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Finite Time Stabilization of An Uncertain Chain of Integrators by Integral Sliding Mode Approach

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Abstract: In this paper, finite time stabilization of an uncertain chain of integrators is studied. The controller proposed in the paper, under some conditions guarantees the convergence of all the states at time exactly t_F and that chosen in advance. The controller is designed based on the integral sliding mode approach, which is the combination of two controls: a nominal control which is designed to obtain desired performances for the disturbance free system and a super-twisting control is designed for the disturbance compensation. The proposed controller also adjusts the chattering because of continuous control. Finally this paper presents the finite time stability proof of super-twisting algorithm by using continuously differentiable Lyapunov function. Thanks to academic example, effectiveness by the proposed method is presented with simulation results. Through the simulations performance of the proposed method is compare with an existing method.

Keywords: Integral sliding mode control, optimal control, finite time convergence, higher order sliding mode control

1. INTRODUCTION

Sliding mode control (SMC) is a powerful tool for the control of an uncertain nonlinear or linear systems (Utkin (1992), Shtessel et al. (2014), Edwards and Spurgeon (1998)). The main advantages of SMC are the finite time convergence and the robustness with respect to the uncertainty, this latter property being obtained during the sliding phase. To obtain robustness from initial time, the integral sliding mode control (ISMC) concept has been introduced (Utkin and Shi (1996), Castaños et al. (2006)). In the current paper main focus is made on the finite time stabilization of some class of systems and time of convergence is defined in advance.

For example, consider the scalar system, $\dot{\sigma} = -K\mathrm{sign}(\sigma)$ with the gain $K > 0$ and σ the sliding variable. It is easy to compute exact time of convergence, which is given as $t_F = \frac{|\sigma(0)|}{K}$. But, for the same system with uncertainty $\dot{\sigma} = -K\mathrm{sign}(\sigma) + d, \quad |d|_{\max} = d_M, \quad K > d_M$, it is just known that convergence time is bounded and bound being given by $t_F \leq \frac{|\sigma(0)|}{\eta}$ with $\eta = K - d_M > 0$. So the interesting problem for an uncertain system consist in designing controller which can stabilize the system exactly at a predefined convergence time. In this paper one considers an uncertain chain of integrators.

Finite time stabilization of an uncertain chain of integrators systems is also known as higher order sliding mode (Levant (1993), Levant (2003), Levant (2005), Kamal et al. (2016), Chalanga et al. (2016), Kamal et al. (2013)). Finite time stabilization of an uncertain chain of integrators is already reported in Laghrouche et al. (2007), Chalanga et al. (2015), Edwards and Shtessel (2014), Taleb et al. (2015). The results presented in Chalanga et al. (2015) and Edwards and Shtessel (2014) use continuous control to achieve finite time stabilization of an uncertain chain of integrators but exact convergence time is not given, only upper bound on the convergence time is discussed.

In Laghrouche et al. (2007), it has been shown that using an integral sliding mode control approach, all the components of the uncertain chain of integrator are stabilized at predefined convergence time. Recall that ISM control introduced by (Utkin and Shi (1996)) is a combination of two control strategies: nominal one which is designed to obtain desired performances for the disturbance free system and a discontinuous one designed for the disturbance compensation. This control approach has been used in Laghrouche et al. (2007) in high order sliding mode and contains discontinuous term which makes an overall discontinuous control. From application point of view, this discontinuous term can excite unmodelled dynamics and can cause failures of the actuator.

To adjusts chattering and to improve the results of Laghrouche et al. (2007), a new scheme based on ISMC approach is presented in this paper, where discontinuous part of the control is replaced by super-twisting control (STC), hence, the overall control becomes continuous. In addition a new stability proof of the super-twisting algorithm is presented based on continuously differentiable Lyapunov function. In Moreno et al. (2014) a similar Lyapunov function proof for super-twisting algorithm is already presented, but in this paper obtained gain condition is less conservative compare to (Theorem 4, Moreno et al. (2014)) and Davila et al. (2005), which induces reduced gains.

1.1 Main Contribution

In this paper, as previously mentioned, two main contributions are presented. In the first contribution, a scheme is presented based on ISMC, in which the nominal control stabilizes the disturbance free system in finite time, and this time being predefined. For the disturbance compensation super-twisting control is used, which makes overall continuous control. As second contribution, a new stability proof of super-twisting algorithm is based on a new continuously differentiable Lyapunov function presented, by this way, less conservative conditions on the control gains are obtained.

1.2 Structure of the Paper

The paper is organized as follows. Section 2 recalls previous works and states the problem. Section 3 details the main contributions of the paper. Application of the proposed control method to an academic example is discussed in Section 4. Section 5 contains simulation results and comparisons with former controller.

2. RECALLS AND PROBLEM STATEMENT

Consider the following linear system

$$\dot{x} = Ax + Bu, \tag{1}$$

with $x = [\, x_1 \; x_2 \; \cdots \; x_{n-1} \; x_n \,]^\top$ the state vector and $u \in \mathbb{R}$ the control input. The matrices A and B are defined as

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \cdots & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \end{bmatrix}_{n\times n}, \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}.$$

The following controller is able to stabilize the system (1) in a finite time t_F, this convergence time being selected in advance.

Theorem 1. Rekasius (1964). Consider the linear system (1) with (A, B) reachable. A control law u minimizing the performance criteria

$$J = \frac{1}{2}\int_0^{t_F} \left(x^\top Q x + u^2\right) dt, \qquad Q = Q^\top > 0,$$

and driving the system trajectories (1) to $x(t) = 0$ at $t = t_F$ for bounded initial condition $x(0)$, reads as

$$u = \begin{cases} -B^\top M x(t) + B^\top \delta(t) & \text{for } 0 \le t \le t_F \\ -B^\top M x(t) & \text{for } t > t_F \end{cases} \tag{2}$$

with $\delta(t)$ and M defined as

$$\dot{\delta} = -(A^\top - MBB^\top)\delta$$
$$0 = MA + A^\top M - MBB^\top M + Q,$$

with an initial condition $\delta(0)$ of $\delta(t)$ selected in order to satisfy the terminal condition $x(t_F) = 0$. ■

The above controller can fulfill the objective only when system is free from the disturbance. But, in reality, system dynamics can be affected by parameter variations and external perturbations. Then, system (1) in the presence of matched disturbance is represented as

$$\dot{x} = Ax + B(u + d(t)) \tag{3}$$

where $d(t)$ is a disturbance. Using the results of Laghrouche et al. (2007), it is possible to stabilize the system (3) in finite time in spite of perturbations.

Theorem 2. Laghrouche et al. (2007). The system (3) is stabilized in a finite time t_F in the presence of disturbance $d(t)$ and $|d(t)|_{\max} = \xi$, by a control input $u = u_n + u_{\text{SMC}}$ defined as

$$u_n = \begin{cases} -B^\top M x(t) + B^\top \delta(t) & \text{for } 0 \le t \le t_F \\ -B^\top M x(t) & \text{for } t > t_F \end{cases} \tag{4}$$

with $\delta(t)$ ($\delta(0)$ selected in order to satisfy the terminal condition $x(t_F) = 0$.) and M as

$$\dot{\delta} = -(A^\top - MBB^\top)\delta$$
$$0 = MA + A^\top M - MBB^\top M + Q$$

and

$$u_{\text{SMC}} = -(GB)^{-1} K \text{sign}(s), \quad K > GB\xi \tag{5}$$

with

$$s = G\left[x(t) - x(0) - \int_0^t (Ax + Bu_n) d\tau\right] \tag{6}$$

with $G \in \mathbb{R}^{1\times n}$ a projection matrix satisfying $GB \neq 0$. ■

Remark 1. This controller achieves the stabilization of the systems state in a predefined finite time, but the overall control is discontinuous because of u_{SMC}. Discontinuous control leads to the chattering problem in real applications. ■

Finite time stabilization of (3) using continuous control is already reported in the literature (see, for example Chalanga et al. (2015), Edwards and Shtessel (2014)), but, in these results, one cannot select the convergence time in advance, only upper bound of the convergence time is given. In the sequel, a control scheme is proposed such that it is possible to stabilize system (3) in a finite time t_F using *continuous control* under some conditions, t_F being chosen in advance and the state converging exactly at t_F. This scheme is detailed in the sequel.

3. MAIN RESULTS

To stabilize the uncertain chain of integrators (18) in finite time, the integral sliding mode approach is used. Integral sliding mode control has two parts, i.e. $u = u_1 + u_2$. The control input u_1 is designed to obtain the desired performance when system is free from the disturbance. In this work $u_1 = u_n$ as Theorem 1 which stabilize (1) in a predefined finite time t_F when there is no disturbance is acting on the system. As previously mentioned the main role of the control input u_2 is to compensate the disturbance from $t = 0$. The new approach has selected u_2 as super-twisting control, which is a continuous control. Consider the following assumptions on the perturbation $d(t)$.

Assumption 1. The derivative of the perturbation $d(t)$ is bounded i.e. $|\dot{d}(t)|_{\max} = \rho$ and $d(0) = 0$. ■

The stability proof of STC is well studied in the literature. Then the question is why we have studied again stability proof of STC? The answer of the question is discussed in the detail in the following subsection.

3.1 New Proof of Super-Twisting Algorithm

Consider the first order system with disturbance

$$\dot{x}_1 = u + d(t) \tag{7}$$

The super-twisting control reading as

$$\begin{aligned} u &= -k_1|x_1|^{\frac{1}{2}}\text{sign}(x_1) + \varsigma \\ \dot{\varsigma} &= -k_2\text{sign}(x_1), \end{aligned} \tag{8}$$

Defining the new variable $x_2 = \varsigma + d(t)$, the closed loop system written as

$$\begin{aligned} \dot{x}_1 &= -\,k_1|x_1|^{\frac{1}{2}}\text{sign}(x_1) + x_2 \\ \dot{x}_2 &= -\,k_2\text{sign}(x_1) + \dot{d}(t) \end{aligned} \tag{9}$$

The gains selection of STC decides the total control effort. If the gains are larger with respect to the disturbance then the control effort is more, and may not be good for the practical applications because of actuator limitations. Here, some of the gain selections methods are

(1) Gain condition in Davila et al. (2005)

$$\begin{aligned} &k_2 > \rho \\ &k_1 > \sqrt{\frac{2}{k_2-\rho}\frac{(k_2+\rho)(1+p)}{1-p}}, \quad 0 < p < 1. \end{aligned} \tag{10}$$

(2) Gain condition in Theorem 4, Moreno et al. (2014)

$$\begin{aligned} &k_2 > \rho \\ &k_1 > \frac{k_2+\rho}{\sqrt{k_2-\rho}} \end{aligned} \tag{11}$$

(3) Gain condition in Theorem 4.1 Kumar P et al. (2017)

$$\begin{aligned} &k_2 > \rho \\ &k_1 > 1.8\sqrt{k_2+\rho} \end{aligned} \tag{12}$$

In gain selections, normally k_2 gain is selected such that it can handle the perturbation. Before controller design, the maximum bound of the perturbations derivative is known, then, k_2 can be selected with a slightly higher value than $|\dot{d}(t)|_{\max} < \rho$. Now, the main question is, **which value of k_1 has to be selected for the given perturbations and generates less control effort?** In this paper, new gain conditions for the STC have been proposed and compared with the mentioned above conditions. It is observed that proposed gain conditions generate the less control effort. The comparison is discussed after the following Lemma.

Lemma 1. Consider the uncertain system (9), under the following gain conditions,

$$k_2 > \rho, \quad k_1 > 1.41\sqrt{k_2+\rho}, \quad \rho = |\dot{d}(t)|_{\max} \tag{13}$$

second order sliding mode is established on x_1 in a finite time and time of convergence is $t \leq \frac{3}{\eta}V^{\frac{1}{3}}(0)$.

Proof. The detailed proof of the Lemma 1 is discussed in the appendix.

For the comparison, consider system (7) with STC (8) and the disturbance $d(t) = 5\sin(2t) + 2\cos(5t) - 2$ with $|\dot{d}(t)|_{\max} = \rho = 20$. It is clear (see Table 1), for the same perturbation, the proposed method gives reduced gains. In STC, large value of k_1 generates more control effort, hence for the practical applications, large k_1 could not respect actuator saturation, in such cases proposed gains are useful.

Method	ρ	k_2	k_1
Moreno 2014	20	1.1ρ= 22	$k_1 > 29.69$
Davila 2005	20	1.1ρ= 22	$k_1 > 51.33$, $p = 0.1$
Ramesh 2017	20	1.1ρ= 22	$k_1 > 11.66$
Proposed	20	1.1ρ= 22	$k_1 > 9.13$

Table 1. Gain Comparisons

Theorem 3. Consider the system (3) with $|\dot{d}(t)|_{\max} = \rho$ and $d(0) = 0$. Define the predefined convergence time t_F then the continuous control input $u = u_n + u_{\text{STC}}$ defined as

$$u_n = \begin{cases} -B^\top Mx(t) + B^\top\delta(t) & \text{for } 0 \leq t \leq t_F \\ -B^\top Mx(t) & \text{for } t > t_F \end{cases} \tag{14}$$

with $\delta(t)$ ($\delta(0)$ selected in order to satisfy the terminal condition $x(t_F) = 0$.) and M as

$$\begin{aligned} \dot{\delta} &= -(A^\top - MBB^\top)\delta \\ 0 &= MA + A^\top M - MBB^\top M + Q \end{aligned}$$

and

$$\begin{aligned} u_{\text{STC}} &= (GB)^{-1}\Big[-k_1|s|^{\frac{1}{2}}\text{sign}(s) + \nu\Big] \\ \dot{\nu} &= -k_2\text{sign}(s), \end{aligned} \tag{15}$$

under the following gain conditions,

$$k_2 > \rho_1, \quad k_1 > 1.41\sqrt{k_2+\rho_1}, \quad \rho_1 = GB|\dot{d}(t)|_{\max} \tag{16}$$

with

$$s = G\left[x(t) - x(0) - \int_0^t (Ax + Bu_n)d\tau\right] \tag{17}$$

with $G \in \mathbb{R}^{1\times n}$ a projection matrix satisfying $GB \neq 0$, allows the establishment of a n-th order sliding mode with respect to x_1.

Proof. To elaborate finite time stabilization and disturbance compensation in details, consider the following integral sliding surface

$$s = G\left[x(t) - x(0) - \int_0^t (Ax + Bu_{\text{n}})d\tau\right]. \tag{18}$$

Sliding surface is chosen such that at $t = 0$, $s = 0$, *i.e.* **system trajectories start from the sliding surface**. The objective of the controller is to maintain the system trajectories on the sliding surface, thanks to the super-twisting algorithm. If it is the case, the system trajectories will be similar as those of a pure chain of integrators controlled by(14). By this way, the trajectories converge to the origin exactly $t = t_F$. From (18) one gets,

$$\begin{aligned} \dot{s} &= G\left[Ax + B(u + d(t)) - Ax - Bu_{\text{n}}\right] \\ &= GB\left[u_{\text{STC}} + d(t)\right] \end{aligned} \tag{19}$$

The super twisting control reading as

$$\begin{aligned} u_{\text{STC}} &= \frac{1}{GB}\Big[-k_1|s|^{\frac{1}{2}}\text{sign}(s) + \nu\Big] \\ \dot{\nu} &= -k_2\text{sign}(s), \end{aligned} \tag{20}$$

after substitution of (20) in (19), one gets

$$\begin{aligned} \dot{s} &= -k_1|s|^{\frac{1}{2}}\text{sign}(s) + \nu + GBd(t) \\ \dot{\nu} &= -k_2\text{sign}(s) \end{aligned} \tag{21}$$

Defining the new variable $z = \nu + GBd(t)$, system (21) can be rewritten as

$$\begin{aligned} \dot{s} &= -k_1|s|^{\frac{1}{2}}\text{sign}(s) + z \\ \dot{z} &= -k_2\text{sign}(s) + GB\dot{d}(t) \end{aligned} \tag{22}$$

Equation (22) is a super-twisting algorithm and sliding surface is chosen as $s(0) = 0$ from starting, the objective consist in maintaining $s = \dot{s} = 0$ from $t \geq 0$. The objective is fulfilled by the following gains (as per Lemma 1),

$$k_2 > \rho_1, \quad k_1 > 1.41\sqrt{k_2 + \rho_1}, \quad \rho_1 = GB|\dot{d}(t)|_{\max} \tag{23}$$

then, the system is govern by the following

$$\dot{x}(t) = Ax + Bu_{\mathrm{n}}, \tag{24}$$

The system (24) is a disturbance free and u_n is selected as Theorem 1, which makes the stabilization of (24) in a predefined finite time and hence objective of finite time stabilization of an uncertain chain of integrators is achieved.

4. ACADEMIC EXAMPLE

To illustrate the performance of the proposed approach consider the pendulum example (Shtessel et al. (2014)). The dynamics of the pendulum system reads as

$$\ddot{\theta} = \frac{1}{J}u - \frac{mgL}{2J}\sin(\theta) - \frac{V_s}{J}\dot{\theta} + d(t) \tag{25}$$

with pendulum mass $m = 1.1kg$, gravitational constant $g = 9.81$, pendulum length $L = 0.9m$, inertia $J = 0.891kg.m^2$, $V_s = 0.18$ is the pendulum viscous friction coefficient and d(t) is a disturbance. Defining $x_1 = \theta$ and $x_2 = \dot{\theta}$ then system (25) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{J}u - \frac{mgL}{2J}\sin(x_1) - \frac{V_s}{J}x_2 + d(t) \end{aligned} \tag{26}$$

The objective of the controller is to force x_1 to $x_{1\mathrm{ref}} = 10\sin(t) + 5$ in the presence of disturbance in predefined time t_F. The disturbance is chosen as $d(t) = 5\sin(2t) + 2\cos(5t) - 2$. Let define the error $e_1 = x_1 - x_{1\mathrm{ref}}$ and $e_2 = x_2 - \dot{x}_{1\mathrm{ref}}$. The tracking error dynamics reads as

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \frac{1}{J}u - \frac{mgL}{2J}\sin(x_1) - \frac{V_s}{J}x_2 + d(t) - \ddot{x}_{1\mathrm{ref}} \end{aligned} \tag{27}$$

The control input is selected as

$$u = J\left(\frac{mgL}{2J}\sin(x_1) + \frac{V_s}{J}x_2 + \ddot{x}_{1\mathrm{ref}} + \bar{u}\right). \tag{28}$$

After substituting the control input (28) in (27), one gets

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \bar{u} + d(t) \end{aligned} \tag{29}$$

The "new" control input $\bar{u}$ is the addition of two components, i.e. $\bar{u} = u_n + u_{\mathrm{STC}}$, with u_n the nominal control and u_{STC} a super-twisting control.

Nominal Control Design

The nominal control u_n is designed to track the x_1 to its reference value in predefined time t_F. In this example we have fixed the convergence time to $t_F = 6$ sec. The nominal control is designed for the disturbance free system, which follows as

$$\dot{e} = Ae + Bu_n \tag{30}$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The nominal controller reads as (Section 2)

$$u_n = \begin{cases} -B^\top Me(t) + B^\top \delta(t) & \text{for } 0 \leq t \leq t_F \\ -B^\top Me(t) & \text{for } t > t_F \end{cases} \tag{31}$$

The matrix M is calculated from the Riccati's equation, stating $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ one gets $M = \begin{bmatrix} 1.7321 & 1 \\ 1 & 1.7321 \end{bmatrix}$. The initial condition of the state is considered as $x(0) = [\,0\ 0\,]$ which implies that $e(0) = [\,-5\ -10\,]$. With the knowledge of $e(0)$ and t_F, initial condition of $\delta(t)$ function is calculated by using the `gramper` function (For details, see Laghrouche et al. (2007), de Larminat (2000)) *i.e.* $\delta(0) = \begin{bmatrix} 0.0005 \\ 0.0018 \end{bmatrix}$.

Super Twisting Control Design

For the design of super-twisting control, integral sliding surface is considered as follows

$$s = G\left[e(t) - e(0) - \int_0^t (Ae + Bu_{\mathrm{n}})d\tau\right], \tag{32}$$

with $G = B^\top$.The time derivative of the disturbance is bounded and its upper bound is $|\dot{d}(t)|_{\max} = \rho = 20$. The gains of the super-twisting control are selected as Theorem 3, $k_2 = 1.1\rho = 22$ and $k_1 = 1.5\sqrt{k_2 + \rho} = 9.72$ which will maintain $s = \dot{s} = 0$ from the beginning of the time i.e. $t > 0$.

In order to compare the performances of the new sceame with former one based on (Laghrouche et al. (2007)) the gain is selected as, bound of the disturbance is $|d(t)|_{\max} = \xi = 8.825$. To maintain the sliding surface at $s = 0$ from the starting i.e. $t > 0$, we have chosen $K = 10$

5. SIMULATION RESULTS

The performance of the discontinuous ISMC and proposed method are compared. First the both methods are able to track the reference trajectory robustly exactly in time t_F which is shown by Figure 1. The performance of the proposed method is depicted in Figure 1 (a),(c), whereas the performance of discontinuous ISMC is depicted in Figure 1 (b),(d). It is clearly from Figure 1 that, the control input is the only difference between proposed method and discontinuous ISMC. As it is known that chattering is a practical phenomenon which can be damageable to the system, the proposed method can give the more promising results compared to discontinuous ISMC.

6. CONCLUSIONS

The paper proposes a new scheme of control. The controller proposed in the paper, guarantees the convergence of the system at exactly a predefined time t_F and this convergence time is chosen in advance. The controller is designed based on the integral sliding mode approach and it also adjusts the chattering because of continuous control. The paper also presents the finite time stability of super twisting algorithm using continuously differentiable Lyapunov function. Using an academic example, effectiveness of the proposed controller is presented with simulation results and results are more promising.

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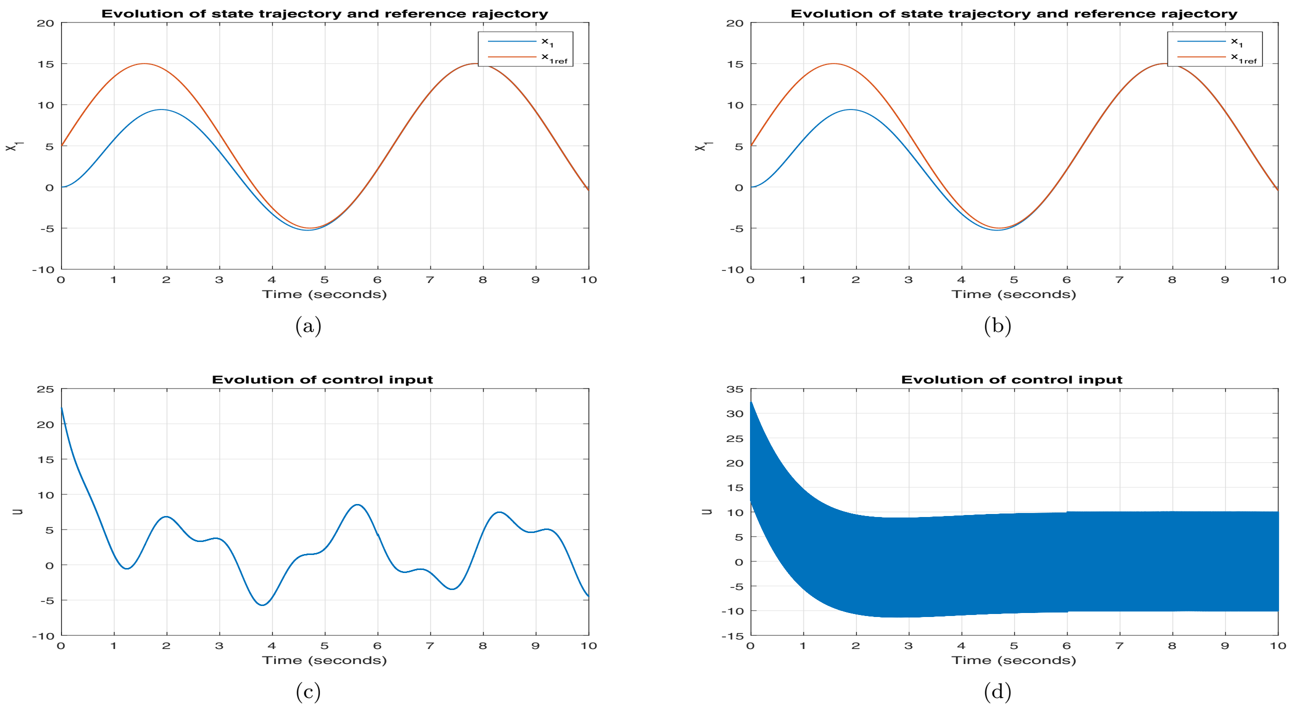

Fig. 1. Comparison: Continuous ISMC (Proposed Method) Vs Discontinuous ISMC (Laghrouche et al. (2007))

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7. APPENDIX

The super twisting algorithm with $|\dot{d}(t)|_{\max} = \rho$.

$$\begin{aligned}\dot{x}_1 &= -\,k_1|x_1|^{\frac{1}{2}}\text{sign}(x_1) + x_2\\ \dot{x}_2 &= -\,k_2\text{sign}(x_1) + \dot{d}(t),\end{aligned} \tag{33}$$

Since the right hand side of (33) is discontinuous, its solutions can be understood in the sense of Filippov (1988). To prove the finite time stability of (33) we have considered a continuously differentiable Lyapunov function as

$$V = \frac{2}{3}k_1\,|x_1|^{\frac{3}{2}} - x_1x_2 + \frac{2}{3{k_1}^2}\,|x_2|^3, \tag{34}$$

which is homogeneous (of degree $\delta_V = 3$) with dilation

$$d_k : V(x_1, x_2) \mapsto V(k^2x_1, kx_2) \quad \text{for every} \quad k > 0.$$

Positive definiteness of V

To show V is a positive definite, we used classical Young's inequality which states the following, for any real values $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and any positive real numbers a, b, c, the inequality $ab \leq c^p\frac{a^p}{p} + c^{-q}\frac{b^q}{q}$ holds. The chosen Lyapunov function (34) can be written as,

$$V \geq \frac{2}{3}k_1\,|x_1|^{\frac{3}{2}} - |x_1||x_2| + \frac{2}{3{k_1}^2}\,|x_2|^3. \tag{35}$$

Using Young's inequality we can write,

$$|x_1||x_2| \leq \frac{2}{3}c^{\frac{3}{2}}|x_1|^{\frac{3}{2}} + \frac{1}{3}c^{-3}|x_2|^3, \quad \text{where} \quad c > 0.$$

After substituting it in (35),

$$V \geq \frac{2}{3}\left(k_1 - c^{\frac{3}{2}}\right)|x_1|^{\frac{3}{2}} + \frac{1}{3}\left(\frac{2}{{k_1}^2} - c^{-3}\right)|x_2|^3$$

It is known that $c > 0$, so k_1 has to be positive to hold the condition $\frac{k_1^{\frac{2}{3}}}{2^{\frac{1}{3}}} < c < k_1^{\frac{2}{3}}$, so V is a positive definite.

Time Derivative of Lyapunov function

$$\dot{V} = k_1|x_1|^{\frac{1}{2}}\text{sign}(x_1)\dot{x}_1 - x_2\dot{x}_1 - x_1\dot{x}_2 + \frac{2}{{k_1}^2}|x_2|^2\text{sign}(x_2)\dot{x}_2$$

Let us define $\Phi = -k_1|x_1|^{\frac{1}{2}}\text{sign}(x_1) + x_2$, then further

$$\begin{aligned}\dot{V} = &-\Phi^2 + k_2|x_1| - |x_1|\text{sign}(x_1)\dot{d}(t) + \frac{2\dot{d}(t)}{{k_1}^2}|x_2|^2\text{sign}(x_2)\\ &- \frac{2k_2}{{k_1}^2}|x_2|^2\text{sign}(x_2)\text{sign}(x_1).\end{aligned}$$

Now it will be shown that $\dot{V}$ is negative definite in the presence of uncertainty. To do that we have analysed $\dot{V}$ in two quadrant because of lack of space. But same results will hold in other quadrant and axis also.

(1) Quadrant I: $\text{sign}(x_1) = 1$ and $\text{sign}(x_2) = 1$

$$\begin{aligned}\dot{V}_{\text{I}} &= -\Phi^2 + k_2|x_1| - |x_1|\dot{d}(t) + \frac{2\dot{d}(t)}{{k_1}^2}|x_2|^2 - \frac{2k_2}{{k_1}^2}|x_2|^2\\ &= -\Phi^2 + (k_2 - \dot{d}(t))|x_1| - \frac{2(k_2 - \dot{d}(t))}{{k_1}^2}|x_2|^2\\ &= -\Phi^2 + \frac{2(k_2 - \dot{d}(t))}{{k_1}^2}\Big[{k_1}^2|x_1| - |x_2|^2\Big]\\ &\quad - (k_2 - \dot{d}(t))|x_1|\\ &= -\Phi^2 + \frac{2(k_2 - \dot{d}(t))}{{k_1}^2}\Big[\Phi^2 - 2{x_2}^2 + 2k_1|x_1|^{\frac{1}{2}}|x_2|\Big]\\ &\quad - (k_2 - \dot{d}(t))|x_1|\\ &= -\left[\frac{{k_1}^2 - 2(k_2 - \dot{d}(t))}{{k_1}^2}\right]\Phi^2\\ &\quad - \frac{2(k_2 - \dot{d}(t))}{{k_1}^2}\left[\frac{k_1|x_1|^{\frac{1}{2}}}{\sqrt{2}} - \sqrt{2}|x_2|\right]^2\end{aligned}$$

If ${k_1}^2 - 2(k_2 - \dot{d}(t)) > 0 \Longrightarrow k_1 > \sqrt{2(k_2 - \dot{d}(t))}$ which further implies $k_2 > \rho$, then $\dot{V}_{\text{I}}$ is negative definite. So, we get conditions on the gains in Quadrant I,

$$k_2 > \rho, \quad k_1 > 1.41\sqrt{k_2 + \rho}.$$

(2) Quadrant II: $\text{sign}(x_1) = -1$ and $\text{sign}(x_2) = 1$

$$\begin{aligned}\dot{V}_{\text{II}} &= -\Phi^2 + (k_2 + \dot{d}(t))|x_1| + \frac{2(k_2 + \dot{d}(t))}{{k_1}^2}|x_2|^2\\ &= -{k_1}^2|x_1| - {x_2}^2 - 2k_1|x_1|^{\frac{1}{2}}|x_2|\\ &\quad + (k_2 + \dot{d}(t))|x_1| + \frac{2(k_2 + \dot{d}(t))}{{k_1}^2}|x_2|^2\\ &= -\Big[{k_1}^2 - 2(k_2 + \dot{d}(t))\Big]|x_1| - (k_2 + \dot{d}(t))|x_1|\\ &\quad - \left[\frac{{k_1}^2 - 2(k_2 + \dot{d}(t))}{{k_1}^2}\right]{x_2}^2 - 2k_1|x_1|^{\frac{1}{2}}|x_2|\end{aligned}$$

If ${k_1}^2 - 2(k_2 + \dot{d}(t)) > 0 \Longrightarrow k_1 > \sqrt{2(k_2 + \dot{d}(t))}$ which further implies $k_2 > \rho$, then $\dot{V}_{\text{II}}$ is negative definite. Here, in Quadrant II we get the same condition

$$k_2 > \rho, \quad k_1 > 1.41\sqrt{k_2 + \rho}.$$

Finite Time Stability

Due to the homogeneity of V and $\dot{V}$, with degrees $\delta_V = 3$ and $\delta_{\dot{V}} = 2$, respectively, it follows that the function $W(x) = \frac{-\dot{V}(x)}{V^{\frac{2}{3}}(x)}$ is homogeneous of degree $\delta_W = 0$. This implies that all the values $W(x)$ takes will contain inside the homogeneous unit ball, i.e., $\mathcal{B}_h = \big\{(x_1, x_2) \in \mathbb{R}| \quad |x_1| + |x_2|^2 = 1\big\}$. On $\mathcal{B}_h$ the function V and $-\dot{V}$ are continuous and different from zero. Therefore, $W(x)$ has a positive minimum, that can be calculated by $\eta = \min_{x\in\mathcal{B}_h} W(x)$. This implies that

$$-\frac{\dot{V}(x)}{V^{\frac{2}{3}}(x)} \geq \eta \Rightarrow \dot{V}(x) \leq -\eta V^{\frac{2}{3}}(x). \tag{36}$$

From this differential inequality it follows immediately that the trajectories converge in finite time to zero and convergence time can be estimated as $t \leq \frac{3}{\eta}V^{\frac{1}{3}}(0)$. ■