

Supplement to Bounds On Treatment Effects On Transitions

Online Appendix C: Average treatment effect on survivors

In this appendix we consider the average effect when averaging over the subpopulation of individuals who would have survived until \( t \) under both treatment and no-treatment. We call this average effect the Average Treatment Effect on Survivors, \( \text{ATES}_t \):

**Definition 1** Average Treatment Effect on Survivors (ATES)

\[
\text{ATES}_t = E \left( Y^1_t | Y^0_{t-1} = 0, Y^0_{t-1} = 0 \right) - E \left( Y^0_t | Y^1_{t-1} = 0, Y^0_{t-1} = 0 \right)
\]

The bounds for \( \text{ATES}_t \) are given in Theorem 1.

**Theorem 1 (Bounds on ATES)** Suppose that Assumption 1 holds. If \( \Pr(Y_{t-1} = 0 | D = 1) + \Pr(Y_{t-1} = 0 | D = 0) - 1 \leq 0 \), then \( \text{ATES}_t \) is not defined.

If \( \Pr(Y_{t-1} = 0 | D = 1) + \Pr(Y_{t-1} = 0 | D = 0) - 1 > 0 \), then we have the following sharp bounds

\[
\max \left\{ 0, \frac{\Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) + \Pr(Y_t = 0 | D = 0) - 1}{\Pr(Y_{t-1} = 0 | D = 1) + \Pr(Y_{t-1} = 0 | D = 0) - 1} \right\} - \text{ATES}_t \leq
\]

\[
\min \left\{ 1, \frac{\Pr(Y_t = 1, Y_{t-1} = 0 | D = 0)}{\Pr(Y_{t-1} = 0 | D = 0) + \Pr(Y_{t-1} = 0 | D = 1) - 1} \right\} - \text{ATES}_t \leq
\]

\[
\max \left\{ 0, \frac{\Pr(Y_t = 1, Y_{t-1} = 0 | D = 0) + \Pr(Y_t = 0 | D = 1) - 1}{\Pr(Y_{t-1} = 0 | D = 1) + \Pr(Y_{t-1} = 0 | D = 1) - 1} \right\}.
\]

**Proof:** First, consider bounds on \( \Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) = \Pr(Y^1_t = 1, Y^0_{t-1} = 0) \) By Assumption 2

\[
\Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) = \Pr(Y^1_t = 1, Y^0_{t-1} = 0).
\]

By the law of total probability

\[
\Pr(Y^1_t = 1, Y^0_{t-1} = 0) = p^0_t(1|0,0)p_{t-1}(0,0) + p^0_t(1|0,\neq 0)p_{t-1}(0, \neq 0)
\]

Therefore,

\[
\Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) = p^0_t(1|0,0)p_{t-1}(0,0) + p^0_t(1|0,\neq 0)p_{t-1}(0, \neq 0)
\]

Solving for \( p^1_t(0|0,0) = \Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) - p^0_t(1|0,\neq 0)p_{t-1}(0, \neq 0) \)

\[
\Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) = \frac{\Pr(Y_t = 1, Y_{t-1} = 0 | D = 1) - p^0_t(1|0,\neq 0)p_{t-1}(0, \neq 0)}{p_{t-1}(0,0)}
\]
The expression on the right-hand side is decreasing in \( p_t^0(1|0, \neq 0) \). The lower bound is obtained by setting \( p_t^0(1|0, \neq 0) \) at 1 and the upper bound by setting \( p_t^0(1|0, \neq 0) \) at 0.

\[
\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - p_{t-1}(0, \neq 0)
\]

\[
\leq \mathbb{E} \left[ Y_t^1|\overline{Y}_{t-1} = 0, \overline{Y}_{t-1} = 0 \right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{p_{t-1}(0, 0)}.
\]

Because

\[
\Pr(\overline{Y}_{t-1} = 0|D = 1) = p_{t-1}(0, 0) + p_{t-1}(0, \neq 0)
\]

we have

\[
\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) - \Pr(\overline{Y}_{t-1} = 0|D = 1) + p_{t-1}(0, 0)}{p_{t-1}(0, 0)}
\]

\[
\leq \mathbb{E} \left[ Y_t^1|\overline{Y}_{t-1} = 0, \overline{Y}_{t-1} = 0 \right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{p_{t-1}(0, 0)}.
\]

The upper bound is decreasing and the lower bound is increasing in \( p_{t-1}(0, 0) \). From the proof of Theorem 1 we have

\[
p_{t-1}(0, 0) \geq \max \{ \Pr (\overline{Y}_{t-1} = 0|D = 1) + \Pr (\overline{Y}_{t-1} = 0|D = 0) - 1, 0 \}.
\]

If \( \Pr (\overline{Y}_{t-1} = 0|D = 1) + \Pr (\overline{Y}_{t-1} = 0|D = 0) - 1 > 0 \) then we are sure that there are survivors in both treatment arms. Upon substitution of this lower bound

\[
\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1) + \Pr (\overline{Y}_{t-1} = 0|D = 0)}{p_{t-1}(0, 0)} - 1
\]

\[
\leq \mathbb{E} \left[ Y_t^1|\overline{Y}_{t-1} = 0, \overline{Y}_{t-1} = 0 \right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 1)}{\Pr (\overline{Y}_{t-1} = 0|D = 1) + \Pr (\overline{Y}_{t-1} = 0|D = 0) - 1}.
\]

By an analogous argument we have

\[
\frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0)}{p_{t-1}(0, 0)} - 1
\]

\[
\leq \mathbb{E} \left[ Y_t^0|\overline{Y}_{t-1} = 0, \overline{Y}_{t-1} = 0 \right] \leq \frac{\Pr(Y_t = 1, \overline{Y}_{t-1} = 0|D = 0)}{\Pr (\overline{Y}_{t-1} = 0|D = 1) + \Pr (\overline{Y}_{t-1} = 0|D = 0) - 1}.
\]

Substitution of these results for \( \mathbb{E} \left[ Y_t^1|\overline{Y}_{t-1} = 0, \overline{Y}_{t-1} = 0 \right] \) and \( \mathbb{E} \left[ Y_t^0|\overline{Y}_{t-1} = 0, \overline{Y}_{t-1} = 0 \right] \) and because both probabilities are bounded by zero and one gives the bounds on ATES_t.