( \varepsilon = R^{-1/5} \). Additional terms (e.g. at \( O(\varepsilon^3) \)) are necessary for a complete match with CL, but they do not affect the analysis presented here. The behaviour of the undisturbed flow in the neighbourhood of the critical layer is
\[ U_0 \sim c + \lambda_1(y-y_c) + \frac{1}{2} \lambda_2(y-y_c)^2 + \frac{1}{6} \lambda_3(y-y_c)^3 + \ldots \]  
(3.3)
The free surface is given by
\[ y = 1 + \varepsilon^2 \eta(x, \tau) + \ldots \]  
(3.2d)
we shall neglect the \( o(\varepsilon^2) \) correction to (3.2d), which remains arbitrary to our order of working, and which, in any case, does not effect the second-order system. Furthermore, in what follows we shall take \( \delta \chi = 0 \) for simplicity. Then, the (streamwise) velocity jump at \( O(\varepsilon^2) \), denoted by \((u_1)_{y_c^+} \), fixes the phase velocity \( c \), while that at \( O(\varepsilon^4) \), \((u_2)_{y_c^+} \), determines the equation for \( \eta \). An analysis of the critical layer, which is required to determine \((u_i)_{y_c^+} \) (i = 1, 2), is given later.

Substituting the expansions (3.2a-d) (and (1.2a,b)) into the Navier-Stokes and continuity equations, we obtain the following solutions for the fundamental disturbance
\[ \Psi_1^+ = -\sigma \eta W(y) \left\{ \frac{1}{\sigma} + \int_1^y \frac{dy'}{W'(y')} \right\} \]  
(3.3a)
\[ \psi_1^- = - \sigma \eta \int_0^y \frac{dy'}{W^2(y')} \]  
\[ \rho_1^+ = \rho_1^- = \sigma \eta \]  

where \( W(y) = U_0(y) - c \). In obtaining (3.3a,b) we have used the kinematic conditions at the free surface [see (i) of §2] and at the wall [see (ii)], while (3.3c) follows from the dynamic condition at the free surface [(i)] and the fact that the pressure jump across CL is \( o(\epsilon^2) \). (The behaviour of \( \psi \) in the neighbourhood of \( y = y_c \), based on the first two terms of (3.2b) and the solution (3.3a,b), is given in appendix B. These expansions will be needed later since they determine the development of the solution in the critical layer.) Notice that if \( c \notin (0,1) \) then the solution (3.3a) holds across the entire layer, and Burns' formula is recovered using the condition \( \psi_1^+ = 0 \) at \( y = 0 \). When \( c \in (0,1) \), however, the result

\[ \langle (u_1) \rangle_{y_1 = 0} = 0 \]  

from the critical-layer analysis given below yields the new formula

\[ \frac{1}{\sigma} = \int_0^1 \frac{dy'}{W^2(y')} \]  

where the stroke through the integral sign denotes the finite part of the integral, which is defined in the usual way, i.e.

\[ \frac{1}{\sigma} = \int_0^1 \left\{ \frac{1}{W(y')} - \frac{1}{\lambda_i^2 (y' - y_c)} + \frac{\lambda_i}{\lambda_i^2 (y' - y_c)} \right\} dy' \]

\[ - \frac{1}{\lambda_i^3} \left( \frac{1}{1 - y_c} - \frac{1}{y_c} \right) - \frac{\lambda_i^2}{\lambda_i^3} \ln \left( \frac{1 - y_c}{y_c} \right) \]

We digress briefly to consider the properties of (3.5). Firstly, as
Burns noticed, velocity profiles satisfying (3.1) have exactly one neutral wavespeed $c > 1$ ($c_+$, say) and one with $c < 0$ ($c_-$) for each value of $\sigma$. Moreover, $c_+$ increases and $c_-$ decreases as $\sigma$ increases. In general, nothing can be said about the existence of a neutral wave with $c \in (0,1)$. If, for example, the shear flow is half-Poiseuillean, with $U_B(y) = 2y - y^2$, then (3.5) gives

$$\frac{1}{\sigma} = \begin{cases} \frac{1}{2c(c-1)} + \frac{1}{2(c-1)^{3/2}} \tan^{-1}\left(\frac{1}{c-1}\right) & \text{for } c > 1 \\ \frac{1}{2c(c-1)} + \frac{1}{4(1-c)^{3/2}} \ln \left| \frac{1+\sqrt{1-c}}{1-\sqrt{1-c}} \right| & \text{for } c < 1. \end{cases} \quad (3.6a)$$

The right-hand side of (3.6a) is sketched in figure 4a. We see that $c_+$ and $c_-$ are the only (real) roots of (3.6a). But if the flow is given by the cubic distribution $U_B(y) = 1-(1-y)^3$, then we have

$$\frac{1}{\sigma} = \frac{1}{3c(c-1)} - \frac{2}{q(c-1)^{3/2}} \left\{ \frac{1}{2} \ln \left| \frac{y+1}{y-1} \right| + \frac{1}{3} \tan^{-1}\left(\frac{3}{2y-1}\right) \right\} \quad (3.6b)$$

where $\gamma = (c-1)^{1/3}$; see figure 4b. In this case there is always one root $c = c_0$, say, with $c_0 \in (0,1)$, just as in the straight-line profile of §2 (cf. figure 2).

The equations governing the $O(\varepsilon^4)$ perturbations in $I^\ast$ can also be solved explicitly. Using the conditions at the free surface and at the wall, we obtain

$$\Psi_{2x}^+ = -W(y) \int_0^y \frac{M^+ dy'}{W^2(y')} - \frac{W(y)}{(1-c)} \left\{ \eta - \frac{2\sigma \eta_x}{(1-c)} \right\} \quad (3.7a)$$

$$\Psi_{2x}^- = -W(y) \int_0^y \frac{M^- dy'}{W^2(y')} - c^{-1} W(y) \Psi_x^* \quad (3.7b)$$
\[ p^+_2 = -\sigma \eta \int_0^y W^2(y') \left\{ \frac{1}{\sigma} + \int_0^{y'} \frac{dy''}{W^2(y'')} \right\} dy' \]  
\[ p^-_2 = -\sigma \eta \int_0^y W^2(y') \int_0^{y'} \frac{dy''}{W^2(y'')} dy' + \eta \int_0^y W^2(y) dy. \]  
(3.7c)

Here we have also used the fact that the pressure jump across CL is \( o(\epsilon^*) \). The operators \( M^\pm \) in (3.7a, b) are the forcing terms in the x-momentum equation arising at this order,

\[ M^\pm = \frac{p^\pm}{\sigma^2} + u^\pm \frac{\partial u}{\partial x} + u_t^\pm \frac{\partial u}{\partial t} - \frac{\Psi^*}{\epsilon^2} \frac{\partial \Psi^*}{\partial y}, \]  
(3.8)

and are functions of \( X, \tau \) and \( y \). The efflux from the viscous Stokes layer \( -\Psi^*_x \) [see (iii) of §2] is again given by (2.10b).

When \( c \neq 0(1) \) the solution (3.7a) holds across the entire region in which \( y = O(1) \), so that the equation governing the evolution of \( \eta \) may be deduced from the condition \( \Psi^*_x = \Psi^* \) at \( y = 0 \), i.e.

\[ \frac{c}{1 - c} \left\{ \frac{\partial \eta}{\partial x} - \frac{2\sigma \eta \eta_x}{1 - c} \right\} - c \int_0^y \frac{M^+ dy}{W^2(y)} = \frac{\Psi^*}{\epsilon^2}. \]  
(3.9)

Using (3.8), (3.3a-c) and (3.7c) in equation (3.9), we obtain

\[ 2 \sigma \eta \int_0^1 \frac{dy}{W^2(y)} - 3 \sigma \eta \frac{\eta_x}{W^2(y)} \int_0^1 \frac{dy}{W^2(y)} \]  
\[ - \sigma \eta \int_0^1 \int_0^y \frac{W^2(y')}{W^2(y)W^2(y'')} dy' dy'' = \frac{\Psi^*}{\epsilon^2}. \]  
(3.10)

A linear analysis of (3.10), along the lines of that given in §2 for the straight-line profile, shows that marginal waves with \( c \neq 0(1) \) are stable to small disturbances on the long \( O(\epsilon^{-3}) \) time scale. So viscous effects are stabilising for such waves, just as they are in the special case of the previous section.
When $c \in (0,1)$, however, the situation is more complicated. The evolution equation for $\eta$ can be determined in terms of $\Lambda$, the jump in the streamwise velocity at $O(\varepsilon^4)$ across the critical layer:

$$\Delta = (u_z)^+_{y_c^+} - (u_z)^-_{y_c^-}.$$ 

Johnson (1986) gives an analysis of the equivalent inviscid problem, in which he assumes $\Lambda = 0$. He shows that (3.10) again holds (although the viscous contribution $\varphi^* = 0$ in his case, of course) but with the integrals being replaced by their finite parts. We go on to consider the critical-layer flow, firstly, to confirm the suggestion (3.4) above, and secondly, to evaluate the velocity jump $\Lambda$. Our analysis of the critical layer follows that given by Johnson (1986), although it is necessary to go to higher order to determine $\Lambda$. Furthermore, we consider the velocity jumps across relatively thin viscous layers that arise within the critical layer.

**The Critical Layer.**

Within the critical layer CL we write $y = y_c + \varepsilon Z$ with $Z = O(1)$. The behaviour of the solutions (3.3) in $I^2$ near $y = y_c$ (see appendix B) implies the development

$$\varphi = \varepsilon c Z + \varepsilon^2 \varphi_0 + \varepsilon^3 \varphi_1 + \varepsilon^4 \varphi_2 + \varepsilon^5 \varphi_3 + \ldots \quad (3.11a)$$

$$u = c + \varepsilon U_0 + \varepsilon^2 U_1 + \varepsilon^3 U_2 + \varepsilon^4 U_3 + \ldots \quad (3.11b)$$

$$p = \sigma(1 - y_c) - \sigma \varepsilon Z + \varepsilon^2 \sigma \eta + \varepsilon^3 p_2 + \ldots \quad (3.11c)$$

in CL, essentially. Additional terms are also strictly necessary to
achieve a complete match with $I^*$, but these do not interact with the main terms above and neither do they alter the essential features of the critical-layer flow. In (3.11a,b)
\[ \psi_c = \frac{1}{2} \lambda_i Z^2 + \frac{\sigma \eta}{\lambda_i}, \quad U_c = \lambda_i Z \] (3.12)
throughout CL, from the Navier-Stokes equations and the match with $I^*$. Thus, to leading order, the streamlines are given by
\[ \frac{1}{2} \lambda_i Z^2 + \frac{\sigma \eta(x, \tau)}{\lambda_i} = \text{constant} \] (3.13)
in the frame of reference moving with the neutral wavespeed $c$. The streamline pattern is changing on the slow ($\tau$) time scale. In the following unsteady critical-layer analysis, we suppose that the initial disturbance to the flow is such that the free surface is in the form of an isolated hump, as depicted in figure 5. Then the initial streamline pattern contains no closed streamlines: all the streamlines extend to infinity (the far-field). Two dividing streamlines separate those which turn back on themselves from those which do not. These streamlines are given by
\[ Z = \pm Z_c(x, \tau) \quad \text{with} \quad \frac{1}{2} \lambda_i Z_c^2 + \frac{\sigma \eta}{\lambda_i} = \frac{\sigma \eta_m(\tau)}{\lambda_i}, \quad Z_c \geq 0 \] (3.14)
where $\eta_m(\tau)$ is the maximum elevation of the free surface. So CL divides into three regions (see figure 5): the interior region $CL_i$ between the two dividing streamlines, and $CL^+$, $CL^-$ above and below. As the free-surface shape evolves, this simple pattern may change, with closed streamlines emerging (if the free surface develops two or more peaks). For simplicity, we suppose that the streamlines remain open, although we expect that the formation of closed streamlines does not alter the main results of the analysis.

It turns out that the vorticity in the far-field plays an important
role in determining the solutions at each order in the three regions. For consistency, the vorticity cannot be prescribed arbitrarily throughout CL at an initial instant, because, to leading order, the vorticity is constant along a streamline. Therefore, we suppose that in CL the vorticity approaches the undisturbed steady-state distribution far ahead of the disturbance (as \( X \to \infty \)), whilst in CL it tends to a constant value:

\[
\epsilon^2 \psi_{zz} \to \frac{\partial}{\partial X} (\psi + \epsilon Z) = \lambda_1 + \epsilon \lambda_1 Z + \frac{1}{2} \epsilon^2 \lambda_1 Z^2 + \ldots \text{ in CL}^+ \\
\implies \lambda_1 \text{ in CL}_i^-, \quad \text{as } X \to \infty .
\] (3.15)

The slight irregularity in the solution at the separating streamlines can be removed by thin viscous regions there; these are considered later.

From the Navier-Stokes and continuity equations, the problem for \((U_i, \tau_i)\) is

\[
\lambda_i Z U_{ix} - \frac{\sigma \eta_i}{\lambda_i} U_{iz} - \lambda_i \psi_{ix} = 0 , \quad U_i = \psi_{iz} .
\] (3.16)

A differentiation with respect to \( Z \) gives the vorticity equation, which is most easily solved by transforming to the new independent variables

\[
\hat{X} = X , \quad \hat{\tau} = \tau , \quad \zeta = \frac{1}{2} \lambda_i Z^2 + \frac{\sigma \eta(X, \tau)}{\lambda_i}
\] (3.17)

which, for want of a better name, we shall call the 'streamline coordinates'. Also, we shall write \( q(X, Z, \tau) = \tilde{q}(\hat{X}, \hat{\tau}, \zeta) \) for any dependent variable \( q \). Under this transformation, (3.16) becomes

\[
\hat{\Omega}_{i\hat{X}} = 0 \quad \text{where} \quad \Omega_{iX} = U_{iZ}
\] (3.18)
which gives
\[ \hat{\Omega}_i = K(\zeta, \tau). \]  
(3.19)

The arbitrary function \( K \) is found by imposing the far-field conditions (3.15); thus
\[ K^\pm(\zeta, \tau) = \pm \lambda_2 \sqrt{\frac{2\zeta}{\lambda_i}} + K^i(\zeta, \tau) = 0. \]  
(3.20)

The solution for \( \Psi_1 \) in each region is therefore given by
\[ \Psi_1^\pm = \beta_1(x, \tau)Z \pm \lambda_2 \int\int_{z_1}^{z_2} sZ^2 + \gamma dZ_2 dZ_1 \text{ in } CL^\pm \]  
\[ \Psi_1^i = \beta_1(x, \tau)Z \text{ in } CL^i \]  
(3.21)

where \( \gamma = 2\sigma\eta(\tau)/\lambda_i^2 \). In obtaining the above solutions, we have imposed the conditions of continuous normal and tangential components of velocity at the separating streamlines; the thin viscous layers, of which more will be said later, are relatively unimportant at this order. The arbitrary function \( \beta_1(x, \tau) \) appearing in (3.21) can be determined by matching the solutions with those in \( I^\pm \). We obtain
\[ \beta_1 = \frac{\sigma\lambda_i\eta}{\lambda_i^3} \left\{ \ln(Z_0 + \gamma_m) + 1 + \frac{Z_0 \gamma_m}{\gamma_1} - \ln 2 \right\} - \sigma\eta\lambda_i D \]  
(3.22a)
\[ \beta_1 = \frac{\sigma\lambda_i\eta}{\lambda_i^3} \left\{ \ln(Z_0 + \gamma_m) + 1 + \frac{Z_0 \gamma_m}{\gamma_1} - \ln 2 \right\} + \sigma\eta\lambda_i E \]  
(3.22b)

from the join with \( I^+, I^- \) respectively. Here \( \gamma_m(\tau) = 2\sigma\eta_m(\tau)/\lambda_i^2 \), and expressions for \( D \) and \( E \) are given in appendix B. Together, (3.22a,b) give \( D + E = 0 \), which is just (3.5) in fact. So our earlier assumption that the velocity jump across \( CL \) is \( o(\varepsilon^2) \) is consistent with the critical-layer flow. The above solution also requires an \( O(\varepsilon^3) \) correction to the outer flow, so that (3.11a) should read
Since the extra term is independent of both \( X \) and \( y \), it does not affect the higher-order solutions.

At the next order, the problem for \( (U_2, \Psi_2, P_2) \) is

\[
\lambda_1 Z U_{2x} - \frac{\sigma \eta \xi}{\lambda_1} U_{2z} - \lambda_1 \Psi_{2x} = -P_{2x} - U_{1x} - U_{1z} + \Psi_{1x} U_{1z} + U_{1zz} \\
P_{2z} = 0, \quad U_2 = \Psi_{2z}.
\] (3.23)

The pressure jump across \( CL \) is therefore \( o(\varepsilon^4) \) as anticipated earlier.

In terms of the streamline coordinates the vorticity equation is then

\[
\hat{\Omega}_{2x} = \frac{1}{\lambda_1 Z} \frac{\sigma \eta \xi}{\lambda_1 Z} \frac{\partial}{\partial \xi} (\lambda_1 Z K_z) \] (3.24)

which may readily be integrated to give

\[
\Omega_{2x}^\pm = -\frac{\lambda_1}{2 \lambda_1 Z^2 + \gamma} \frac{\int_X^{x} \hat{\gamma}_\tau \, d\xi}{\sqrt{Z^2 + \gamma - \hat{\gamma}}} \\
+ \frac{\lambda_1}{\lambda_1 Z} \frac{\partial}{\partial \xi} \left\{ \frac{1}{\sqrt{Z^2 + \gamma - \hat{\gamma}}} \int_X^{x} \sqrt{Z^2 + \gamma - \hat{\gamma}} - \sqrt{Z^2 + \gamma} \, d\xi \right\} \\
\pm \frac{\lambda_1}{\lambda_1 Z^2 + \gamma} \Psi_{1x}^\pm + K_2^\pm (Z^2 + \gamma) \] (3.25a,b)

(where \( \gamma = \gamma(X, \tau), \hat{\gamma} = \gamma(\xi, \tau) \) above and below the separating streamlines. Some care has been taken in integrating the viscous term so that the integrals in (3.25) are convergent. The solution in the interior \( CL^i \) is found by replacing \( K_2^\xi \) by \( K_2^i = 0 \) in the above:

\[
\Omega_{2x}^i = K_2^i (Z^2 + \gamma). \] (3.25c)
The far-field conditions (in the limit as $X \to \infty$) now fix $K_2$ in each region:

$$K_2^i(\zeta) = -\frac{\lambda_i}{\lambda}(\beta_1 - \frac{i}{\lambda} \lambda_2 \beta) - \frac{2 \lambda_1^3 \beta_0}{3 \lambda_2 \lambda_3} + \left(\frac{\lambda_1}{\lambda} - \frac{\lambda_3^3}{3 \lambda_1^3}\right)\zeta$$

(3.26)

We see, therefore, that $\Omega_2$ is even in $\zeta$, whereas $\Omega_1$ is odd. The evenness or oddness of the solutions is a major simplifying feature of the analysis. For only the even part of $\Omega$ contributes to the velocity jump across $CL$.

The solutions in each region may be written

$$\Psi_2^+ = \alpha_i^+ + \beta_2^+ \zeta + \int_{\frac{\zeta}{\lambda}}^{\zeta} \Omega_2^+ d\zeta d\zeta$$

$$\Psi_2^- = \alpha_i^- + \beta_2^- \zeta \zeta$$

(3.27)

Continuity of normal velocity at the separating streamlines gives the relations

$$\alpha_2^+ + \beta_2^+ \zeta = \alpha_2^- + \beta_2^- \zeta$$

(3.28)

between the unknown functions appearing in (3.27). Furthermore, a relationship can be found between the two unknowns $\beta_2^\pm$ by considering the match with $I^\pm$. From (3.27) we see that the finite part of the $O(\epsilon^3)$ velocity perturbation just above $CL$ is

$$\beta_2^+ + \int_{\omega}^{\infty} \Omega_2^+ d\zeta$$

whilst that below $CL$ is

$$\beta_2^- + \int_{\omega}^{-\infty} \Omega_2^- d\zeta = \beta_2^- - \int_{\omega}^{\infty} \Omega_2^+ d\zeta$$
since $\Omega_2$ is even. But both expressions must be zero to match with the solutions in $I^*$, giving

$$\beta_2^+ + \beta_2^- = 0.$$  \hfill (3.29)

The discontinuity in the above solution is smoothed out by the thin viscous layers surrounding $Z = \pm Z_0$.

The $O(\epsilon^*)$ velocity jump across CL is determined by the system governing $(U_3, \Omega_3)$, which, in terms of the streamline coordinates, is

$$\begin{align*}
\hat{\Omega}_{3\hat{x}}^+ &= \pm \sqrt{2\lambda_i \frac{d}{\dot{\xi}}} \left( \hat{\Omega}_{2\xi} \sqrt{\lambda_i - \frac{\eta \xi}{\lambda_i}} \right) \\
\hat{\Omega}_{3\hat{x}}^- &= \mp \sqrt{2\lambda_i \frac{d}{\dot{\xi}}} \left( \hat{\Omega}_{2\xi} \sqrt{\lambda_i - \frac{\eta \xi}{\lambda_i}} \right) \\
\hat{\Omega}_{3\hat{x}} &= 0.
\end{align*}$$ \hfill (3.30)

Using the oddness and evenness of the lower-order solutions, we see that the solutions of (3.30) in the three regions may be written

$$\begin{align*}
\Omega_3^+ &= K_3^+ \left( Z^2 + \gamma \right) \pm \frac{\alpha_2^+}{\lambda_i \sqrt{Z^2 + \gamma}} + \text{terms odd in } Z \\
\Omega_3^- &= K_3^- \left( Z^2 + \gamma \right) \\
\Omega_3^i &= K_3^i \left( Z^2 + \gamma \right).
\end{align*}$$ \hfill (3.31)

The far-field conditions give $K_4^f = 0$ and

$$K_3^f \left( Z^2 + \gamma \right) = \mp \frac{\alpha_2^f}{\lambda_i \sqrt{Z^2 + \gamma}} + \text{Odd} \quad (3.32)$$

(and $K_3^f$ is independent of $\tau$). So we have

$$\begin{align*}
\Omega_3^f &= \text{Odd} \\
\Omega_3^i &= 0.
\end{align*}$$ \hfill (3.33)
The contribution to the $O(\varepsilon^4)$ velocity jump from the inviscid parts of CL is therefore zero. The solution at this order may be written

\[
\begin{align*}
\Psi_{3}^{-} &= \alpha_{3}^{+} + \beta_{3}^{+} Z + \int_{Z_{0}}^{Z} \int_{Z_{1}}^{Z_{0}} \Omega_{3} dZ dZ, \\
\Psi_{3}^{+} &= \alpha_{3}^{-} + \beta_{3}^{-} Z. 
\end{align*}
\] (3.34)

Matching the finite part of the stream function to the outer flow gives

\[
\alpha_{3}^{+} + \int_{Z_{0}}^{Z_{1}} \int_{Z_{0}}^{Z} \Omega_{3} dZ dZ = 0 = \alpha_{3}^{-} + \int_{Z_{1}}^{Z_{0}} \int_{Z_{0}}^{Z} \Omega_{3} dZ dZ
\]

so that

\[
\alpha_{3}^{+} + \alpha_{3}^{-} = 0
\] (3.35)

since $\alpha_{3}$ is odd.

Next, we need to consider the $O(\varepsilon^4)$ velocity jump across the thin viscous layers. From the above solution for $U_3$ (from (3.34)) this is simply

\[
(\beta_{3}^{+} - \beta_{3}^{-}) + (\beta_{3}^{-} - \beta_{3}^{+}) = \beta_{3}^{+} - \beta_{3}^{-}. \quad (3.36)
\]

Now, within the viscous layers $\hat{z} = 0(1)$, where

\[
y = y_c \pm \varepsilon Z_0 (\chi, \tau) + \varepsilon^\frac{3}{2} \hat{z}. \quad (3.37)
\]

Here and below, whenever * or * occur, the upper sign refers to the upper viscous layer surrounding $Z = Z_0$, and the lower sign to that surrounding $Z = -Z_0$. The development in each viscous layer is mainly a continuation of the outer critical-layer flow:

\[
\begin{align*}
u &= c \pm \varepsilon \lambda_i Z_e + \varepsilon^\frac{1}{2} \lambda_i \hat{z} + \varepsilon \beta_i + \varepsilon^\frac{1}{2} \hat{u}_i + \varepsilon^\frac{1}{2} \hat{u}_2 + \varepsilon^\frac{1}{2} \hat{u}_3 + \varepsilon^\frac{1}{2} \hat{u}_4 + \ldots, \\
\psi &= \pm \varepsilon c Z_c + \varepsilon^\frac{3}{2} c \hat{z} + \varepsilon^2 \left(\frac{1}{2} \lambda_i Z_0 + \frac{c q}{\lambda_i}\right) \pm \varepsilon^\frac{3}{2} \lambda_i Z_0 \hat{z} + \varepsilon^2 \left(\frac{1}{2} \lambda_i \hat{z}^2 \pm \beta_i \hat{z}\right) + \varepsilon^\frac{3}{2} \beta_i \hat{z} + \varepsilon^2 \hat{\psi}_1 + \varepsilon^\frac{3}{2} \hat{\psi}_2 + \varepsilon \hat{\psi}_3 + \varepsilon^\frac{3}{2} \hat{\psi}_4 + \ldots
\end{align*}
\] (3.38a)

\[
\begin{align*}
u &= c \pm \varepsilon \lambda_i Z_e + \varepsilon^\frac{1}{2} \lambda_i \hat{z} + \varepsilon \beta_i + \varepsilon^\frac{1}{2} \hat{u}_i + \varepsilon^\frac{1}{2} \hat{u}_2 + \varepsilon^\frac{1}{2} \hat{u}_3 + \varepsilon^\frac{1}{2} \hat{u}_4 + \ldots \\
\psi &= \pm \varepsilon c Z_c + \varepsilon^\frac{3}{2} c \hat{z} + \varepsilon^2 \left(\frac{1}{2} \lambda_i Z_0 + \frac{c q}{\lambda_i}\right) \pm \varepsilon^\frac{3}{2} \lambda_i Z_0 \hat{z} + \varepsilon^2 \left(\frac{1}{2} \lambda_i \hat{z}^2 \pm \beta_i \hat{z}\right) + \varepsilon^\frac{3}{2} \beta_i \hat{z} + \varepsilon^2 \hat{\psi}_1 + \varepsilon^\frac{3}{2} \hat{\psi}_2 + \varepsilon \hat{\psi}_3 + \varepsilon^\frac{3}{2} \hat{\psi}_4 + \ldots
\end{align*}
\] (3.38b)
\[ \rho = \sigma (1 - y_c) \pm \sigma e Z_c - \sigma \varepsilon \frac{1}{2} \tilde{Z} + \varepsilon^2 \sigma \eta + \varepsilon^4 \rho + \ldots \quad (3.38c) \]

Again, the expansions (3.38) are not complete, but the additional terms are not required. The match with the outer solutions gives

\[
\begin{align*}
\tilde{u}^\pm_1 &\sim \pm \lambda_2 Y \tilde{Z}^2 + O \\
\tilde{u}^\pm_2 &\sim \frac{\lambda_2 Z C_0}{2 Y_m} \tilde{Z}^2 + \beta_1^\pm \\
\tilde{u}^\pm_3 &\sim \pm \frac{\lambda_1 Y}{b Y_m} \tilde{Z}^3 + A \tilde{Z} + O \\
\tilde{u}^\pm_4 &\sim - \frac{\lambda_1 Y Z_c}{8 Y_m} \tilde{Z}^4 \pm B \tilde{Z}^2 + \beta_3^\pm \\
\tilde{u}^\pm_i &\to 0, \quad \tilde{u}^\pm_2 \to \beta_2^i, \quad \tilde{u}^\pm_3 \to 0, \quad \tilde{u}^\pm_4 \to \beta_3^i \quad \text{as} \quad \tilde{Z} \to \pm \infty \quad (3.39a)
\end{align*}
\]

\[
\tilde{u}^\pm_i \to 0, \quad \tilde{u}^\pm_2 \to \beta_2^i, \quad \tilde{u}^\pm_3 \to 0, \quad \tilde{u}^\pm_4 \to \beta_3^i \quad \text{as} \quad \tilde{Z} \to \pm \infty. \quad (3.39b)
\]

The functions A and B are derived from the behaviour of \( \Omega_2 \) in the neighbourhood of the viscous layers, and are rather complicated; fortunately, we will not need to know them.

The total 'viscous' velocity jump (3.36) can be found from the equations governing \( \tilde{u}^\pm_i \), which are

\[
\begin{align*}
\tilde{u}^\pm \frac{\partial}{\partial \tilde{Z}} &= \pm \lambda_1 Z C_0 \tilde{u}^\pm_{ix} + \lambda_1 Z C_\tau \tilde{Z} \tilde{u}^\pm_{iZ} \pm \lambda_1 Z C_{\tau \xi} \tilde{u}^\pm_i \\
&\quad + \tilde{u}^\pm_{iX} - (\beta_1 \tilde{u}^\pm_1) - \beta_1 \tilde{Z} \tilde{u}^\pm_{iZ} + \tilde{u}^\pm_{ix} \tilde{u}^\pm_{iZ} - \tilde{\psi}^\pm \tilde{U}^\pm_{iZ} \\
&\quad + \left[ Z C_\tau + (\beta_1 Z C)_{\tau i} \right] \tilde{u}^\pm_{iZ} + \lambda_1 \tilde{Z} \tilde{u}^\pm_{3Z} - \lambda_1 \tilde{\psi}^\pm \tilde{U}^\pm_{iZ} \quad (3.40)
\end{align*}
\]

from the Navier-Stokes equations. Substituting the asymptotic forms (3.39) into (3.40) we find

\[
\begin{align*}
\mp 2B &= \pm \lambda_1 Z C_0 \beta_3^\pm \pm \lambda_1 Z C_\tau \beta_3^\pm + \beta_{3\tau}^\pm + (\beta_1 \beta_{2\tau}^\pm) \\
&\quad + \left[ Z C_\tau + (\beta_1 Z C)_{\tau i} \right] A \mp \lambda_1 \alpha_{3\tau}^\pm \quad (3.41)
\end{align*}
\]

which, added together, yield
\[ \left[ \lambda_1 Z_0 (\beta_3^+ - \beta_3^-) \right]_\chi = 0 \] (3.42)

using (3.29) and (3.35), so that

\[ \beta_3^+ - \beta_3^- = \frac{C(\tau)}{Z_\chi(\chi, \tau)} . \] (3.43)

The arbitrary function \( C(\tau) \) can be determined by evaluating (3.43) at \( \chi = \infty \), far ahead of the disturbance, where it is reasonable to impose a zero velocity jump across CL, so that \( C = 0 \).

In summary, the total velocity jump across the critical layer is effectively zero (more precisely, \( o(\epsilon^4) \)). As a result, the evolution equation for the free surface is simply

\[ \sigma \eta \int_0^\infty \frac{d\eta}{\bar{W}(\eta)} - 3 \sigma \eta \int_0^\infty \frac{d\eta}{\bar{W}(\eta)} - \sigma \eta \int_0^\infty \int_0^\infty \frac{\bar{W}(\eta') \, d\eta'}{\bar{W}(\eta) \bar{W}(\eta')} \]

\[ = \begin{cases} 
- \frac{\sigma}{\pi \lambda^2 \epsilon^{3/2}} \int_0^\infty \frac{\eta \, d\xi}{(\xi - \eta)^{3/2}} & \text{for } \epsilon > 0 \\
- \frac{\sigma}{\pi \lambda^2 \epsilon^{3/2}} \int_\infty^\chi \frac{\eta \, d\xi}{(\xi - \eta)^{3/2}} & \text{for } \epsilon < 0 
\end{cases} \] (3.44)

which is essentially identical to the special case of the straight-line profile. The curvature of the shear flow has no effect on its stability properties, therefore, apart from the global influence through the integrated quantities in (3.44).

Hence the conclusions in §2 concerning the alternative stabilising or destabilising effects of viscosity carry over to the present more general basic flow. In particular, we see that the middle mode \( \epsilon = c_0 \in (0,1) \) of the cubic velocity studied earlier (see (3.6b) ff.) does yield growth on the long time scale since the coefficient of \( \eta_\tau \) in (3.44) is positive.
\[
\int_0^1 \frac{dy}{W(y)} = \frac{1}{6(1-c)} \left( \frac{\xi}{\sigma} + \frac{1}{c^2} \right)
\]

when \( U_8(y) = 1-(1-y)^2 \) (using the relation (3.5)).
To derive (2.20) from (2.19) we first use the convolution theorem to write the right-hand side of (2.19) as

\[ \sqrt{2\pi\mu} \int_{0}^{\infty} \alpha^{\frac{1}{2}} \hat{\eta}_i(\alpha, \tau) e^{i\alpha x} d\alpha. \]  

(A1)

Then, multiplying (2.19) by \( \eta_1(x, \tau) \) and integrating over \( x \) from \(-\infty\) to \( \infty \), we obtain

\[ \frac{d}{d\tau} \int_{-\infty}^{\infty} \eta_i^2 dX - \int_{-\infty}^{\infty} \left( \eta_i^2 \eta_i x + \eta_i \eta_i xx \right) dX \]

\[ = \sqrt{2\pi\mu} \int_{-\infty}^{\infty} \int_{0}^{\infty} \alpha^{\frac{1}{2}} \hat{\eta}_i(\alpha, \tau) \eta_i(x, \tau) e^{i\alpha x} d\alpha dX. \]  

(A2)

The second integral on the left-hand side of (A2) may be evaluated by parts, and is found to be zero using the conditions (2.17). We may interchange the order of integration of the double integral on the right-hand side of (A2) (provided that the integrals are uniformly convergent, of course) so that (A2) reduces to

\[ \frac{d}{d\tau} \int_{-\infty}^{\infty} \eta_i^2 dX = \sqrt{2\pi\mu} \int_{0}^{\infty} \alpha^{\frac{1}{2}} \hat{\eta}_i(\alpha, \tau) \hat{\eta}_i(c)(\alpha, \tau) d\alpha \]  

(A3)

(where the superscript \( (c) \) denotes the complex conjugate) which is (2.20).
Appendix B

In terms of the critical-layer variable \( Z = \epsilon^{-1}(y - y_c) \) the first two terms of the expansion (3.2b) for \( \psi \) yield the asymptotes

\[
\psi \sim \epsilon c Z + \left( \frac{1}{2} \lambda Z^2 + \frac{\sigma_\eta}{\lambda_i} \right) \epsilon^2 + \frac{\sigma_\eta \lambda_i}{\lambda_i^3} Z \ln \epsilon
\]

\[
\quad + \left\{ \frac{1}{6} \lambda_i Z^3 + \frac{\sigma_\eta \lambda_i}{\lambda_i^3} Z \ln Z + \left( \frac{\lambda_i^3}{2 \lambda_i^3} - \lambda_i D \right) \sigma_\eta Z \right\} \epsilon^3
\]

\[
\quad + \frac{\sigma_\eta \lambda_i^3}{2 \lambda_i^3} Z^2 \epsilon^4 \ln \epsilon + O(\epsilon^4) \quad \text{as} \quad Z \to \infty, \quad (B1)
\]

\[
\psi \sim \epsilon c Z + \left( \frac{1}{2} \lambda Z^2 + \frac{\sigma_\eta}{\lambda_i} \right) \epsilon^2 + \frac{\sigma_\eta \lambda_i}{\lambda_i^3} Z \ln \epsilon
\]

\[
\quad + \left\{ \frac{1}{6} \lambda_i Z^3 + \frac{\sigma_\eta \lambda_i}{\lambda_i^3} Z \ln |Z| + \left( \frac{\lambda_i^3}{2 \lambda_i^3} + \lambda_i E \sigma_\eta Z \right) \epsilon^3
\]

\[
\quad + \frac{\sigma_\eta \lambda_i^3}{2 \lambda_i^3} Z^2 \epsilon^4 \ln \epsilon + O(\epsilon^4) \quad \text{as} \quad Z \to -\infty \quad (B2)
\]

where

\[
D = \int_{y}^{y_c} \left\{ \frac{1}{W(x)} - \frac{1}{\lambda_i^2(x-y_c)} + \frac{\lambda_i}{\lambda_i^3(x-y_c)} \right\} dy
\]

\[
\quad + \frac{1}{\lambda_i^2(1-y_c)} + \frac{\lambda_i}{\lambda_i^3} \ln (1-y_c) + \frac{1}{\sigma} \quad ,
\]

\[
E = -\int_{0}^{y_c} \left\{ \frac{1}{W(y)} - \frac{1}{\lambda_i^2(y-y_c)} + \frac{\lambda_i}{\lambda_i^3(y-y_c)} \right\} dy
\]

\[
\quad + \frac{1}{\lambda_i^2 y_c} - \frac{\lambda_i}{\lambda_i^3} \ln y_c \quad .
\]

The expansions (B1) and (B2) imply that \( \psi \) expands in the asymptotic sequence \( \epsilon, \epsilon^2, \epsilon^3 \ln(\epsilon), \epsilon^4, \epsilon^4 \ln(\epsilon), \epsilon^5, \ldots \). Johnson (1986) calculates further terms in (B1,2) – see his appendix for more details. The logarithmic contributions to the expansions have been ignored in the critical-layer analysis of §3 since they do not interact with the terms in \( \epsilon, \epsilon^2, \epsilon^3 \), etc.
Captions

Figure 1. Schematic diagram of a long wavelength, small amplitude disturbance on the surface of a flow with a simple shearing motion, showing also the regions I⁺, I⁻ and II defined in §2.

Figure 2. σ vs. c from equation (2.7).

Figure 3. Structure of the flow with a critical layer [§3], indicating the Rayleigh zones I⁺, I⁻, the critical layer CL, and the viscous wall layer II.

Figure 4. σ vs. c from (a) equation (3.6a) (no critical layer), and (b) equation (3.6b) (always just one critical layer).

Figure 5. Internal structure of the critical layer under a single-crest free-surface elevation. Thin viscous layers surround the streamlines which separate the three inviscid regions CL⁺, CL⁻ of the critical layer.
**Figure 1**

**Figure 2**
FIGURE 4
CHAPTER SIX

The stability of a two-fluid shearing flow.
In our discussion of the stability properties of a liquid layer in an arbitrary shearing motion over a rigid surface given in the previous chapter, we concentrated on the combined effects of a non-zero mean flow and small viscosity on long gravity waves. In a more general analysis, we may also consider the effects of surface tension and the dynamical response of an external fluid. In some circumstances, we would expect these two additional effects to be negligible. In a situation where, for example, a flow of water under air is driven by some upstream source, the motion induced in the air above is likely to be so small dynamically that it results in only slight variations of pressure, which would only be significant if the densities of the two fluids were almost equal. Again, surface tension does not significantly affect long-wavelength disturbances such as those studied in chapter 5. But, given that, it seems important nevertheless to know when they can safely be neglected, and the consequences when they cannot; both effects may be important in other two-fluid motions. In this chapter, therefore, we consider the effects of all the above on the stability of a two-fluid system in a general shearing motion. The fluids are taken to be incompressible, and the motion laminar and two-dimensional.

For definiteness, we suppose that both fluids are driven by an upstream source, that the upper fluid is moving with speed $U_\infty$ far from the wall, and that the interface between the two fluids lies within the boundary layer (so that the depth of the lower fluid $d_\ast$ is comparable with the boundary-layer thickness of the combined
system; see figure 1). The physical properties of the two fluids (the densities \( \rho_0^f, \rho_1^f \) and viscosities \( \mu_0^f, \mu_1^f \), where the suffices refer to the upper and lower fluids) are taken to be fixed, as is the interfacial tension \( \gamma^* \), although the latter may easily be varied in an experiment by simply adding surfactant (such as washing-up liquid) to the interface. The Reynolds number of the flow \( R = \rho_1^f U_1^* \epsilon^*/\mu_1^f \) (where \( \epsilon^* \) is a representative length scale of the flow – see below) is based on the lower-fluid properties and is taken to be large throughout. The coordinate system we use is \( \xi(x,y) \) (with the wall given by \( y = 0 \)) with corresponding velocity components \( U_1^*(u,v) \) and pressure \( \rho_1^f U_1^* \tau_1 \), and time is written \( (\epsilon^*/U_1^*)t \). In non-dimensional form the equations governing the motion are

\[
\begin{align*}
\frac{\partial u^+}{\partial t} + u^+ u_x^+ + v^+ u_y^+ &= -\frac{p^+}{\rho^+} + \frac{\mu^+}{R^+} (u_{xx}^+ + u_{yy}^+) \\
\frac{\partial v^+}{\partial t} + u^+ v_x^+ + v^+ v_y^+ &= -\frac{p^+}{\rho^+} - \sigma + \frac{\mu^+}{R^+} (v_{xx}^+ + v_{yy}^+) \\
u_x^+ + v^+_y &= 0
\end{align*}
\]

\( (1.1a) \)

\[
\begin{align*}
\frac{\partial u^-}{\partial t} + u^- u_x^- + v^- u_y^- &= -\frac{p^-}{\rho^-} + \frac{1}{R^-} (u_{xx}^- + u_{yy}^-) \\
\frac{\partial v^-}{\partial t} + u^- v_x^- + v^- v_y^- &= -\frac{p^-}{\rho^-} - \sigma + \frac{1}{R^-} (v_{xx}^- + v_{yy}^-) \\
u_x^- + v^-_y &= 0
\end{align*}
\]

\( (1.1b) \)

in the upper fluid and the lower fluid respectively. Here, \( \mu = \mu_0^f/\mu_1^f, \rho = \rho_0^f/\rho_1^f, \) and \( \sigma = g \epsilon^*/U_1^* \). If the interface is given by \( y = \eta(x,t) \), the requirements of continuity of the components of the velocity and stress there are

\[
\begin{align*}
u^+ &= u^- , & u^+_y &= u^-_y \\
p^+ - \frac{2\mu^+}{R^+ (1 + \eta_x^2)} (u_{x}^+ \eta_x^2 - (u_x^+ + u_y^+) \eta_x^+ + u_y^+) &= -p^- + \frac{2}{R^+ (1 + \eta_x^2)} (u_{x}^- \eta_x^2 - (u_x^- + u_y^-) \eta_x^- + u_y^-) + \frac{\gamma \eta_{xx}}{(1 + \eta_x^2)^{3/2}}
\end{align*}
\]

\( (1.2a,b) \)

\( (1.2c) \)
\[ 2\mu(u^+_x - u^-_y)\eta_x - \mu(u^+_y + u^-_x)(1 - \eta^2_x) = 2(u^+_x - u^-_y)\eta_x - (u^+_y + u^-_x)(1 - \eta^2_x) \]  \hspace{1cm} (12c)

where \( \gamma = \gamma^*/\rho^* \xi^* U^*_{\infty}^2 \), and superscripts +/- refer to the upper/lower fluid.

The propagation of small-amplitude waves of wavelength \( \xi^* \) depends in part on the relative magnitudes of three main length scales (if not more): the boundary-layer thickness, \( O(d^*) \) here; a length scale associated with capillary effects, \( \gamma^*/\rho^* \xi^* U^*_{\infty}^2 \); and a length scale associated with gravity effects which result from the density stratification, \( U^*_{\infty}^2/g \). Thus, to retain all the important physical effects, we suppose \( \xi^* \sim \gamma^*/\rho^* \xi^* U^*_{\infty}^2 \sim U^*_{\infty}^2/g \) i.e. \( \gamma, \sigma = O(1) \) as \( R \to \infty \).

It remains to determine at what location a viscous disturbance of such a wavelength can become neutrally stable, for this fixes \( d^* \).

Perhaps the easiest way to fix the neutral-stability characteristics of the large-Reynolds-number flow is to consider the correspondence with the equivalent one-fluid system, which is a special case of the present flow with \( \rho = \mu = 1, \gamma = 0 \). It is known that the triple-deck structure governs Tollmien-Schlichting instabilities on the "lower branch" of the Orr-Sommerfeld neutral stability curve for Blasius-type boundary layers (see Smith 1979b and figure 2). The length scale of the triple-deck structure is \( O(\hat{\lambda}^{-3/4}R^{-3/8}) \) at a point where the (non-dimensional) wall shear is \( \hat{\lambda} \). The stability is mainly governed by the flow in the viscous lower deck, of lateral extent \( O(\hat{\lambda}^{-3/4}R^{-3/8}) \), which itself interacts the the flow in the upper deck of lateral extent \( O(\hat{\lambda}^{-3/4}R^{-3/8}) \). So the triple-deck length scale rises to \( O(1) \) when \( \hat{\lambda} \sim R^{-3/10} \), at which point the relative boundary-layer thickness is \( O(d^*/\xi^*) = O(R^{-1/5}) \). Finally, a nonlinear-viscous-unsteady response is provoked in the lower deck when the amplitude of the disturbance (relative to the streamwise component of the
velocity, say) is $O(R^{-1/5}) = O(R^{-1/5})$. The frequency of the disturbance is fixed by the requirement that unsteadiness appears explicitly only in the lower deck; then $t = \tau R^{-1/5}$ with $\tau = O(1)$ i.e.

the above structure governs comparatively low-frequency waves. For such a disturbance, the flow in the main deck (the majority of the boundary layer), in which $\bar{Y} = O(1)$ where $y = \varepsilon \bar{Y}$ and $\varepsilon = R^{-1/5}$, is given by

\begin{align*}
\upsilon^+ &= U_b^+(\bar{Y}) + \varepsilon A(x, \tau) U_b^+(\bar{Y}) + \ldots \\
\upsilon^- &= -\varepsilon^2 (\delta A/\delta x) U_b^+(\bar{Y}) + \ldots \\
p^+ &= -\sigma \varepsilon (\rho \bar{Y} + (1-\rho) d) + \varepsilon^2 p^+(x, \tau) + \ldots \\
p^- &= -\sigma \varepsilon \bar{Y} + \varepsilon^2 p^-(x, \tau) + \ldots
\end{align*}

(1.3)

where $U_b^+(\bar{Y})$ is the basic undisturbed flow in each fluid, and $d = d^*/\varepsilon^2$. The displacement function $A(x, \tau)$ is the same in both fluids as a result of the continuity of the normal velocity at the interface (from (1.2a,b)). Moreover, the condition (1.2c) of continuous tangential component of stress fixes the position of the interface as

$$\bar{Y} = d - \varepsilon A(x, \tau) + \ldots$$

(1.4)

Then (1.2d), the condition of continuity of normal stress, yields the relation

$$p^+ = p^- + \sigma (1-\rho) A - \gamma A_{xx}$$

(1.5)

between the pressure perturbations above and below the interface. In the lower deck, $y = \varepsilon^2 Y$ with $Y = O(1)$, and

\begin{align*}
\upsilon &= \varepsilon U(x, Y, \tau) + \ldots \\
\upsilon &= \varepsilon^3 V(x, Y, \tau) + \ldots \\
p &= -\sigma \varepsilon^2 Y + \varepsilon^2 p(x, \tau) + \ldots
\end{align*}

(1.6)
as implied by the main deck expansions. Using these expansions in (1.1b) we obtain

\begin{align}
U_t + U U_x + V U_y &= -P_x + U_{yy} \\
U_x + V_y &= 0
\end{align}  \hspace{1cm} (1.7a)

while the match with the main deck gives

\begin{align}
\bar{P}(x, \tau) &= P(x, \tau) \\
U &= \lambda (Y + A(x, \tau)) \quad \text{as} \quad Y \to \infty
\end{align}  \hspace{1cm} (1.7c)

where \( \lambda = \lambda R^{3/10} \) is the scaled \( O(1) \) skin friction of the basic flow.

We also have the no-slip condition at the wall:

\begin{align}
U = V = 0 \quad \text{at} \quad Y = 0.
\end{align}  \hspace{1cm} (1.7d)

Finally, the flow in the upper deck, where \( y = O(1) \), determines the relationship between the unknowns \( P(x, \tau) \), \( A(x, \tau) \). Here, we have

\begin{align}
\begin{cases}
\bar{u} = 1 + \varepsilon^2 \bar{U}(x, y, \tau) + \ldots \\
\varepsilon^2 \bar{v}(x, y, \tau) + \ldots \\
\bar{p} = -\sigma \bar{p} y - \sigma \varepsilon d(1-\rho) + \varepsilon^2 \bar{p}(x, y, \tau) + \ldots.
\end{cases}
\end{align}  \hspace{1cm} (1.8)

The governing equations (1.1a) and the match with the main deck reduce to the following system for \( \bar{p} \):

\begin{align}
\bar{p}_{xx} + \bar{p}_{yy} &= 0 \quad (0 < y < \infty) \\
\bar{p} &= P + \sigma(1-\rho) A - \gamma A_{xx}, \quad \bar{p}_y = \rho A_{xx} \quad \text{at} \quad y = 0.
\end{align}  \hspace{1cm} (1.9a-c)

We also need an outer condition on \( \bar{p} \):

\begin{align}
|\bar{p}| < \infty \quad \text{as} \quad y \to \infty.
\end{align}  \hspace{1cm} (1.9d)

Although we shall mainly use equations (1.9) as they stand in the analysis of the unsteady flow given below, we can easily obtain the
pressure-displacement law explicitly. We find that

\[ P = -\sigma(1-\rho)A + \sqrt{A_{xx}} + \frac{\rho}{\pi} \int_0^\infty \frac{A'(\xi,\zeta)}{\xi^2 - \zeta} d\zeta. \]  

(1.10)

Thus, the effects of density stratification, surface tension, and the motion of the external fluid are all contained in the above law in a remarkably simple way, each effect contributing one term to (1.10). In the limit \( \rho \to 0, \gamma \to 0 \) we recover the supercritical law derived in chapter 4. For \( \rho = 0, \gamma \neq 0 \) the effect of surface tension counteracts that of the curvature of the streamlines, possibly preventing a free interaction. The law for external flow is recovered in the limit \( \rho \to 1, \gamma \to 0 \). Other limits may be of interest, but in fact we shall consider the most general case in the following.

A simplifying feature of the central problem (1.7a-d) with (1.9a-d) is that the solution only depends on two independent combinations of the four parameters \( \lambda, \rho, \sigma, \gamma \). If we set

\[ (x, y) = \frac{\gamma}{\rho} (x^*, y^*), \quad y = \frac{y^*}{\rho y^*}, \quad \tau = \frac{\gamma y^*}{\rho y^*}, \quad \bar{u} = \frac{\rho^{3/2}}{\gamma^{3/2}} u^*, \quad u = \frac{\rho^{3/2}}{\gamma^{3/2}} v^*, \quad (p, \bar{p}) = \frac{\rho^{3/2}}{\gamma^{3/2}} (p^*, \bar{p}^*), \quad A = \frac{\gamma^{3/2}}{\rho^{3/2}} A^*. \]  

(1.11a)

then we see that the only remaining parameters in the equations are

\[ \lambda_* = \frac{\lambda y^{3/2}}{\rho^{1/2}}, \quad S = \frac{\gamma \sigma (1-\rho)}{\rho^2}. \]  

(1.11b)

The new equations are obtained from (1.7), (1.9) by replacing \( \lambda \) by \( \lambda_* \), \( \gamma \) by 1, and \( \rho \) by 1 except where \( \rho \) appears in the combination \( \sigma (1-\rho) \), which is replaced by \( S \).

In section 2 below, we shall investigate the linear stability of (1.7) with (1.9). Neutral waves are found to exist as long as

\[ S < \frac{1}{4}. \]  

(1.12)
in terms of the original dimensional flow quantities, this condition is

\[
\frac{g y^* (\rho^* - \rho_u^*)}{\rho_u^* U_{\infty}^*} < \frac{1}{4}.
\]

For the common example of air blowing over shallow water, a viscous instability then occurs when

\[
U_{\infty}^* > 6.6 \text{ m/sec}
\]

(or \( U_{\infty}^* > 15 \text{ mph} \)), taking \( g = 981 \), \( g^* = 72.8 \), \( \rho_u^* = 1.21 \times 10^{-3} \), \( \rho_f^* = 0.998 \) in cgs units (Batchelor 1967). This corresponds to wind speeds of at least force 4 measured on the Beaufort scale. Broadly speaking, the effect of surface tension is to stabilise short wavelength disturbances, as we would expect physically, and there is a 'cut-off' wavelength below which waves cannot grow. The density stratification, on the other hand, prohibits long wave instabilities (provided the stratification is 'stable' i.e. \( \rho < 1 \)). The growth or decay of waves depends on the local boundary-layer thickness, which increases as \( \lambda \) decreases, and it is found that, as long as (1.12) is satisfied, there is a unique location at which just one mode is neutrally stable, all other modes decaying, and upstream of which all modes decay. This important case of marginal stability is analysed further in §3, where a Ginzberg-Landau evolution equation is derived for the amplitude of the fastest-growing modes near (i.e. just downstream of) marginal stability. Such an equation cannot be derived in quite the same way for the one-fluid system, where marginal stability (of the sort found here) does not exist.
We suppose first that the disturbance is relatively small, of order $h$ say, with $h < 1$. Then the flow variables may be expanded as follows

$$\begin{align*}
(U, V, P, A, \rho) &= (Y, 0, 0, 0, 0) + h(U, V, P, A, \rho) + \ldots \\
\text{(2.1)}
\end{align*}$$

This may be conveniently re-expressed as $S = S_0 + hS_1 + O(h^2)$ where the 'solution' vectors $S$ etc. correspond to (2.1). We seek simple-wave solutions of (1.7), (1.9) by separating the $x$ and $\tau$ dependence as follows:

$$S_i = \tilde{S}_iE + \tilde{S}_i^{(c)}E^{-i} \quad \text{(2.2)}$$

where

$$E = \exp\left[i(\alpha x - \omega \tau)\right] \quad \text{(2.3)}$$

and the superscript $(c)$ denotes the complex conjugate. The wavenumber $\alpha$ of the disturbance is taken to be real and positive, while in general the frequency $\omega$ will be complex. Substituting the expansions (2.1) into the lower-deck equations (1.7), we obtain

$$\begin{align*}
i(\alpha \lambda Y - \omega)\tilde{U}_i + \lambda \tilde{V}_i &= -i\alpha \tilde{P}_i + \tilde{U}_{iyy}, \quad i\alpha \tilde{U}_i + \tilde{V}_y = 0 \\
\tilde{U}_i &= \tilde{V}_i = 0 \quad \text{at } Y = 0, \quad \tilde{U}_i \rightarrow \lambda \tilde{A}_i \text{ as } Y \rightarrow \infty. \quad \text{(2.4a,b,c,d)}
\end{align*}$$

By differentiating (2.4a) with respect to $Y$, using (2.4b) to eliminate $\tilde{V}_i$, and transforming to the variable

$$\xi = \xi_o + \Delta Y \quad \text{where } \xi_o = -i\omega \Delta^2 \text{ and } \Delta = (i\alpha \lambda)^{\frac{3}{2}} \quad \text{(2.5)}$$

we obtain Airy's equation

$$\left(\frac{d^2}{d\xi^2} - \xi\right)\tilde{U}_i\xi = 0 \quad \text{(2.6)}$$
for $\tilde{U}_1$, which has the (bounded) solution

$$\tilde{U}_1(\xi) = B_i A_i(\xi) \quad (2.7)$$

The amplitude $B_1$ remains undetermined in the linear theory, of course. The boundary conditions (2.4c,d) now give

$$i\alpha \tilde{P}_1 = B_i A_i' \quad , \quad \lambda \tilde{A}_i = B_i C_i \quad (2.8a,b)$$

where $\kappa = \int_{\xi_0}^{\infty} A_i(\xi) d\xi$ and the suffix $o$ denotes evaluation at $\xi = \xi_0$. Elimination of $B_1$ between (2.8a,b) yields one relation between $\tilde{P}_1$ and $\tilde{A}_1$. Another comes from the upper-deck equations, which have the solution

$$\tilde{P}_1 = B e^{-\omega y} \quad (2.9a)$$

From the conditions at $y = 0$ we find

$$b = \tilde{P}_1 + \sigma (1 - \rho) \tilde{A}_1 + \gamma \alpha^2 \tilde{A}_1 \quad , \quad b = \rho \alpha \tilde{A}_1 \quad (2.9b,c)$$

So (2.8a,b) and (2.9b,c) combine to give the dispersion relation

$$\lambda ^2 \tilde{A}_i = \kappa \Delta (\rho \alpha - \sigma (1 - \rho) - \gamma \alpha^2) \quad (2.10)$$

which fixes the frequency $\omega$ in terms of the wavenumber $\alpha$. From (2.10) we may obtain the neutrally-stable modes, which have $\omega$ real. These occur when $\xi_0 = -2.298^{1/3}$ and $\text{Ai}'_o/k = 1.001^{1/3}$ (see e.g. Reid 1965, Lin 1955) so that

$$\omega = 2.298 (\alpha \lambda)^{2/3} \quad (2.11a)$$

$$1.001^{2/3} = \alpha^{2/3} (\rho \alpha - \sigma (1 - \rho) - \gamma \alpha^2) \quad (2.11b)$$

In the special case of one-fluid flow ($\gamma = 0, \rho = 1$) we recover the classical results of Lin (1955) for which there is just one neutral mode for each value of $\lambda$ (see also Smith 1979b). The variation of the
wavenumber of the neutral mode with \( \lambda \) corresponds to the lower branch of the Orr-Sommerfeld neutral-stability curve. For \( \gamma \neq 0 \), (2.11) may have two root (for \( \alpha \)), none, or possibly one double root (marginal stability) depending on the values of \( S = \gamma \sigma (1-\rho)/\rho^2 \) and \( \lambda \).

In figure 3a we sketch the growth rate \( \text{Im}(\omega) \) of the disturbance as a function of \( \alpha \) (using the well-known behaviour of the functions appearing in (2.10): see, for example, Miles 1960) for various values of the controlling parameters. For each value of \( \lambda \), there is, at most, only a finite band of unstable modes. In figure 3b we sketch the neutral wavenumbers as a function of \( \lambda \), or equivalently of distance along the wall, since \( x \) increases as \( \lambda \) decreases. For non-zero values of \( \gamma \) there are no neutral modes for sufficiently large values of \( \lambda \), and all modes decay. From (2.11b) we see that if

\[
\rho \alpha - \sigma (1-\rho) - \gamma \alpha^2 = 0
\]

has a (real) solution, i.e. if

\[
S \leq \frac{1}{4}
\]

(cf. (1.12)), then neutral modes do exist.

We shall concentrate on the interesting and important case of marginal stability, where (2.11a,b) hold together with

\[
7 \gamma \alpha^2 - 4 \rho \alpha + \sigma (1-\rho) = 0
\]

from (2.11b), using the fact that \( d\lambda/d\alpha = 0 \) at marginal stability. Suppose, then, that \((\alpha_1, \omega_1, \lambda_1)\) satisfy (2.11a-c). A small \( O(\delta^2) \) perturbation to \( \lambda_1 \) gives rise to a band width of \( O(\delta) \) of unstable modes. Expanding

\[
\lambda = \lambda_1 + \delta^2 \lambda + \ldots
\]

\[
\alpha = \alpha_1 + \delta \alpha_1 + \delta^2 \alpha + \ldots
\]

\[
\omega = \omega_1 + \delta \omega_1 + \delta^2 \omega + \ldots
\]
we find that the dispersion relation gives

\[ \omega_2 = \frac{2 \omega}{3 \alpha_1} \alpha_2 \]

which is real so that the wave packet of unstable modes propagates with the group velocity \( c_g = \frac{\omega}{\alpha} = \frac{2c_1}{3} \) (where the phase velocity \( c_1 = \frac{\omega_1}{\alpha_1} \)) on an \( O(\delta^{-1}) \) time scale. Again, the dispersion relation gives \( \omega_3 \) complex in general, so that the amplitude of the disturbance varies on a slow \( O(\delta^{-2}) \) time scale. In the next section we investigate the subsequent evolution of the wave packet as nonlinearity comes into play. At present, the largest disturbance that can be incorporated into a weakly-nonlinear theory is such that \( h \sim \delta \); see Stewartson & Stuart (1971). Our nonlinear analysis below closely follows that given by Smith (1979b) of near-neutral (rather than near-marginal) stability of boundary-layer flow.
§3 Weakly-nonlinear theory of marginal stability.

To treat both the slow spatial and temporal modulations of the perturbations to the marginal state, we introduce the multiple-scales transformation

\[
\begin{align*}
\frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x} + h \frac{\partial}{\partial x} \\
\frac{\partial}{\partial \tau} &\rightarrow \frac{\partial}{\partial \tau} - h c_3 \frac{\partial}{\partial x} + h^2 \frac{\partial}{\partial \tau}
\end{align*}
\] (3.1)

according to the argument given in the previous paragraph. As before, the x and \( \tau \) dependence appears only through the factor

\[ E = \exp(i(\alpha_1 x - \omega_1 \tau)) \]

where \( \alpha_1 \) and \( \omega_1 \) (and the corresponding wall shear \( \lambda_1 \)) satisfy (2.11a-c) (and also \( \alpha_1 \) and \( \lambda_1 \) are both positive, of course). We should emphasise that marginal stability only exists for the range of values of \( \rho, \gamma, \sigma \) satisfying (1.12), but that this range is quite extensive nevertheless. We seek solutions of (1.7), (1.9) of the form

\[ S = S_0 + h S_1 + h^2 S_2 + h^3 S_3 + O(h^4). \] (3.2)

In addition, in the neighbourhood of marginal stability the wall shear will vary by a small amount, so that

\[ \lambda = \lambda_1 + h^2 \lambda_2 + \ldots. \] (3.3)

Substituting the above expansions (3.2), (3.3) (and (3.1)) into the lower-deck equations (1.7), we obtain at successive orders

\[
\begin{align*}
U_{1x} + V_{1y} &= 0 \\
U_{2x} + V_{2y} + U_{1x} &= 0 \\
U_{3x} + V_{3y} + U_{2x} &= 0
\end{align*}
\] (3.4)
while the boundary conditions are

\[
\begin{align*}
U_i &= V_i = 0 \quad \text{at} \quad y = 0 \quad \text{for} \quad i = 1, 2, 3 \\
U_i &\to \lambda_1 A_i, \quad U_2 \to \lambda_1 A_2, \quad U_3 \to \lambda_1 A_3 + \lambda_2 A_i \quad \text{as} \quad y \to \infty.
\end{align*}
\]  

\[\text{(3.6)}\]

In the upper deck we have

\[
\begin{align*}
p_{1xx} + p_{1yy} &= 0 \\
p_{2xx} + p_{2yy} + 2p_{1xx} &= 0 \\
p_{3xx} + p_{3yy} + 2p_{2xx} + p_{1xx} &= 0
\end{align*}
\]  

\[\text{(3.7)}\]

with the boundary conditions

\[
\begin{align*}
p_1 &= P_1 + \sigma(1 - \rho)A_i - y A_{1xx}, \quad p_{1y} = p A_{1xx}, \\
p_2 &= P_2 + \sigma(1 - \rho)A_2 - y A_{2xx} - 2y A_{1xx}, \quad p_{2y} = p A_{2xx} + 2p A_{1xx} \\
p_3 &= P_3 + \sigma(1 - \rho)A_3 - y A_{3xx} - 2y A_{2xx} - y A_{1xx}, \\
p_{3y} &= p A_{3xx} + 2p A_{2xx} + p A_{1xx}
\end{align*}
\]  

\[\text{(3.8)}\]

at \( y = 0 \); also \( |p_1| < \epsilon \) as \( y \to \infty \).

The solution \( S_1 \) of the first-order system is given by the linear theory of the previous section, only now the undetermined amplitude \( B_1(X,T) \) is a function of the slow variables. The forcing terms in the
second-order system then require

$$S_2 = \sum_{n=2}^{\infty} S_{2n} E^n \quad \text{with} \quad S_{1,-n} = S_{2n}^{(c)}.$$  \hspace{1cm} (3.9)

By substituting (3.9) into (3.4-8) and equating powers of $E$, we obtain the controlling equations for each $S_{2n}$. First

$$U_{22 \xi \xi \xi} - 2 \xi U_{22 \xi} = \frac{1}{\Delta} \frac{d}{d \xi} \left( \tilde{\nu}, \tilde{U}_{\xi}, \tilde{U}_{\xi} \right)$$  \hspace{1cm} (3.10)

where $\xi$ is given by (2.5) again, only $\xi_0$ and $\Delta$ are evaluated at marginal stability. The bounded solution of (3.10) is

$$\frac{d U_{22}}{d \xi} = B_{22} A_i(\xi) + \frac{i \alpha B^2}{\Delta^2} \left\{ A_i'((\xi)) \int_{\xi_0}^{\xi} A_i(q) \, dq \right.$$  

$$+ A_i(\xi) \int_{\xi_0}^{\xi} \frac{dq_1}{A_i''(q_1)} \int_{q_1}^{q_2} A_i(q_2) R(q_2) \, dq_2 \right\}$$  \hspace{1cm} (3.11)

where

$$R(\xi) = - 2^{-2} \left[ 2 A_i(\xi) A_i''(\xi) + A_i'(\xi) A_i'(\xi) \right]$$

and $\xi = 2^{1/3} \xi$, $\xi_0 = 2^{1/3} \xi_0$. The amplitude function $B_{22}$ depends on the slow variables, as do all similarly named functions below. The boundary conditions (3.6) then give $P_{22}, A_{22}$ in terms of $B_1, B_{22}$, for

$$2i \alpha, P_{22} = \Delta^2 \left( U_{22 \xi \xi \xi} \right)_0, \quad \lambda, A_{22} = \int_{\xi_0}^{\infty} U_{22 \xi} \, d\xi \quad .$$  \hspace{1cm} (3.12)

The upper-deck equations determine the relation between $P_{22}$ and $A_{22}$:

$$P_{22} = \left( 2 \rho \alpha, - \sigma (1 - \rho) - 4 \gamma \alpha^3 \right) A_{22} \quad .$$  \hspace{1cm} (3.13)

So $P_{22}, A_{22}, B_{22}$ can all be found in terms of $B_1$, and we see from (3.11) that $S_{22} = B_{22}^2$. Next, the solution $S_{20}$ gives the $O(h^2)$ mean-flow correction to the basic flow $S_0$. We find
\[ \Delta^2 V_{20} \xi = \lambda_i V_{10} = f(\xi), \quad V_{20} = 0 \]  

where

\[ f(\xi) = \lambda_i^\prime \Delta^2 \xi \left( A_i(\xi) \right)^{(i)} \left\{ A_i(\xi) - A_i(0) - \xi \int_0^\xi A_i(q) dq \right\} + c.c. \]

which has the solution

\[ V_{20} = 0, \quad U_{20} = \Delta^2 \int_{\xi_0}^{\xi} d\xi \int_{\xi_0}^{\xi} f(\xi) d\xi. \]

The upper-deck equations have the solution

\[ \rho_{20} = \rho_{10} + \sigma (1 - \rho) A_{20}. \]

\( \rho_{20} \) remains arbitrary since it is absent from the lower-deck equations, but \( A_{20} \) may be determined from the outer boundary condition:

\[ \lambda_i A_{20} = \Delta^2 \int_{\xi_0}^{\xi} d\xi \int_{\xi_0}^{\xi} f(\xi) d\xi. \]

Finally, at this order, the components of \( S_{21} \) satisfy

\[ \left( \frac{\partial}{\partial \xi} \right)^2 \frac{d U_{21}}{d \xi} = \Delta^2 \left( \lambda_i (\xi - \xi_2) - \Delta c_3 \right) A_i(\xi) B_{1x} \]

\[ U_{21} = \Delta^2 \left( \beta \rho_{21} + \lambda_i A_i^\prime B_{1x} \right) \quad \text{at} \quad \xi = \xi_c \]

\[ U_{21} \rightarrow \lambda_i A_{21} \quad \text{as} \quad \xi \rightarrow \infty. \]

This system is different from that found in Smith's (1979b) study of near-neutral modes, where there is no modulation on the \( O(h^{-1}) \) length scale. The differential operator acting on \( U_{21} \) in (3.16) is identical with that acting on \( \tilde{U}_{11} \xi \) (see (2.6)) so that a solution of
(3.16) can only be found if a certain solvability condition is satisfied. But this condition is identically satisfied since \( c_9 = 2c_1/3 \), confirming that the disturbance simply propagates with the group velocity and without change of form on the (intermediate-) slow \( O(h^{-1}) \) time scale. In fact, the solution may conveniently be expressed as

\[
\frac{\partial U_{21}}{\partial \xi} = B_{21} A_i(\xi) + \frac{\lambda_i}{3\Delta^3} \left( A_i(\xi) + (\xi - \xi_o) A_i(\xi) \right) B_{1x}. 
\]

Then the boundary conditions give

\[
\lambda_i A_{21} = \kappa B_{21}, \quad \Delta P_{21} = \lambda_i B_{21} A_{i_0}' - \frac{\lambda_i^2}{3\Delta^3} B_{1x} A_{i_0}'. 
\]

In the upper deck we obtain the solution

\[
P_{21} = b_{21} e^{-\alpha_i y} + i\alpha_i \rho \kappa \lambda_i' B_{1x} y e^{-\alpha_i y}; 
\]

the boundary conditions at \( y = 0 \) yield the two relations

\[
b_{21} = \rho \kappa A_{1x} + i\rho \lambda_i' \kappa B_{1x}, \quad \sigma(1 - \rho) A_{21} + \gamma \alpha_i^2 A_{21} - 2i\alpha_i y \lambda_i' \kappa B_{1x}.
\]

The complete solution \( S_{21} \) can thus be found from (3.17) and (3.19) in terms of \( B_{21} \) and \( B_{1x} \). The amplitude function \( B_{21} \) remains undetermined at this order, however.

The evolution equation governing \( B_1(X,T) \) is found by considering the third-order system for \( S_3 \). The forcing terms in (3.5) dictate that the solution has the form

\[
S_3 = \sum_{n=-3}^{3} S_{3n} E^n \quad \text{with} \quad S_{3n} = S_{3n}^{(c)}.
\]

We need only consider the terms proportional to \( E \). The lower-deck equations are now
\[
\Delta^3 \left( U_{31,\xi} - \xi U_{31} \right) - \lambda_i V_{31} - i\alpha_i P_{31} = G(\xi, X, T) \tag{3.21a}
\]

\[
i\alpha_i U_{31} + \Delta V_{31,\xi} + U_{21,\xi} = 0 \tag{3.21b}
\]

\[
U_{31} = V_{31} = 0 \text{ at } \xi = \xi_0, \quad U_{31} \to \lambda_i A_{31} + \lambda_2 \tilde{A}_i \text{ as } \xi \to \infty \tag{3.21c}
\]

where the function \( G(\xi, X, T) \) contains all the forcing terms proportional to \( E \) in the \( x \)-momentum equation:

\[
G(\xi, X, T) = P_{21,\xi} + (\lambda_i Y - c_i) U_{21,\xi} + \tilde{U}_{1T} + i\alpha_i \lambda_i Y \tilde{U}_i + \lambda_2 \tilde{V}_i + i\alpha_i \tilde{U}_i^{(c)} U_{22} + i\alpha_i \tilde{U}_i U_{20} + \tilde{V}_i U_{10Y} + \tilde{V}_i^{(c)} U_{21Y} + \tilde{V}_{12} \tilde{U}_i^{(c)}. \tag{3.22}
\]

The upper-deck equations may be solved to give the following relation between \( P_{31} \) and \( A_{31} \):

\[
P_{31} = \left( \rho \alpha_i - \sigma(1 - \rho) - \gamma \alpha_i^3 \right) A_{31} - \frac{i\alpha_i K}{\lambda_i} B_{21,\xi} + \frac{\gamma K}{\lambda_i} \left( B_{11,\xi} + 2i\alpha_i B_{21,\xi} \right). \tag{3.21d}
\]

Differentiation of (3.21a) with respect to \( \xi \), and use of (3.21b) yields

\[
\left( \frac{\partial^2}{\partial \xi^2} - \xi \right) \frac{dU_{31}}{d\xi} = \Delta^2 \frac{dG}{d\xi} - \Delta^3 \lambda_i U_{21,\xi}. \tag{3.23}
\]

Again, a solution of (3.23) only exists if a certain solvability condition is satisfied. This is found by multiplying (3.23) by \( A_i(\xi)(A_i L_i)/L_0 \) and integrating with respect to \( \xi \) from \( \xi_0 \) to \( \xi \); cf. Smith (1979b). Here \( L(\xi) \) is the unique function satisfying \( L'' = \xi L = 1 \), and \( L_0 = L(\infty) = 0 \); specifically

\[
L(\xi) = A_i(\xi) \int_{\xi_0}^{\xi} \frac{dq_i}{A_i^2(q_i)} \int_{q_i}^{q_1} A_i(q_2) dq_2.
\]

In this way, we obtain

\[
\int_{\xi_0}^{\infty} \left( A_i(\xi) - \frac{A_i(\xi)}{L_0} L(\xi) \right) \left( \frac{dG}{d\xi} - \frac{\lambda_i}{\Delta} U_{21,\xi} \right) d\xi
\]

\[
= -A_i \left( i\alpha_i P_{31} + P_{21,\xi} \right) - \frac{\Delta^2 A_i}{L_0} \left( \lambda_i A_{31} + \lambda_2 \tilde{A}_i \right). \tag{3.24}
\]
We may eliminate $P_{31}$ and $A_{31}$ from (3.24) using (3.21d) and the dispersion relation (2.10). Moreover, terms involving $B_{21}$ also cancel. So (3.24) leads to the nonlinear differential equation

\[ B_{1t} + a_1 B_{1xx} = \alpha_2 B_{1} + \beta B_{1} |B_{1}|^2 \]  

(3.25)

for the (leading-order) amplitude $B_{1}$ of the disturbance. The coefficients in (3.25) are given by the expressions

\[ a_1 = \frac{\lambda_1^2}{3 \lambda_i} \left( \frac{2}{3} \xi_0 A_{i_0} \left( \xi_0 + \frac{A_{i_0}'}{\kappa} \right) \right) - \frac{i \lambda_i \kappa y}{\lambda_1} \]  

(3.26a)

\[ a_2 = \frac{\lambda_1 \lambda_i^2}{\lambda_1} \left\{ \frac{2}{3} \xi_0 A_{i_0} \left( \xi_0 + \frac{A_{i_0}' \xi}{\kappa} \right) - \frac{5}{3} A_{i_0} \right\} \]  

(3.26b)

\[ \beta = \frac{\int_{\xi_c}^{\infty} \left( A_i(\xi) - \frac{A_{i_0}'}{\xi_c} L(\xi) \right) \frac{dQ}{d\xi} d\xi}{A_{i_0} \left( \xi_c + \frac{A_{i_0}'}{\kappa} \right)} \]  

(3.26c)

where

\[ B_{1} |B_{1}|^2 Q(\xi) = i \alpha_1 \tilde{U}_{11} U_{22} + i \alpha_1 \tilde{U}_{11} U_{22} + \tilde{V}_{11} U_{220} + \tilde{V}_{i} U_{220} + \tilde{V}_{i} U_{220} + V_{22} \tilde{U}_{11} U_{220} \]

The amplitude of the disturbance is therefore governed by the so called Ginzberg-Landau equation. The coefficients $a_1$, $a_2$ of the linear terms in (3.25) can be found directly from the dispersion relation (2.10), in fact. The sign of $a_{2r}$ (where subscripts $r,i$ refer to the real and imaginary parts) - see table 1 - confirms that the flow becomes linearly unstable when $\lambda_2$ becomes negative. The interaction
between the growing modes is contained in the cubic term. The subsequent evolution of $B_i$ according to (3.25) depends crucially on the values of the coefficients (3.26). It is worthwhile mentioning briefly the variety of solutions that have been found for (3.25) in some recent analytical and numerical studies. Steady, time-periodic (or Stokes wave) solutions can be found, with

$$B_i = C e^{i(\alpha_c X - \omega_c T)} \quad (3.27a)$$

provided that the amplitude $C$, wavenumber $\alpha_c$ and frequency $\omega_c$ are real and satisfy the relations

$$\omega_c + a_{i+i} \alpha_c^2 = -a_{i+i} - \beta_i C^2 \quad (3.27b)$$

$$a_{i+r} \alpha_c^2 = -a_{i+r} - \beta_r C^2.$$

The solution (3.27) therefore gives a periodic modulation of the fundamental plane wave solution (2.2). A linear stability analysis of this steady solution shows it to be unstable to side-band instabilities (Stuart & DiPrima 1978) in some circumstances. Keefe (1985) and Moon, Huerre & Redekopp (1983) address the full nonlinear evolution of an initial Stokes wave, and find a rich variety of solutions which exhibit limit cycles, period doubling, two-tori, three-tori, frequency-locked two-tori, and chaos in phase space, depending on the wavenumber of the initial disturbance; such behaviour can occur when $a_{i+r} < 0$, $\beta_r < 0$. The limit $a_{i+i}, \beta_i \to 0$ gives the cubic Schrodinger equation, which also has chaotic solutions: see, for example, Nozaki & Bekki (1983) who investigate the evolution of a localised disturbance governed by a slightly perturbed cubic Schrodinger equation. Smith (1986) shows that when the disturbance is controlled by an unperturbed cubic Schrodinger equation its amplitude continues to
grow exponentially (as in the linear growth) and, moreover, an initially localised disturbance spreads exponentially. He finds no evidence of chaos, however. When $\beta_r > 0$ the form of the equation suggests that the growth of an initially small, linearly unstable, disturbance will be enhanced by the nonlinear interaction. Hocking & Stewartson (1972) show that a finite-time breakdown of (3.25) may occur in that case, depending on the relative values of $a_{11}/a_{1r}$ and $\beta_1/\beta_r$, but not on the initial conditions. They find that two distinct types, 1 and 2 (see below), of breakdown structure are possible, and give numerical evidence in support of the existence of both.

In the present case, we may work in terms of the starred variables defined in (1.11a,b), so that the values of the coefficients (3.26) depend on only one parameter, to wit, $S$. The conditions (2.11a-c) at marginal stability then fix $(\alpha_{*1}, \omega_{*1}, \lambda_{*1})$ where

$$\alpha_{*1} = \frac{\chi}{\rho} \alpha, \quad \omega_{*1} = \frac{\chi}{\rho^2} \omega, \quad \lambda_{*1} = \frac{\chi}{\rho^2} \lambda,$$

(cf. (1.11)). The differential equations determining Airy's function and all the other functions needed in the evaluation of $\beta$ were solved numerically by us. The equations were central-differenced on a uniform grid in the real variable $\zeta = i^{-1/3}(\xi - \xi_0)$, and the discretised systems were solved as two-point boundary-value problems by Gaussian elimination. The trapezium rule was used to evaluate the various integrals involved. The results are given in table 1 for the full range of values of $S$ that yield marginal stability. Different grid widths and integration ranges were used to check the accuracy of our results, and as such we believe them to be accurate to at least three significant figures. In addition, checks were made in certain special cases against independent calculations, and the agreement was found to be good.
We observe first that $\beta_r > 0$, $a_{1r} < 0$ throughout the range of $S$, so there is a similarity in that respect between the present flow and plane Poiseuille flow (Stewartson & Stewart 1971). The equation is therefore of the type studied by Hocking & Stewartson (1972), and since also $\beta_i < 0$, $a_{1i} > 0$ three distinct types of behaviour can occur. Summarising Hocking & Stewartson (and using their terminology), we note that if

$$\frac{|\beta_i|}{\beta_r} < -\frac{2a_{ii}}{|a_{ir}|} + \sqrt{3 + \frac{4a_{ii}^2}{a_{ir}^2}}$$

(3.28a)

then a finite-time breakdown (or bursting) of type 1 occurs (at $\tau = \tau_0$, say); that is to say, $|B_1| \to \infty$ as $\tau \to \tau_0$, and there is a corresponding focussing on a short length scale of order $[(\tau-\tau_0)\ln(\tau_0-\tau)]^{-1/2}$. If

$$-\frac{2a_{ii}}{|a_{ir}|} + \sqrt{3 + \frac{4a_{ii}^2}{a_{ir}^2}} < \frac{|\beta_i|}{\beta_r} < \frac{a_{ii}}{|a_{ir}|} + \frac{a_{ii}^2}{a_{ir}^2}$$

(3.28b)

then a finite-time breakdown of type 2 occurs, with the length scale of the disturbance decreasing like $(\tau_0-\tau)^{-1/2}$. In this case, the amplitude function has the similarity form

$$B_i \sim \frac{f(x^{2/3/2}(\tau_0-\tau))}{[2(\tau_0-\tau)]^{1/4}} \quad \text{as} \quad \tau \to \tau_0^-$$

where $\nu$ and $q$ are both real constants (and $q > 0$) which depend on $a_{1i}/|a_{ir}|$ and $|\beta_i|/\beta_r$. Finally, if

$$\frac{|\beta_i|}{\beta_r} > \frac{a_{ii}}{|a_{ir}|} + \frac{a_{ii}^2}{a_{ir}^2}$$

(3.28c)
then bursting does not occur; instead the amplitude remains finite for all time. Hocking & Stewartson show that, in this case, quasi-steady solutions can be found, but conjecture that they are unstable since they could not be obtained by a numerical integration of (3.25). Indeed, the numerical solutions have an apparently random behaviour.

The calculated values of \( a_1 \) and \( \beta \) show that the type 2 breakdown will occur in the present flow for \( 0 < S < S_c \), and that (3.28c) holds (giving bounded solutions) for \( S_c < S < \frac{1}{2} \), where (from the calculations) \( S_c \approx 0.24 \). So the magnitude of the surface tension, or of the density difference between the two fluids, or of the external 'wind' speed \( U^* \), can affect the ultimate behaviour of the instability quite substantially.
Summary

The analysis of sections 2 and 3 shows that the inclusion of both surface tension and density stratification in unsteady two-fluid flows can have a dramatic effect on its stability properties. On the whole, the two effects combine to stabilise Tollmien-Schlichting waves through part of the flow. Nevertheless instabilities can still arise (if (1.12) is satisfied) and initially an unstable disturbance has weak nonlinear growth and weak modulation. The subsequent development of the disturbance often gives rapid nonlinear growth leading to bursting and, ultimately, a breakdown of the initial structure, although, again, the existence of bursts depends on the relative fluid densities, the interfacial tension and the external fluid velocity, and need not occur. But in any event an ultimate steady-state finite-amplitude wave motion is not attained in the present regime. This is in marked contrast to Blennerhassett's (1980) general conclusion that nonlinear modal interactions have a stabilising effect in two-fluid systems (although Keefe (1985) suggests that the evolution equations derived by Blennerhassett may also have chaotic solutions). However, Blennerhassett restricted attention to confined two-fluid flows (between parallel plates in relative streamwise motion) and it may be that the confinement prevents bursting.

To place the present work in a broader context, we note that a similar analysis can be given when the interface lies in the upper deck or lower deck of the triple-deck structure. A brief investigation of the linear stability of two-fluid systems in these two cases has been carried out by us, in fact. The algebra is considerably more awkward than the work presented above (especially when the interface lies in the lower deck), and analytical progress has only been
made in certain limiting cases. For example, in the limit when the interface moves out of the upper deck three neutral modes exist, two of which correspond to the inviscid Kelvin-Helmholtz neutral modes, and the third corresponds to the single viscous Tollmien-Schlichting mode of the lower fluid. A comprehensive study of the linear stability of a Couette flow of a two-fluid system bounded by a wall, which is similar to the present boundary-layer flow with the interface in the lower deck, has been made by Hooper (1986). Also, other instabilities may exist in the present flow, corresponding, say, to the upper branch of the neutral stability curve of the one-fluid flow, although the lower-branch instabilities studied here seem to be more important since these are encountered 'first' (i.e. furthest upstream) in a developing boundary-layer flow. Finally, and most important from a practical point of view, three dimensionality is almost certain to have an overriding effect on the stability of such flows, and this should be taken into account eventually.
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<td>0.568</td>
<td>2.087</td>
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<td>2.087</td>
<td>1.567</td>
<td>2.122</td>
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Table 1. Coefficients appearing in (3.25) for $0 < S < k$. $\lambda = \frac{\lambda_2 \cdot \beta}{\beta}$. $\lambda_2 = -\frac{\lambda_3 \cdot \alpha_1}{\alpha_1^2} \cdot (1.133 - 0.02134i)$. $a_i = \frac{\alpha_i}{\alpha_1 \cdot \beta / \beta}$. $\beta = \frac{-\alpha_2}{\alpha_1}$.
Captions

Figure 1. Schematic diagram of a two-fluid system flowing over a horizontal wall with the interface lying within the combined boundary layer.

Figure 2. Indicating the size of Tollmien-Schlichting waves of non-dimensional length $O(1)$ propagating in the two-fluid system, and the relative thickness of the boundary layer and of the viscous wall layer.

Figure 3. (a) Sketch of the typical behaviour of the growth rate $\text{Im}(\omega)$ as a function of the wavenumber $\alpha$ of a small disturbance according to the dispersion relation (2.10) (with $\lambda$ fixed). Note that if $\gamma = 0$ then there is always an infinite range of unstable modes; $\gamma \neq 0$, $S > \frac{1}{4}$, no unstable modes; $\gamma \neq 0$, $S < \frac{1}{4}$, at most a finite band of unstable modes. (b) Neutral modes as a function of $\lambda$. Two typical cases are shown, for $S > \frac{1}{4}$, $S < \frac{1}{4}$. The point marked $x$ corresponds to marginal stability. (The behaviour of the one-fluid system is shown by the dotted line.)
FIGURE 3(a)

FIGURE 3(b)
PART III

Channel flow.
CHAPTER SEVEN

High-Reynolds-number flow
in an asymmetric branching channel.
CHAPTER SEVEN

Branching Channel Flow.

Our concern is with the laminar high-Reynolds-number flow in an asymmetrically blocked or bifurcating channel (see figure 1), it being supposed that the fluid is incompressible and the motion two-dimensional, steady and fully-developed upstream, and in particular we are concerned with the possible onset of separation. Previous related studies include symmetric branching (Smith 1977b), symmetrically and asymmetrically constricted channels (Smith 1976a), entry-flow effects (Smith 1976b), the symmetric merging of two channels (Badr et al 1985, Bates 1978), and three-dimensional effects in some of the above (Bennett 1986, Smith 1976c). In all these studies, and also in the present work, the distortions (e.g. constrictions or blockages) are taken to be just large enough to provoke regular boundary-layer separation by means of a viscous-inviscid interaction. The "correctness" of this idea in describing the large-Reynolds-number limit of steady internal flows tends to be borne out by full Navier-Stokes calculations; the agreement between theory and computations is often gratifyingly close, as for example in Badr et al (1985), Dennis & Smith (1980).

The approach here is to suppose that the flow can be analysed in terms of three distinct regions comprising a core flow (region I), where the perturbations to the oncoming Poiseuille flow are relatively small and are governed mainly by inviscid dynamics, and two thin viscous wall layers. The core flow suffers a simple lateral displacement of the relative order of the thickness of the wall layers, and the resulting curvature of the streamlines in the core causes a
pressure gradient across the channel. Thus the two pressures driving the wall layers can interact, and they do so on a long $O(R^{1/7})$ length scale. Moreover, the wall layers respond nonlinearly (so that separation is a possibility) when the disturbance in the core, to the streamwise velocity component, say, is $O(R^{-2/7})$. So we are led to consider a bifurcation described by the shape

$$
y = \begin{cases} 
    c + \varepsilon^2 h S(X) & \text{upper surface} \\
    c - \varepsilon^2 h T(X) & \text{lower surface}
\end{cases}
$$

in $X > 0$, where $\varepsilon = R^{-1/7} \ll 1$, $X = \varepsilon x = O(1)$, $S(0) = T(0) = 0$, and the scale factor $h = O(1)$ for now. The core flow, region I, is then given by

$$
\begin{align*}
    u &= U_B(y) + \varepsilon^2 A(X) U_B'(y) + \ldots \quad (2a) \\
    \psi &= \psi_B(y) + \varepsilon^2 A(X) U_B(y) + \ldots \quad (2b) \\
    p &= \varepsilon^4 \bar{P}(X,y) + \ldots \quad (2c)
\end{align*}
$$

from the $x$-momentum and continuity equations; here the basic Poiseuille flow is given by

$$
U_B(y) = 6(y - y^2), \quad \psi_B(y) = 3y^2 - 2y^3 \quad (0 < y < 1) \quad (3)
$$

so that the nondimensional flow rate is 1 in the channel of width 1. Also in (2) the $O(\varepsilon^2)$ displacement of the streamlines $-A(X)$ is unknown at this stage. The condition of tangential flow along the surface of the flow divider requires

$$
\psi = \psi_B(c) + \varepsilon^2 K U_B(c) + \ldots \quad (4)
$$

there, for some unknown constant $K$, so that

$$
\begin{align*}
    A^+(X) &= K - h S(X) \quad \text{in } X > 0, \quad c < y < 1 \quad (5a) \\
    A^-(X) &= K + h T(X) \quad \text{in } X > 0, \quad 0 < y < c \quad . \quad (5b)
\end{align*}
$$

The superscripts $\pm$ have been introduced to distinguish the displacement...
ments in the upper and lower downstream channels in \( X > 0 \). The constant \( K \) is therefore one of the principle unknowns of the problem. The solution (2a,b) with (5a,b) implies the existence of a developing Blasius-type boundary layer of thickness \( O(\varepsilon^2) \) on the flow divider, which responds passively to the core flow provided that there is no flow separation from the divider (e.g. due to sharp edges). Upstream of the bifurcation, \( A(X) \) is a single-valued function of \( X \). From the \( y \)-momentum equation the expansions (2) give

\[
\tilde{P}(X, y) = P(X) + A''(X) \int_0^y U_B(y') dy' \quad (6)
\]

where \( P(X) \) is the unknown pressure on the lower wall. If \( \tilde{P}(X) \) is the pressure at the upper wall, then (6) gives

\[
\tilde{P}(X) = P(X) + qA''(X) \quad \text{in} \quad X < 0 \quad (7)
\]

where the momentum flux across the channel \( q = \int U \tilde{g}(y) dy \).

In region II, the viscous layer adjacent to the lower wall, the development of the solution is given by

\[
(u, \psi, p) = (\varepsilon^2 U(X, Y), \varepsilon^4 \psi(X, Y), \varepsilon^4 P(X)) + \ldots \quad (8a, b, c)
\]

where \( Y \neq \varepsilon^{-2} y = O(1) \), in view of the core-flow properties. The \( y \)-momentum equation and the match with I has been used to obtain (8c). The \( x \)-momentum and continuity equations now give

\[
UU_X - \psi_X U_Y = - P'(X) + U_{YY}, \quad U = \psi \quad (9a, b)
\]

while the no-slip condition at the wall and the match with I require

\[
U = \psi = 0 \quad \text{on} \quad Y = 0, \quad (9c)
\]

\[
U \sim 6(Y + A) \quad \text{as} \quad Y \to - \infty \quad (9d)
\]

in which \( A = A^- \) in \( X > 0 \). Similarly, in the upper viscous layer, region III, we expand
\[(u, 1-\psi, p) = (\varepsilon^2 \tilde{u}(X, \tilde{y}), \varepsilon^4 \tilde{\psi}(X, \tilde{y}), \varepsilon^4 \tilde{p}(X)) + \ldots \quad (10a, b, c)\]

where \(\tilde{y} = \varepsilon^{-2} (1-y) = O(1)\), and we obtain the system

\[
\begin{align*}
\tilde{u}_x - \tilde{y} \tilde{u}_y &= - \tilde{p}'(X) + \tilde{u}_{yy} , \quad \tilde{u} = \tilde{\psi}_y , \quad (11a, b) \\
\tilde{U} = \tilde{\psi} &= 0 \quad \text{on } \tilde{y} = 0 , \quad (11c) \\
\tilde{U} &\sim 6(\tilde{y} - A) \quad \text{as } \tilde{y} \to 0 , \quad (11d)
\end{align*}
\]

in which \(A = A^+ \) in \(X > 0\).

The coupled system (5), (7), (9), (11) (supplemented by suitable up- and downstream conditions) determines the leading-order flow perturbation, and it can admit separated-flow solutions in view of the nonlinearity and the unknown pressure gradients in (9a) and (11a). The system requires a numerical treatment for general \(O(1)\) values of the parameter \(h\), but analytical progress can be made if \(h < 1\), when linearisation is possible. Although the linear theory strictly cannot give separated-flow solutions, it can often predict the main features of the full nonlinear problem for quite a range of values of \(h\) (possibly even beyond separation, in practice), as we have seen in chapter 2 in the context of liquid-layer flows.

Accordingly, we suppose \(h < 1\), and set

\[U = 6Y + h\tilde{U} , \quad \psi = 3Y^2 + h\tilde{\psi} , \quad P' = h\tilde{G} , \quad A = hA \quad (12)\]

in (9) (and similarly in (11)). The solution of the linearised system can be found for the Fourier transforms in terms of Airy's function, in the usual way, and we obtain the relations

\[
\begin{align*}
\tilde{G}^* + \tilde{G} &= -\theta (i\alpha)^2 \left(\tilde{A}^* + \tilde{A}^\dagger*\right) \quad (13a) \\
\tilde{G}^* + \tilde{G} &= \theta (i\alpha)^2 \left(\tilde{A}^* + \tilde{A}^\dagger*\right) \quad (13b)
\end{align*}
\]

with \(\theta = -6^\circ / 3 \text{Ai}'(0)/2\), and where, for a general function \(f(X)\), the Fourier transform is defined
\( f^*(\alpha) = f^+_*(\alpha) + f^-*(\alpha) \), with
\[ f^+_*(\alpha) = \int_{-\infty}^{0} f(x)e^{-i\alpha x} \, dx \quad , \quad f^-*(\alpha) = \int_{0}^{\infty} f(x)e^{-i\alpha x} \, dx \ . \]

The subscripts +/- therefore label functions which are analytic in the upper/lower half of the complex \( \alpha \)-plane. Also in (13), the function \( (i\alpha)^n \) has a branch cut along the positive imaginary axis of the \( \alpha \)-plane, and its argument lies in \( (-n\pi,n\pi) \). The relations (5) and (6) yield
\[ A^+* = \frac{-K}{i\alpha} - S^-* \quad , \quad A^-* = \frac{-K}{i\alpha} + T^-* \quad \quad \text{(13c,d)} \]
\[ G^+_* = \frac{-\alpha}{\alpha} + q(i\alpha)^3A^+_* \quad \quad \text{(13e)} \]
in which \( K = hK \). If we also define \( \beta(X) = \tilde{G}(X) - \tilde{G}(X) \) in \( X > 0 \), \( \beta(X) = 0 \) in \( X < 0 \), then
\[ \beta^-* = G^-* - G^-* \quad \quad \text{(13f)} \]

Equations (13a-f) combine to give
\[ (i\alpha)^3(2\theta - q(i\alpha)^3)A^+_* = \beta^- - \frac{2\theta K}{(i\alpha)^{1/3}} + \theta(i\alpha)^3(S^-* - T^-*) \quad \quad \text{(14)} \]
for the determination of \( A^+_* \) and \( \beta^+_* \) by the Weiner-Hopf technique. Writing \( \gamma = (2\theta/q)^{3/7} = (-180\text{Ai}'(0))^{3/7}/6^{1/7} \approx 4.016 > 0 \), we see that the function
\[ \gamma^3 - (i\alpha)^3 \]
which appears in (14) has only one zero in the \( \alpha \)-plane satisfying \( |\text{arg}(i\alpha)^{7/3}| < 7\pi/3 \), and that is at \( \alpha = -i\gamma \). So we may write (14) in the form
\[ q(\gamma - i\alpha)A^+_* = D_+ (\alpha) \left[ \beta^-* - \frac{2\theta K}{(i\alpha)^{1/3}} + \theta(i\alpha)^3(S^-* - T^-*) \right] \quad \quad \text{(15)} \]

where
and $D_\alpha$ is analytic in $\text{Im}(\alpha) < 0$. If we now suppose that $\overline{A_{+}}^*$ is analytic in $\text{Im}(\alpha) > -i\gamma$ (which may be justified a posteriori) then both sides of (15) are analytic in the strip $-i\gamma < \text{Im}(\alpha) < 0$, and so, by analytic continuation, are equal to a polynomial in $\alpha$. Now, continuity of velocity across $X = 0$ in the core requires

$$\overline{A} \to \overline{K} \text{ as } X \to 0^-$$

from (2) and (5), so that

$$\overline{A_+}^* \sim -\frac{\overline{K}}{i\alpha} \text{ as } |\alpha| \to \infty \text{ in } 0 > \text{Im}(\alpha) > -i\gamma.$$  

Then both sides of (15) must be equal to the constant $q\overline{K}$. Inverting, we find

$$\overline{A}(X) = \overline{K}e^{\gamma X} \text{ in } X < 0. \quad (16)$$

The pressures on the two outer walls may now be obtained from (13a,b). At the lower wall

$$\overline{G}(X) = \begin{cases} -\theta\overline{K}\frac{X}{27\pi}e^{\gamma X} & \text{in } X < 0, \\ \frac{-\sqrt{3}\theta}{2\pi} \left[ \overline{K} \frac{X}{t^{1/3}(\gamma + t)} \right. & + \left. \Gamma\left(\frac{4}{3}\right) \frac{X}{(\xi - t)^{2/3}} \right] & \text{in } X > 0; \end{cases} \quad (17)$$

a similar expression holds at the upper wall for $\overline{G}(X)$ with $\theta$ replaced by $-\theta$, and $T$ by $-S$. So if the downstream channels are ultimately parallel sided, with $T(X) \to T_\infty$, $S(X) \to S_\infty$ reasonably fast as $X \to \infty$, say, then (17) (and its equivalent for the upper-wall pressure gradient) gives

$$\overline{G}(X) \sim \frac{\sqrt{3}\theta\Gamma(2/3)}{2\pi X^{2/3}} (\overline{K} + T_\infty) \quad (18a)$$

$$\overline{G}(X) \sim \frac{\sqrt{3}\theta\Gamma(2/3)}{2\pi X^{2/3}} (\overline{K} - S_\infty) \quad (18b)$$
The pressure difference between the two channels remains bounded and tends to zero far downstream on the present length scale if

$$\bar{K} = \frac{S_w - T_w}{2}. \quad (19)$$

Thus $\bar{K}$, which gives the upstream level of the dividing streamline, therefore depends on, and indeed is equal to, the ultimate level of the centre-line of the flow divider, and is independent of its detailed shape (provided separation does not take place on the divider, e.g. due to sharp corners). In particular, a divider that is ultimately symmetric about $y = c$ gives no upstream influence (since $\bar{K} = 0$ then), on the $O(R^{1/7})$ length scale at least. An important implication of this result is that even when $h = O(1)$ no significant upstream response will be induced (and therefore there is no possibility of upstream separation) if the flow divider is symmetrical, despite the asymmetry of the the geometry. We observe also that the downstream behaviour (18a,b) with (19) implies that the pressure falls rapidly as $X \to \infty$, except in the special case when $S_w = -T_w$, reflecting the fact that the fluid must speed up slightly to compensate for the slight decrease in the total cross-sectional area of the channel.

We may also find the $O(h)$ perturbations to the shear at the outer walls. If $h_T = (u_y - U_b)$ at $y = 0$ (or 1) then on the lower wall

$$\bar{\tau}(X) = \begin{cases} 
3.6^4 \text{Ai}(0) \bar{K}_y^2 e^{\gamma X} & \text{in } X < 0 , \\
\frac{3\sqrt{3}}{2\pi} \left[ \bar{K}_y \frac{e^{-t/3}}{t^{2/3}(\gamma + t)} + \Gamma\left(\frac{1}{3}\right) \int_0^X \frac{T'((\xi)d\xi}{(X-\xi)^{1/3}} \right] & \text{in } X > 0
\end{cases} \quad (20)$$

and similarly at the upper wall, with $T$ replaced by $-S$. An asymmetric divider with $S_w > T_w$ (i.e. $\bar{K} > 0$) then causes the flow to become more attached at the lower wall upstream of the divider, and less attached on the upper wall (the situation being reversed if
Sm < T), which is in line with one's physical expectations since the core flow is displaced downwards (i.e. towards the lower wall) in the upstream channel when \( \tilde{K} > 0 \). We would expect the same features to emerge at higher \( O(1) \) values of \( h \), with regular separation taking place on the upper wall for sufficiently large \( h \) when \( S > T \), and the flow becoming increasingly strongly attached along the lower wall as \( h \) increases; see the sketch in figure 2. We note incidentally that the lower wall pressures and shears are independent of the precise lateral location, at \( y = c \), of the flow divider (as long as \( c = O(1) \)), and that these quantities depend only on the shape of the lower surface of the divider, and not on the shape of the upper surface apart from its ultimate level; similarly for the properties on the upper wall. Another quantity of physical interest, the pressure difference induced across the divider, does, of course, depend on both \( S(X) \) and \( T(X) \) as well as on \( c \).

Beyond the current \( O(R^{1/7}) \) length scale, the flow returns to a Poiseuille form in each downstream channel on a much longer \( O(R) \) length scale. The boundary-layer equations then hold across the entire channel, and since there is no upstream influence on this length scale the oncoming flow is given by the undisturbed Poiseuille flow (3) at \( \tilde{X} = 0 \), if \( x = R \tilde{X} \) and the bifurcation starts at \( \tilde{X} = 0 \), and moreover the two flows then develop separately without interaction.

It as a simple matter to extend the above theory to include distortions of the outer channel walls. One special case which may be of interest is when the two downstream channels are ultimately parallel sided (as above) but diverging. Another geometry of some interest is when the flow divider is close to a wall, within one of the wall layers. In that case the same flow structure holds, with the long upstream response and the two viscous wall layers. The analysis is
more awkward then, mainly because the full no-slip condition must be imposed on the flow divider; the simple Airy-function solution in the wall layers no longer holds because the disturbance is nonlinear. We find that there is a continuous adjustment of the solution to that presented above as the divider moves out of the wall layer (schematically); this is equivalent to increasing the Reynolds number, in fact. Also, as the divider moves closer still to the wall, to within a distance \( d R^{-2/7} \), say, with \( d \ll 1 \), some analytical progress is again possible. The flow features in the main channel are then equivalent to the flow in a slightly constricted channel, and can be obtained from linear theory, whilst the flow in the thin side-channel is controlled largely by lubrication-type properties. A nonlinear adjustment takes place on a relatively short \( O(d^3 R^{-1/7}) \) length scale in the vicinity of the mouth of the side-channel, so that separation, if it takes place, is restricted to that region.
Captions

**Figure 1.** Schematic diagram of the fully-developed flow through a branching channel, showing non-dimensionalisation and the high-Reynolds-number structure.

**Figure 2.** Likely form of the flow with separation (with $S_\infty > T_\infty > 0$) on the basis of the results of the linear theory.
FIGURE 1.

FIGURE 2.
The main conclusions of each chapter of this thesis are summarized as follows.

**Chapter 2.** (i) Separation is pushed increasingly far upstream ahead of a forward-facing step in a supercritical stream as the step height increases, and the free interaction structure emerges there. (ii) The downstream separation of a subcritical flow over a hump takes the form of a removable Goldstein singularity in the limit as the hump-height parameter becomes asymptotically large. The length of the downstream reversed-flow region and the process of reattachment remain undetermined as yet in that same limit, although the two-parameter problem associated with shorter humps provides an alternative approach there.

**Chapter 3.** The main features of the stationary hydraulic jump observed experimentally, and in particular its initial shape, compare favourably with the predictions of viscous-inviscid interaction theory.

**Chapter 4.** An apparently self-consistent description of the large-scale separation of a non-uniform liquid-layer flow over bottom topography is given in terms of Kirchhoff's free-streamline theory. The theoretical predictions are shown to be not inconsistent with experimental observations.

**Chapter 5.** Long waves on a running stream in shallow water are shown to be subject to a viscous instability, in some circumstances, which can lead to rapid nonlinear growth.
Chapter 6. In a two-fluid system with an interface near a fixed wall, the destabilising effect of small viscosity combined with the stabilising influences of surface tension and density stratification can result in weak nonlinear growth of a wave packet, leading to bursting within a finite time.

Chapter 7. The asymmetric branching of a channel provokes a long $O(R^{1/7})$ upstream and $O(R)$ downstream response. The wall pressure and shear-rate distributions are calculated (assuming the disturbance is relatively small) and it is shown that the major upstream response depends only on the ultimate downstream levels of the bifurcation walls.
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