Effective degrees of freedom & RFT resel counts

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Summary
In one of neuroimaging’s key papers, Worsley et al. (1992) introduced the concept of a resolution element (resel) as an intuitive parametrisation of the roughness-adjusted search volume underlying random field theory’s expression for corrected p-values. In another key development, Worsley and Friston (1995) showed that temporal correlation in a linear model reduced the effective degrees of freedom (eDF). Here, we illustrate a surprising connection between these quantities, which may have application to multiple comparison correction on data with elaborate statistical dependence that can make RFT over-conservative.

Theory
Resolution elements (resels)
An estimate of the (stationary) resels-per-voxel (RPV) is computed from the covariance of the spatial gradient of the normalised random fields, following Kiebel et al. (1999). In D dimensions, this can be expressed in units of full-width half-maximum (FWHM) as:

\[ \text{FWHM} = \text{RPV}^{-1/D} \]

(1)

Effective degrees of freedom (eDF or \( \nu_e \))
The sum of squares of \( \nu \) independent standard normal variables defines the Chi-square distribution on \( \nu \) degrees of freedom (DF); its mean and variance are \( \nu \) and \( 2\nu \). Now consider \( n \)-vector \( x - N(0, V) \), where covariance \( V \) has eigenvalues \( d_i \); \( x' \) is a \( d_i \)-weighted sum of unit Chi-square variables. The result is not Chi-square distributed, but its mean and variance are easily computed:

\[ E(x' e) = \sum_{i=1}^{\nu} d_i E(x_i^2(1)) = \sum_{i=1}^{\nu} d_i = \text{tr}V \]

(2)

\[ V(x' e \cdot e' x) = \sum_{i=1}^{\nu} d_i^2 = 2 \text{tr}V^2. \]

(3)

Matching these moments with a Chi-square on \( \nu \), DF scaled by \( s \), whose moments are \( s\nu \) and \( 2s^2\nu \), yields

\[ \nu = \frac{\text{tr}V^2}{4s^2}. \]

(4)

Simulations
Considering a simple 1-dimensional random process, we simulate 5000 realisations of a standard normal vector, each of which is smoothed with a 12 voxel FWHM Gaussian kernel and cropped to 100 elements to avoid edge-effects.

Results
Figure 1: Comparison of the covariance matrix estimated from smoothed data to the sampled theoretical Gaussian covariance function.

Theoretical covariance matrix corresponding to the four largest eigenvalues. They can be seen to resemble sinusoidal basis functions, so their eigenvalues should relate to the Fourier transform of the covariance function, which, according to the Wiener-Khinchin theorem, is the power spectral density.

In the field of genomics, Gao et al. (2008) also proposed an estimator of the effective number of independent tests, defining \( M_{\text{ddf}} \) such that the first \( M_{\text{ddf}} \) summed eigenvalues of the correlation matrix amount to 99.5% of the total. Here, this results in an over-estimate of eDF as 14 (whether from theoretical or estimated correlation or covariance matrices).

Figure 2: Eigenvectors of the theoretical covariance matrix corresponding to the four largest eigenvalues. Effective degrees of freedom (eDFs) from the theoretical covariance matrix, over varying applied smoothness (FWHM).

Figure 3: (a) The eigenvalues and spectral samples, showing reasonable agreement. The eDF is very close to the true resel count, and the equivalent from the spectrum is only slightly lower. (b) The continuous spectrum and its square; the ratio of the squared area to the area under the squared spectrum analytically yields eDF proportional to the expected number of resels, with scale factor near 94%.

Figure 4: Properties of the effective DF computed from the theoretical covariance matrix, over varying applied smoothness (FWHM).

Figure 5: Properties of the estimators from empirical data over varying numbers of realisations, showing the break-down in the accuracy of the effective DF as the covariance matrix becomes less precisely estimated. Here, the applied FWHM was set to 10, the vector length remained at 100, and the number of realisations varied from 1 to 100,000 logarithmically.

Discussion
In data with complicated covariance structure, such as source reconstructed EEG or MEG, it will typically not be possible to estimate the full covariance accurately. Random field theory resolves this by focusing on the (possibly nonstationary) local smoothness, which is known to be valid but conservative in the presence of longer range correlations (Taylor and Worsley, 2007). The link between RFT and eDF shown here thus motivates considering eDF as a potentially less conservative estimate of the equivalent number of independent tests. Further research will investigate the performance of this measure in Bonferroni and related correction procedures.

References