

**STUDENTS' UNDERSTANDING OF THE FUNDAMENTAL  
THEOREM OF CALCULUS: AN EXPLORATION OF  
DEFINITIONS, THEOREMS AND VISUAL IMAGERY**

**CLAUDIA COELHO DE SEGADAS VIANNA**

**Thesis submitted for the PhD degree of the University of London**

**INSTITUTE OF EDUCATION, UNIVERSITY OF LONDON**

**1998**



## ABSTRACT

The aim of this research was to investigate students' understanding of the Fundamental Theorem of Calculus (FTC). The FTC was chosen as the basis of this research because it is one of the most important topics taught in calculus, establishing the link between the concepts of differentiation and integration.

Data was collected from first year undergraduate students at the Federal University of Rio de Janeiro, in Brazil. The sample comprised students from three areas: mathematics, computer sciences and engineering. A pilot study was applied in 1994 and the main study in 1995 and 1996. Questionnaire, interviews based on responses to the questionnaire and computer-task based interviews were used. The data were analysed using both quantitative and qualitative methods.

The results show that some of the students' obstacles to understanding the FTC are related to difficulties with the concepts of function, continuity, derivative and integral. The definitions of these concepts were not clear in their minds and they frequently made use of images that contained only partial aspects of the definitions or were based on some particular examples. This hampered the students when they met new examples that did not fit with their pre-formed images. It was also found that definitions and theorems were so fragmented in the students' minds that there was no way they could appreciate a proof, exemplified by the proof of the FTC. Their conceptions of proof reflected the fact that they were not used to thinking of proving as fundamental to generalising a proposition, and in examining a proof, it was difficult for them to see the central ideas behind it. These results are closely associated with students' habits: they tend not to pay attention to theoretical aspects, memorising algorithmic procedures without reflecting on their applicability.

## **TABLE OF CONTENTS**

<b>LIST OF TABLES</b>	<b>7</b>
<b>LIST OF FIGURES</b>	<b>9</b>
<b>ACKNOWLEDGEMENTS</b>	<b>10</b>
<b>CHAPTER 1 - INTRODUCTION</b>	<b>11</b>
<b>CHAPTER 2 - BACKGROUND TO THE STUDY</b>	<b>13</b>
2.1 Introduction	13
2.2 The Fundamental Theorem of Calculus	13
2.2.1 The statement	14
2.1.2 The proof	15
2.3 The calculus course	18
2.4 Research on the concept of function: a challenge from secondary school onwards	19
2.5 The different definitions of continuity	21
2.6 Limits, derivatives and integrals	24
2.7 Research on proof	26
2.8 Research into students' understanding of the FTC	30
2.9 Summary	32
<b>CHAPTER 3 - DESIGN OF THE STUDY</b>	<b>33</b>
3.1 Introduction	33
3.2 The sample	32
3.3 The curriculum of Calculus I	34
3.4 Data collection	35
3.5 The pilot study	36
3.6 The main study	36

<b>CHAPTER 4 - THE PILOT STUDY</b>	<b>38</b>
4.1 Introduction	38
4.2 The questionnaire	38
4.2.1 The sample and the setting	38
4.2.2 Structure and aims	39
4.2.3 Analysis of the questions	40
4.3 The interviews based on responses to the questionnaire	59
4.3.1 The sample	59
4.3.2 The schedule	59
4.3.3 Analysis of the interviews	60
4.4 The computer- based interviews	66
4.4.1 The sample	66
4.4.2 The schedule	66
4.4.3 Analysis of the interviews	69
4.5 Preparing the main study	71
4.5.1 The questionnaire	71
4.5.2 The interviews around the computer	72
 <b>CHAPTER 5 - THE QUESTIONNAIRE</b>	 <b>75</b>
5.1 Introduction	75
5.2 The sample	75
5.3 Methodology of analysis	77
5.4 Comparing the classes	78
5.4.1 Analysis of the results of part I	78
5.4.2 Analysis of the results of part II	79
5.4.3 Conclusion	80
5.5 Aims of the questions	80
5.6 Results per questions: part I	80
5.6.1 Evaluating integrals: question 1	80
5.6.2 Understanding continuity: question 3	85
5.6.3 Finding the derivative of the integral function: questions 2 and 5	100
5.6.4 Visualising the functions $F(x)$ and $F'(x)$ : question 4	109

5.6.5 Stating the FTC: question 6	119
5.7 Results per questions: part II	125
5.7.1 Generalising in mathematics: question 1	126
5.7.2 The proof in mathematics: question 2	133
5.8 Summary	138
<b>CHAPTER 6 - THE INTERVIEWS BASED ON RESPONSES TO THE QUESTIONNAIRE</b>	143
6.1 Introduction	143
6.2 The sample	143
6.2.1 Procedures	143
6.2.2 Performance on the questionnaire	144
6.3 Interview Schedule	167
6.4 Methodology of analysis	168
6.5 Results	170
6.5.1 Estela	171
6.5.2 Amanda	174
6.5.3 Gérson	177
6.5.4 Geisa	179
6.5.5 Alice	182
6.5.6 Elisa	183
6.5.7 Fernando	185
6.5.8 Roberto	187
6.5.9 Fabiana	190
6.5.10 Heitor	192
6.5.11 Hugo	194
6.5.12 Marcelo	195
6.5.13 Beatriz	198
6.5.14 Ricardo	200
6.5.15 Cristina	202
6.5.16 Ivo	205
6.5.17 Raul	207
6.6 Summary	210

<b>CHAPTER 7 - THE COMPUTER-TASK BASED INTERVIEWS</b>	<b>217</b>
7.1 Introduction	217
7.2 The sample	217
7.3 Description of the activities	218
7.4 Development of the interviews	223
7.5 Methodology of analysis	224
7.6 Results	227
7.6.1 Estela	227
7.6.2 Gérson	230
7.6.3 Alice	233
7.6.4 Roberto	236
7.6.5 Heitor	239
7.6.6 Ricardo	243
7.6.7 Cristina	245
7.7 Summary	247
<b>CHAPTER 8 - DISCUSSION OF THE RESULTS</b>	<b>251</b>
8.1 The main findings	252
8.2 Summary and calculus remarks	263
8.3 General implications for teaching	264
8.4 Limitations of this research	266
8.5 Implications for future research	266
<b>REFERENCES</b>	<b>268</b>
<b>APPENDIX 1 - The pilot study questionnaire</b>	<b>272</b>
<b>APPENDIX 2 - The main study questionnaire</b>	<b>276</b>
<b>APPENDIX 3 - The computer-based activities</b>	<b>280</b>
<b>APPENDIX 4 - Tables used for statistical analysis</b>	<b>284</b>

## LIST OF TABLES

### Chapter 4:

Table 4.1	Distribution of categories of responses to question 1 - part 1	41
Table 4.2	Distribution of categories of responses to question 2- part 1	44
Table 4.3	Distribution of students' choices in question 4a	47
Table 4.4	Distribution of categories of justifications for the choices of (c) and (d) in question 4a	48
Table 4.5	Distribution of students' choices in question 4b	49
Table 4.6	Distribution of categories of justifications for the choice of (f) in question 4b	50
Table 4.7	Distribution of categories of responses to question 5	52
Table 4.8	Distribution of categories of responses to question 1 - part II	55
Table 4.9	Distribution of categories of responses to question 2 - part II	58

### Chapter 5:

Table 5.1	Sample data to part I	78
Table 5.2	Frequency in percentage of each score per class.	79
Table 5.3	Absolute frequency of each score per class.	74
Table 5.4	Means and standard deviations found for question 1 - part 1	81
Table 5.5	Distribution of categories of responses to question 1 - part I	82
Table 5.6	Distribution by groups of the main mistakes made in question 1b	84
Table 5.7	Means and standard deviations found for question 3	87
Table 5.8	Distribution of categories of responses to function $f_1$ in question 3	93
Table 5.9	Distribution of categories of responses to function $f_2$ in question 3	94
Table 5.10	Distribution of categories of responses to function $f_3$ in question 3	95
Table 5.11	Distribution of categories of responses to function $f_4$ in question 3	97
Table 5.12	Means and standard deviations found for question 2 - part 1	101

Table 5.13	Means and standard deviations found for question 5	102
Table 5.14	Distribution of categories of responses to question 2a and 5a	105
Table 5.15	Distribution of categories of responses to question 2a and 5a in group 1	106
Table 5.16	Distribution of categories of responses to question 2b and 5b	107
Table 5.17	Distribution of categories of responses to question 5	108
Table 5.18	Means and standard deviations found for question 4	110
Table 5.19	Distribution of students' choices in question 4a	111
Table 5.20	Distribution of categories of justifications for the choices of (b), (c) and (d) in question 4a	115
Table 5.21	Distribution of students' choices in question 4a	116
Table 5.22	Distribution of categories of justifications for the choice of (f) in question 4b	119
Table 5.23	Means and standard deviations found for question 6	120
Table 5.24	Distribution of categories of responses to question 6	124
Table 5.25	Frequency of scores found for question 1 - part II	127
Table 5.26	Distribution of categories of responses to question 1- part II	131
Table 5.27	Frequency of scores found for question 2 - part II	134
Table 5.28	Distribution of categories of responses to question 2 - part II	136

## **Chapter 6:**

Table 6.1	Scores obtained in the questionnaire by the students interviewed	145
-----------	--	-----



## LIST OF FIGURES

### Chapter 4:

Figure 4.1	Question 4	46
Figure 4.2	Questions 1 - part II	52
Figure 4.3	Questions 2 - part II	56

### Chapter 5:

Figure 5.1	Question 3	86
Figure 5.2	Questions 2 and 5	100
Figure 5.3	Question 4	109
Figure 5.4	Question 1 - part II	126
Figure 5.5	Question 2 - part II	133

### Chapter 6:

Figure 6.1	Frame of one of the tables used to analyse students' interviews.	169
------------	--	-----

### Chapter 7:

Figure 7.1	Set 1 of the computer-based activities	219
Figure 7.2	Set 2 of the computer-based activities	221
Figure 7.3	Set 3 of the computer-based activities	223

## ACKNOWLEDGEMENTS

First of all I would like to express my gratitude to my supervisor, Professor Celia Hoyles, for her guidance, insightful comments and advice in this study. Thanks as well to Professor Richard Noss, who has followed the process of this study.

I would like to thank the friends of Project Fundação (Institute of Mathematics/ Federal University of Rio de Janeiro) and its director, Prof. Maria Laura Leite Lopes, for giving me all the support during the time I was in Brazil doing this research. Special mention is due to Lilian Nasser and Lucia Tinoco, who have been more directly involved in this research since the data collection and whose suggestions and words of friendship and encouragement have been essential.

I am indebted to the teachers and students at the Federal University of Rio de Janeiro who agreed to take part in this research. I wish also to give my thanks to Flávia Landim, who helped me in the statistical analysis and to Kenneth Payne, who kindly helped me with my English.

I am very grateful to my family, especially my parents who helped me with my little children, André and Pedro, the joy of my life at this time. Many thanks as well to Guida, Beth, Márcia, Valéria and other friends from England and from Brazil, who in different moments were involved in this process. I would like to dedicate this thesis to the memory of a very special friend, Prof. Mario Tourasse Teixeira, who was my supervisor in my Master's degree and always supported me.

It would not have been possible to carry out this course without the financial support from my sponsors, CNPq.

## CHAPTER 1

### INTRODUCTION

The calculus course is usually the first contact that the undergraduate student has with mathematics at an advanced level. It is, therefore, very important that students form correct concepts of the mathematical ideas in this course, some of which involve a high degree of generalisation and abstraction. The calculus course may also be the first contact that a student has with deductive reasoning outside the context of Euclidean geometry. Spivak (1967) argues that calculus should be presented as “not merely a prelude to but as the first real encounter with mathematics” (p.vii).

Research in mathematics teaching, however, has shown that this first real encounter of students with mathematics in the undergraduate teaching has not been so successful. Several reasons have been suggested. Some are linked with inherent difficulties of specific concepts. Others are more general, explaining the difficulties with the way the course itself is structured. In view of this situation there has been an increasing demand for research aiming at identifying the obstacles students face in calculus in order to provide suggestions for improving the teaching of it.

The purpose of this study is to help to identify some of these obstacles. A particular theorem was chosen as its focus: the Fundamental Theorem of Calculus (FTC). The FTC plays an important role in any calculus course, since it establishes the link between differentiation and integration. One of the consequences of its playing such a unifying and central role is its wide scope: by investigating students’ understanding of the FTC, it is possible as well to identify difficulties with other concepts related to it.

With this purpose in mind, the study aims to investigate students’ understanding of the FTC; examine the obstacles that most frequently hinder students from understanding it; identify the role of visual imagery in understanding the concepts of continuity, differentiation and integration; identify students’ main difficulties in trying to understand the formal definitions and concepts in mathematics which might prevent them from

appreciating the central ideas behind the theorem; and, finally, identify students' conceptions of the processes of proof as exemplified in the FTC. In order to achieve these aims, a questionnaire and interviews were specifically designed to probe the understanding of a sample of undergraduate Brazilian students from the Federal University of Rio de Janeiro.

The thesis is divided into 8 chapters, as follows. Chapter 2 presents the background to the study and includes a review of the relevant literature. Chapter 3 comprises a description of the research methodology adopted, including a characterisation of the sample to which the study was applied.

Chapter 4 presents the pilot study. It gives an outline description of it and an analysis of the results. Chapters 5, 6 and 7 describe and analyse the results obtained in the different phases of the main study. In these three chapters there is a careful presentation of the rationale behind each of the steps chosen for analysis of the main study.

Chapter 8 comprises the results of the overall analysis, discussed in the light of the review of the literature, as well as the conclusions and some implications for further research. There are three appendices presenting respectively: the pilot study questionnaire (Appendix 1), the main study questionnaire (Appendix 2), the computer-task based activities (Appendix 3) and some of the tables used for statistical analysis (Appendix 4).

## **CHAPTER 2**

### **BACKGROUND TO THE STUDY**

#### **2.1 Introduction**

This chapter aims at providing the reader with the relevant background for this research. Since the focus of this study is the understanding of the FTC by undergraduate students in the calculus course, first the FTC itself will be examined and then some general problems related to the course, since they may lie behind many of the obstacles students have to face. The examination of the FTC includes a description of its statement and its proofs and discussion on some ideas behind the proof which will help to indicate the concepts prerequisite for learning the FTC: function, limit, continuity, differentiation and integration. Difficulty in understanding these concepts might cause difficulty in understanding the FTC. Next follows an examination of previous research on these concepts. Another area on which research has been done and which is also related to the aims of this study is that on proof and proving, and so literature relevant to this will also be examined. Finally, research in mathematics education focusing specifically on the FTC will be discussed.

Although there are a number of different topics to be examined, the link between them is the FTC itself, and when selecting the literature to be analysed, it was constantly borne in mind that they should give insights into how difficulties in different fields might interfere in the learning of the FTC.

#### **2.2 The Fundamental Theorem of Calculus**

The Fundamental Theorem of Calculus is certainly one of the most important topics of any calculus course and is stated in every book on calculus and analysis. This section, besides stating some of the conventions adopted in this study, presents some of the proofs of the FTC found in most textbooks. Furthermore, a visual interpretation of the FTC will be discussed.

### 2.2.1 The statement

In this research the convention adopted is that the Fundamental Theorem of Calculus is the following theorem:

**Theorem:** Let  $f$  be an integrable function defined on  $[a, b]$ . Let  $F$  be a function on  $[a, b]$  defined by:

$$F(x) = \int_a^x f(t)dt .$$

If  $f$  is continuous at  $x$  in  $[a, b]$ , then  $F$  is differentiable at  $x$ , and

$$F'(x) = f(x).$$

If  $x = a$  or  $b$ , then  $F'(x)$  is understood to mean the right- or left-hand derivative of  $F$ .

The following corollary follows:

**Corollary:** If  $f$  is continuous on  $[a, b]$  and  $f = g'$  for some function  $g$ , then

$$\int_a^b f(x)dx = g(b) - g(a).$$

In some textbooks the name FTC is given to the corollary above. However, as the function  $F(x) = \int_a^x f(t)dt$  will be one of the focuses of attention in this research, the first theorem will be considered as the FTC. It is also a more general result with a stronger visual appeal. Thus if reference is made to the corollary of the FTC, the words “corollary of the FTC” will be explicitly stated.

Another convention refers to the function  $F(x) = \int_a^x f(t)dt$ . In this study it will be called “integral function” or “area function”. The term area will be used here, although it can also have a negative value. The notation  $F(x)$  in this study will always mean the integral function, although in the questionnaire other letters will be used to designate it as well.

### 2.2.2 The proof

The proofs found in some books can be roughly divided into two models according to the general argumentation, with the second found more frequently in the textbooks used in calculus in Brazil. They are the following:

#### Proof 1:

Assume  $x$  is in  $(a,b)$ . The cases  $x = a$  and  $x = b$  are similar (considering the right- or left-hand derivative of  $F$ ). By definition of derivative we have that,

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}.$$

If  $h > 0$  then

$$F(x+h) - F(x) = \int_x^{x+h} f(t) dt.$$

Let  $m_h$  and  $M_h$  be the inf and sup of  $f$  in the interval  $[x, x+h]$ , respectively. It follows that:

$$m_h \cdot h \leq \int_x^{x+h} f(t) dt \leq M_h \cdot h$$

and therefore,

$$m_h \leq \frac{F(x+h) - F(x)}{h} \leq M_h.$$

If  $h < 0$ , then a similar argument leads to the same formula above.

Since  $f$  is continuous at  $x$  we have:

$$\lim_{h \rightarrow 0} m_h = \lim_{h \rightarrow 0} M_h = f(x)$$

and this proves that

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

### Proof 2:

Assume  $x$  is in  $(a, b)$ . The cases  $x = a$  and  $x = b$  are similar. According to the mean value theorem for integrals we have:

$$F(x+h) - F(x) = \int_x^{x+h} f(t) dt = hf(\xi),$$

where  $\xi$  is some value in the interval  $[x, x+h]$ . For  $h$  tending to zero the value  $\xi$  must tend to  $x$  so that:

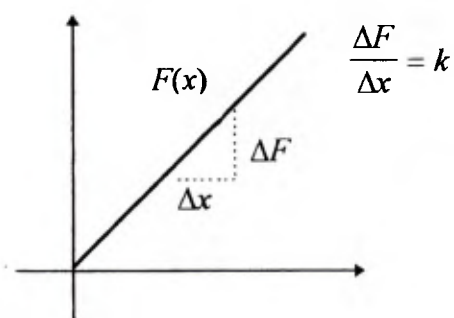
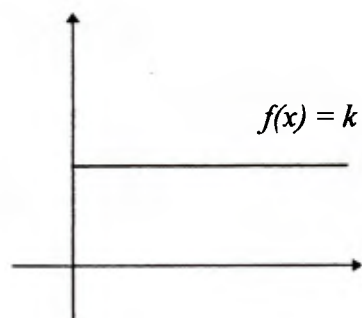
$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} f(\xi) = f(x)$$

since  $f$  is continuous. Hence  $F'(x) = f(x)$  as stated by the theorem.

Examining the proofs above, neither of them can be considered very complicated at first sight. Proof 2 is shorter, but proof 1 seems to be more constructive and visual. Proof 2 makes use of the mean value theorem for integrals and it is shorter because the proof of that theorem already uses the following theorem which is used in proof 1: if  $f$  is continuous on  $[a, b]$  and  $m \leq f(x) \leq M$  then  $m(b-a) \leq \int_a^b f \leq M(b-a)$  for all  $x$  in the interval  $[a, b]$ . Of course if the students have a good visualisation of the mean value theorem for integrals and its proof, both can be considered to have the same visual appeal. One must, however, keep in mind the extra requirement carried by proof 2 that the students need a good visualisation of the mean value theorem for integrals.

The proofs do not have any tricky arguments, but they do not seem to throw very much light on how these two concepts, derivative and integral, can be connected geometrically. In order to help this to be understood, a very simple case will be examined first. Since nothing can be more simple than a constant function, let us start with a function  $f$  defined by  $f(x) = k$ , where  $k$  is a positive real number. Let  $F$  be a function giving the area of the region bounded by the graph of  $f$  and the horizontal axis. The graph of  $f$  and the graph of  $F$  look like:





The slope of the tangent in the graph of  $F(x)$ , i.e., the derivative of  $F(x)$  is equal to  $k$ , which is the value of the function  $f(x)$  for any value of  $x$ . It is certainly easier to understand how the FTC works, geometrically speaking, when  $f$  is just a constant function.

If  $f$  is not constant but it is continuous, it is true that  $\lim_{x \rightarrow a} f(x) = f(a)$  for every point in the domain of the function. Thus, a continuous function can be thought of as a 'nearly' constant function at very small intervals. In both proofs this idea is present:

$$\lim_{h \rightarrow 0} m_h = \lim_{h \rightarrow 0} M_h = f(x) \quad (\text{Proof 1})$$

and,

$$\lim_{h \rightarrow 0} f(\xi) = f(x) \quad (\text{Proof 2})$$

where  $h$  is the length of the interval .

The idea illustrated with a constant function of how to relate slope of the tangent line and area, can now be applied to any continuous function.

The proofs helps us to decide some points that may hinder students from acquiring a good understanding of the FTC. They need to understand that both  $F(x)$  and  $F'(x)$  are functions. Since visualisation is a point of interest in this research, it is important for them to relate derivative with slope of the tangent line and integral with area. Continuity is essential, as noted above. Of course, understanding the proof of a theorem implies not only understanding each line of the proof, but what it is more important, the main idea(s) behind the proof that connect it as a whole. For a proof to be able to help students to understand a theorem, they need to know how to read the proof properly.

## 2.3 The calculus course

The calculus course is usually the first course in mathematics that a university student has to take. In Brazil, students in scientific and technological areas will have, during their first two undergraduate years, to take calculus I, II III and in some careers calculus IV as well. Calculus I is very important in that it will help to form the opinion the undergraduate students will have about mathematical ideas at a higher level. It is also when the student will start to learn to study at the rate required at the university, which will be new to most of them.

The reason it is called new is that there are certain characteristics of university teaching that justify the use of this word. With special reference to the subject of mathematics, Robert and Schwarzenberg (1991) say that students have to learn more concepts in less time. Consequently, students are supposed to study on their own much more, complementing what has been taught. Dreyfus (1990) emphasises that the degree to which students are expected to be able to formalise, define and prove is higher and of more central importance than in the more elementary courses. Tall (1993), besides citing the point of students' having to cope with more complex subjects in a short time, also stresses the difficulty they face in dealing for the first time with the concept of limit. These characteristics of the calculus course justify to some extent the use of the expression "advanced mathematics".

However, the students themselves do not change very much from secondary school to university. Robert and Schwarzenberg (1991) say that although at university students are no longer attending compulsory teaching of mathematics but rather going to classes related to a profession that they themselves have selected, the fact is that their attitudes towards study do not change very much. The teaching itself also does not encourage students to form a more mature attitude. In fact, after the teachers have presented concepts from a theoretical point of view, all too often they go on to teach algorithmic procedures not related to the theoretical concepts introduced. This attitude encourages students to learn only what they need for the application of the algorithms and ultimately what they need to do well in their exams. Artigue (1991) criticises this concentration on

teaching algebraic techniques to the detriment of the more theoretical aspects in a very suggestive passage that deserves to be quoted:

“In mathematics the means of justifications is classically that of proof. However, from the start, education distorts real difficulties concerning limits, functions, basic tools of approximating (such as inequalities, absolute values, reasoning with sufficient conditions, etc), and the understanding and manipulation of quantified statements. Instead it conceals them by using powerful algorithms (calculation of derivatives, partial derivatives, Jacobian matrices, primitives) and potent theorems which reduce theoretical considerations to algebraic techniques (such as theorems involving the sum, product and composition of  $C^1$  functions). Regrettably this premature algebraic algorithmization, on the one hand gives too privileged a role to the algebraic setting, and, on the other tends to drain the differential and integral procedures of their real meaning.” (Artigue, 1991, p.186)

In a nutshell, the calculus course tends not to be very stimulating and is easily transformed into a collection of recipes of algorithms and procedures.

One point of particular interest for this research concerns the way the course itself is structured and how students' attitudes towards it can influence their performance in the diverse phases of this study, which might partly explain some of their difficulties in understanding the FTC.

## **2.4 Research on the concept of function: a challenge from secondary school onwards**

Research in mathematics education has given special attention to the learning of the concept of a function. It constitutes a challenge from secondary school onwards and difficulties with it seriously affect the learning of many of the definitions students meet in calculus. Ferrini- Mundy and Lauten (1994) say that:

“The function concept is a central organizing idea for the study of calculus, and the ways in which students understand the function concept have substantial influence on the ways in which they understand such calculus ideas as limit, continuity, and the slope of a tangent line.”

(Ferrini-Mundy and Lauten, 1994, p.115)

In this study the function concept plays an important role, because besides its influence on the understanding of the calculus topics cited by Ferrini-Mundy and Lauten in the

above quotation, which are ultimately related to the FTC, it is also important in helping students to understand the way  $F(x)$  and  $F'(x)$  are defined.

However, research has shown that there are several problems affecting the learning of the function concept. One of these is that students' images of a function do not always fit with the formal definition. To analyse these images and the way they interfere in the use of this concept and in the concepts of limit, continuity, differentiation and integration which will be discussed below, it is useful to introduce the ideas of concept image and concept definition as they are described in Tall and Vinner (1981). By concept image the authors mean all the pictures that students hold in their minds and are associated with a concept, while by a concept definition they mean the words used by the student to refer to a concept. In their words:

“We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.” (Tall and Vinner, 1981, p.152)

and

“We shall regard the concept definition to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or less degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition.”  
(Tall and Vinner, 1981, p.152)

Some concept images of function are firmly fixed in students' minds. Thompson (1994), for example, calls attention to what he names the students' figural image of a function, that is a concept image of a rule:

“....an image of a short expression on the left and a long expression on the right separated by an equal sign.” (Thompson, 1994, p.268)

This image is strongly connected with the dominance of an analytical approach and the great difficulty in connecting different representations of a function, particularly connecting graphical and analytical representations. Based on research conducted by different people, Eisenberg (1991) points out some problems concerning the use of graphs by students: they do not understand the graphs they themselves draw, do not

know how to justify results obtained graphically and do not see the graph of a function as really part of its essence. In summary, they are not used thinking in graphical terms.

The strongly rooted images the students have and their lack of habit in using different ways to represent a function hampers the students when they meet new definitions of a function. Tall (1991a), in reference to one of his findings, reports that the students do not see the derivative function  $f'(x)$  as a function itself with an associated graph. A similar problem was found in relation to the understanding of the integral function  $F(x)$  (Thomas (1995)), which is an object of this study. Another problem is the lack of certain images, such as seeing the function relating two quantities, which is important for forming the concept of rate of change used later when the students meet the derivative of a function.

The ability to use and link different representations of the same concept is strongly related to the understanding of the definition of this concept and this reflects as well in the understanding of definitions of limit, continuity, differentiation and integration, which will be discussed below.

## 2.5 The different definitions of continuity

The concept of continuity is of vital importance in the FTC as already emphasised in discussion on the visual interpretation on the FTC, section 2.2.2. Tall and Vinner (1981) found that poorly formed concept images of continuity can cause difficulties in properly understanding further theory development. They analysed texts of the School Mathematics Project (SMP) in England and noted, although in the SPM Book 1 a footnote states that a function is not continuous or discontinuous at points where it is not defined (this is exemplified for the function  $f(x) = \frac{1}{x-1}$ , not defined at  $x = 1$ ), it is also stated on the same page that  $x = 1$  is an example of discontinuity. They administered a questionnaire to a sample of students with an A or B grade in A-level mathematics in order to see how this text could lead to the formation of concept images that might conflict with the formal definition. The students were required to classify certain functions as continuous or discontinuous, and of 41 students, 6 classified  $f(x) = \frac{1}{x}$  as a

continuous function and 35 classified  $f(x)$  as a discontinuous function. From the students' justifications Tall and Vinner noted that most of them thought that a continuous function could have no "gaps".

In this study, an analysis of several books on calculus was carried out. This showed that there is some confusion over definitions used and that there are basically two different definitions of continuity, as exemplified by the following extracts:

"The function  $f(x)$  is continuous at a point  $x_0$  of its domain if for every positive  $\varepsilon$  we can find a positive  $\delta$  such that

$$|f(x) - f(x_0)| < \varepsilon$$

for all values  $x$  in the domain of  $f$  for which  $|x - x_0| < \delta$ ."

(Courant and John, 1965, p.33)

and

"The function  $f$  is said to be continuous at the number  $a$  if and only if the following three conditions are satisfied:

- (i)  $f(a)$  exists;
- (ii)  $\lim_{x \rightarrow a} f(x) = f(a)$  exists;
- (iii)  $\lim_{x \rightarrow a} f(x) = f(a)$ ."

(Leithold, 1981, p.112)

This latter definition will be named in this work "the three-condition definition" and, of course, it can be written in  $\varepsilon$ -  $\delta$  terms as well. Leithold's book is very popular in Brazil (translated into Portuguese: Leithold (1994)) and is currently used in many calculus courses. In Hardy (1952) and Spivak (1967) there is also this idea that a function can be discontinuous even for a point that is not in the domain of the function while Lang (1968) and Lima (1989) only consider points in the domain of the function.

There are also inconsistencies evident in the texts. Despite having defined continuity as it is quoted above, Courant and John write further on:

"We can illuminate the definition of continuity by contrast with examples of discontinuity, examples which do not fit the definition above."

(Courant & John, 1965, p.36)

In page 37, they exemplify using  $f(x) = \frac{1}{x}$ , as an example of “infinite discontinuity” at the point  $x = 0$ . This is clearly contradictory with the given definition (as the function is not defined at 0).

In a personal letter, Tall (1995) discussed these different views, saying that nowadays most mathematicians, on the basis of the ideas of sets and functions would define continuity only considering the domain in which the function is defined. However, for engineers and mathematicians of earlier generations, a function like  $f(x) = \frac{1}{x}$  is considered discontinuous at the origin. He explains these different views using the different modes of mental representations: enactive, iconic and formal suggested by Bruner. In his words:

“Early ideas about continuity are *enactive*, the graph of a function is drawn in a “continuous” movement without taking the pencil from the paper, so naive continuity means enactively “going through all intermediate values”. However, when the picture of the graph is drawn and viewed as a set of points it becomes a static *iconic* representation. The naive idea of continuity is now to do with *connectedness* - is it in one piece? Finally, when the whole thing is formulated in, say  $\epsilon$ - $\delta$  terms, it becomes more a *formal symbolic* concept” (Tall, 1995)

Thus, it is not really a question of saying that one is correct and the other is wrong, it is more that one definition is more modern. Having said that, any author should take great care to be consistent: once you have defined continuity, stick with your definition. Of course, there is also the problem of communication between people using two non-equivalent definitions.

Besides this problem of books using different definitions simultaneously, which may lead students to form conflicting concept images, there is another problem related to how students use definitions. Vinner (1991) claims that most students do not use the formal definition, but use concept images which are not correct: they can be applied in some cases, but not in all. One of the examples he gives is the image of a tangent line at a point as a line which touches a curve only at that point, this is only true in certain cases, such as that of the circle, which is usually used to introduce students to the concept of tangent.

As already said, the concept of continuity of a function is of vital importance to the understanding of the proof of the FTC, as well as to giving a visual interpretation for it. Thus, one of the aims of this study will be, following suggestions in the literature and always bearing in mind the understanding of the FTC, to discover if the students know what a continuous function is, if they understand the graphical meaning of its definition, what concept images are associated in their minds with continuity and if they make use of the definition.

## **2.6 Limits, derivatives and integrals**

Limit is regarded as the powerful tool that is going to enable one to solve problems which, without it, could only be solved approximately. It also constitutes the foundation on which the definitions of continuity, derivative and integral of a function will be based. It is in the interests of this research to see how difficulties with the concept of limit might interfere with these definitions.

First of all, as Cornu (1991) pointed out, the teaching of limits is often obscured by algebraic manipulations. After quickly learning the definition, students tend to concentrate on solving exercises in which they are required to perceive what the algebraic trick is that they will have to apply in order to find the limit. This procedure distances them from dealing with and thinking about conceptual issues.

Besides this problem there is another related to students' dynamic view of the limit, a problem discussed by various authors, e.g.: Tall and Vinner (1981), Orton (1983a), Heid (1988) and Cornu (1991). In different ways these authors point out that the dynamic view that a limit gets closer may lead the students to interpret the limit as a process rather than a number. It is interesting to note that in spite of students being able to evaluate limits which will result in numbers, this ability seems to be dissociated in their minds from their view of limit. This view will directly affect students' interpretation of the definitions of a derivative and of a definite integral. The works of Orton (1983a and 1983b) and Heid (1988) address this question.

Orton did his research with 100 students in the age range 16-22 years. He wanted to investigate students' understanding of integration and differentiation. The students were



interviewed individually and were asked to solve some tasks presented to them. Heid (1988) compared the performance of two classes of calculus students who used graphical and symbolic manipulator computer programs with a class which was being taught in the traditional manner.

Among their findings, one is related to the association of a derivative with the slope of a tangent line. According to Orton (1983b), students have great difficulty in understanding the tangent as the limit of the set of secants; they tend to think of lines getting shorter. Heid says that derivatives are interpreted as approximations of slopes of tangent lines instead of being equal to the slopes themselves. Orton (1983a) also discusses the fact that, in relation to the association of integration with the Riemann sum there is a similar problem: in analysing a sequence whose limit will give the area under a curve, the students thought they would get better values each time but did not state that the limit would give the exact area under the curve. Orton appropriately says that “the students had little conception of the power of the limiting process in mathematics” (Orton, 1983a, p.7).

This problem is also reflected in students’ understanding of rate of change, which in the view of both authors is very difficult for them to understand. There are some specific issues with rate of change that are not directly involved with the concept of limit, as for example, interpreting rate of change as the  $y$ -value. But there are some others directly related to the dynamic view of limit, while making the distinction between rate of change at a point and average rate of change over an interval. In the case of a straight line it is true that the average rate of change over any interval is equal to the rate of change at any point; however, it is not true for every curve and the limit in evaluating the rate of change at a point is often misunderstood.

These are problems which will interfere directly in the understanding of the FTC and in the ability to interpret it graphically. In more than one part of the proof the concept of limit is used. In order to connect the expression  $F'(x) = f(x)$  with the graphical relation between slope and area (always thinking of a what happens at a point), it will be important for the students to have a clear view in their minds of the power of the limiting process.

## 2.7 Research on proof

The process of proving has received considerable attention from researchers in mathematics education. It is becoming clearer that mathematics will not be made easier or more accessible, or its teaching more creative for students by simply avoiding mathematical formalism. It has been a challenge to try to incorporate deductive reasoning into educational practice in such a way that students can perceive it as a necessary ingredient of mathematics, which distinguishes it from the other sciences. Leron (1985) says that it is important to consider “how the formalism itself might be improved to become more communicative of the ideas behind it” (p.321). However, in order to improve it, by giving suggestions about how to enhance deductive reasoning in the teaching of mathematics, some more basic questions invite research into mathematics education. They are: how students regard proving in mathematics, how they construct their proofs, what their main misconceptions are about proof, and what the role of proving is in mathematics classes.

Research on proof is very extensive and in order to select studies to be examined, a number of criteria were taken into account. First of all, examination was made not only of studies conducted with undergraduate students, but also of studies carried out with secondary school. There were two reasons for this: one is that in Brazil, although deductive reasoning should be introduced, in the context of Euclidean geometry when students are about 13-14 years old, the fact is that this teaching is very often neglected, and in other contexts proofs are not usually presented, so perhaps they are at the same level as students in other countries who are introduced to proofs earlier. Another reason is that it is considered in this study that some of the processes used by secondary school students may be used by undergraduate students as well: it is not only curriculum differences that influence the learning; some of the difficulties might persist into undergraduate years.

The studies by Balacheff (1988), Chazan (1993) and more recently Hoyles and Healy (described in Hoyles (1997) and Healy and Hoyles (1998) ) are among those which were carried out with secondary school students, while the study of Harel and Sowder (1996 and 1998) relates to undergraduate students.

Besides these studies, which show a concern for analysing the processes of proof from results obtained with students, some others were selected in which the authors, despite not describing in detail any particular experiments or surveys, discuss the meaning of proving, and provide a general view of the role of proving in the classroom and its importance in the understanding of mathematical ideas and topics. The studies of Tall (1989), Hanna (1989), Hersh (1993) and Leron (1983 and 1985), are among those which were selected on this basis.

In the first set of studies cited, the authors were concerned to classify the processes of proving. Balacheff and Chazan compared students' preferences regarding empirical arguments and deductive ones. In a study carried out in France, Balacheff (1988) asked 28 students aged thirteen and fourteen to calculate the number of diagonals of a polygon given the number of vertices. The students worked in pairs and were required, at the end of their task, to write answers as messages to other students of their age. In order to analyse the types of proofs his students produced, he classified them into four types: naive empiricism, crucial experiment, generic example and thought experiment. He used the word proof for all these types because they were recognised as such by the students who produced them.

The first two types could be considered closer to empirical arguments, while in the last two there is a concern for giving reasons for establishing the truth of the results found. Balacheff recognises that some of the obstacles for passing from the first two levels to the last two are the language, which has to become a tool for deductive arguments, and students' understanding of the mathematical ideas involved.

Chazan (1993) conducted interviews with 17 high school students in the United States from geometry classes that had used a specific software in the classroom: the Geometric Supposer. One of the aims of Chazan's research was to investigate to what extent the use of the software might have influenced students in the direction of thinking that measurement of examples was enough to prove geometrical statements, so that they would not appreciate deductive proofs and not realise the need to write them.

The interviewees were asked to compare two types of arguments: a deductive one taken from their textbook and one based on measurement of examples. Previous literature had

indicated that the students' conceptions could be basically divided into two sets of beliefs: evidence is proof and proof is simply evidence. Chazan wanted to understand why this was so.

Several interesting observations of students' attitudes towards deductive reasoning were made. Some students did not understand the role of the 'givens' in a proof. Some did not believe that a proof guarantees that no counter-example exists. Some thought that if they tested the proposition for each kind of triangle (acute, obtuse, right, equilateral and isosceles triangles) and it worked, it would work for all triangles.

As one of the conclusions of his work, Chazan says that he does not believe that the extensive use of measurement of examples in geometry classes would have prejudiced students' appreciation of a proof. He thinks that it gives the students the possibility to compare the two types of arguments: deductive and experimental. He also deduces from the students' comments the importance of emphasising in the classroom the explanatory aspect of proofs, instead of only valuing them as a guarantee that no counter-examples can exist.

Healy and Hoyles (1998) conducted their study with 2549 students aged about 15 years old from 90 schools throughout England and Wales. Two questionnaires were designed: a student questionnaire and a school questionnaire. They aimed at investigating how the students analysed and valued different arguments in geometry and algebra, how they constructed their proofs and what the reasons were why they made such analyses and constructions. This last point is particularly important in their research, since as Hoyles (1997) states, in previous literature on proof, the curriculum emphasis and the previous experience of the students has been given little weight in explaining the results concerning students' view of proving.

Healy and Hoyles found many results, since they not only compared students' preferences regarding deductive and empirical arguments but also analysed the influences of such variables as sex, school, curriculum and teachers to explain some of these results. One of the conclusions of their study is that deductive reasoning is rarely used when students construct a proof, and when they try to make use of it they come up against several obstacles. As they explain:

“It is too clear that many students were not even able to begin to construct a proof and, if they did make a start, could only indicate some relevant information unconnected by logical reasoning.” (Healy and Hoyles, 1998, p.42)

They also concluded that most of the students recognise that proofs are general and see their role much more as one of convincing than of explaining, although many students have no idea what proofs are for. In relation to the influence of schools and curriculum they concluded that, increasing the time devoted to mathematics teaching and the stress the school places on students writing proofs, reduces the possibility of students choosing empirical arguments in algebra and increases the expectation of writing formal proofs in geometry problems.

The studies quoted above were carried out with secondary students. Harel and Sowder (1998) investigated a sample of 128 students who were mainly undergraduate students. They classified students' justifications according to categories and subcategories of proof schemes: the external conviction proof schemes (divided into ritual, authoritarian and symbolic); the empirical proof schemes (divided into inductive and perceptual) and analytical proof schemes (divided into transformational and axiomatic). One of the points of interest of their study is the presence of the external conviction proof schemes. While the last two classes of schemes could be very roughly described as empirical and deductive sorts of argumentation, in the external conviction proof scheme students are not convinced on the basis of any of these sorts of arguments, but on the basis of an external factor, which may be the appearance of the argument itself (ritual scheme), the fact that the teacher or textbook presented it (authoritarian scheme) or even the presence of mathematical symbols (symbolic scheme).

They worked with undergraduate students in two different universities in the United States: San Diego and Purdue Universities. They make a point of frequently emphasising the settings in which the study was applied. They are conscious that some of the difficulties come from high school, as previous research demonstrates.

The studies quoted above suggest that students are using very different sorts of argumentation in order to convince themselves and in order to justify the truth of a proposition. Some of these arguments would be acceptable, depending which level the study is concerned with. However, it is important to bear in mind that the present study

is concerned with undergraduate students from the calculus course. Calculus I is regarded as the first course in which undergraduate students from the scientific and technological areas will meet a more formal mathematics. Thus the results have to be compared and analysed with this particular sample in mind. At this stage they will find definitions and theorems clearly stated and, as Tall (1989) stresses, they need to have in mind the definitions and statements necessary to prove the proposition in question. These are the tools for them to appreciate a proof.

Hanna (1989) and Hersh (1993) say the role of proving is explaining and convincing and that in class it should be primarily one of explaining. Leron (1983 and 1985) proposes a way to improve the way in which proofs are presented in the classroom. He criticises the “linear style”, where the proof follows step by step, from hypothesis to conclusion, and proposes a “structural style”, where the proof is organised in levels, proceeding from the top down, the main idea of the proof coming in the top level. In the case of the FTC, the main idea of its proof was discussed in section 2.2.2 and has to do with the fact that  $f$  has to be continuous.

There is still research to do with undergraduate students and this study will concentrate on basically two issues. The first is students’ view of proving, whether they hold one, both or neither of the beliefs already mentioned, that is that proof is simply evidence or evidence is proof. The second is their appreciation of proof and whether students regard proof as important in mathematics and mathematics teaching and whether they understand how proofs are constructed. The theorem which will be examined is the FTC, but the points raised certainly go beyond its scope.

## **2.8 Research into students’ understanding of the FTC**

Three studies have specifically looked at the learning and teaching of the FTC. The first of them, by Tall (1991b) does not involve any teaching experiment or observations but gives some ideas on visualising the FTC using the computer. The other two by Thompson (1994) and Thomas (1995) describe a teaching experiment with students.

Tall (1991b) discusses the formal aspects linked to the FTC. Of particular importance for this research are his ideas of how thinking about the meaning of the differential in



integration will help in visualising the FTC in a computer graphical program. Discussion of this visualisation leads to the proof of the FTC and why the function  $f(x)$  has to be continuous.

Thompson (1994) worked on a teaching experiment with 19 senior and graduate mathematics students. He explored with students problem situations dealing with calculus and comparisons of speed and distance, volume and surface of area, among other problems. He emphasises that the comprehension of these notions are essential for students to understand  $F'(x)$  as a rate of change. He attributes some difficulties the students had with the FTC to difficulties related to the notions of accumulation, rate of change and rate of accumulation, as well as other problems as a poor image of a function.

Evidences of problems with the concept of a function were also found by Thomas (1995). She followed the progress of 27 undergraduates in sciences, engineering and mathematics in a non-traditional calculus course in which computers were used and the students worked in groups. She followed one of the groups closely and at the end interviewed 3 students from this group and 3 from other groups. The computer system used in her research was the programming language ISETL (Interactive Set Language) and the computer algebra system Maple V. Her aim was to examine how these students learnt the FTC and how the use of computers would affect this learning. Besides citing problems with the function concept she also discussed what she called “a fundamental misconception”, that is the difficulty the students had with the use of the variables in the FTC (the role of the each of the variables expressed in the function  $F(x)$ ). She attributed that to the way the students constructed their function schema, which did not include different types of functions as such defined by integrals.

These studies examine students’ understanding of the FTC and give ideas which will help in improving this understanding. The two latter do not address directly the question of how students’ view and use of more formal aspects in calculus (definitions, theorems and proof) interfere in this understanding. The present study, therefore, will aim at giving closer attention to this aspect.

## 2.9 Summary

Examination of the literature indicates the main points which are relevant for this research. Some of them are related to problems with the concepts of function, limit, continuity, derivative and integral. While the first concept is not new for them, the others, in Brazil, are introduced for the first time in calculus I. Many of the difficulties with these concepts are related to poor images associated with them, which will ultimately interfere in their use. Since the FTC establishes the link between differentiation and integration, under the hypothesis that  $f$  has to be continuous, students' use of graphical images associated with continuity, differentiation and integration and how this will be reflected in their understanding of the FTC will be investigated. Students' difficulties with functions and limits will not be directly addressed, but if they are found to lie behind obstacles students face, reference will be made to them.

The other relevant field, given the aims of this research, is that of students' attitudes to proof. A large number of studies have been conducted with secondary school students and some with undergraduate students, aiming at identifying their conceptions of the processes of proof. It seems that some of the difficulties which are found in secondary school persist in undergraduate teaching. Given that, in Brazil, most of the students are not accustomed to studying proofs before entering university, it will be interesting to examine in this particular sample students' views of proving and how they influence their appreciation of proof, as exemplified by the proof of the FTC.

Research on the learning of the FTC mentioned in the previous section has already pointed out some of the students' obstacles to understanding the FTC. However, there are still some other issues to be further investigated. One of them is related to how students cope with the formal aspects of calculus: definitions, theorems and proofs, and how the way they deal with these aspects may influence them in dealing with the FTC. This study will investigate these issues. The kind of experiment and the methodological procedures to be used in this study are very different from the teaching experiments on the FTC already mentioned (Thompson (1994) and Thomas (1995)), as can be seen in Chapter 3, in which the methodology of the present study is described.



## **CHAPTER 3**

### **DESIGN OF THE STUDY**

#### **3.1 Introduction**

The main method used to research students' understanding of the FTC was questionnaire analysis, followed by interviews with a sub-sample of the students. In addition, activities around the computer were designed along with an interview schedule to probe further students' views and attitudes. All these approaches were designed to be used with first year undergraduate students at the Federal University of Rio de Janeiro (UFRJ), in Rio de Janeiro, Brazil. This chapter contains a general description of the design of the study, including the sample, the curriculum adopted in Brazil (which gives more data to characterise the background of the sample), and a rationale for the choice of methodology adopted. Details of the instruments used in each of the phases will be given in chapters 4, 5, 6 and 7.

#### **3.2 The sample**

The sample who took part in this study were undergraduate students finishing calculus I. In Brazil, in order to qualify for study at the university, students have to pass the "vestibular", an entrance examination. This exam is usually held in November or December, the academic year beginning in the following March. In UFRJ, for some courses, there are two periods of entry in each academic year, one in March and the other in August. The best qualified students begin in March, although even those who begin in August are still considered good students, since the entrance examination in UFRJ is one of the most difficult in Rio de Janeiro.

The classes in which the study was applied were from three courses: mathematics, computer science and engineering. Calculus for the students of these courses is usually divided into three courses: calculus I, II III. Depending on the course, some may have calculus IV as well. Each of these courses lasts four months and a half, 90 hours each,

from March to July or from August to December. In one of the classes in which the main study was applied, a mathematics class held during the evenings, the structure is a little different and will be explained in the description of the sample in chapter 5.

### **3.3 The curriculum of Calculus I**

In Brazil, students who begin a university course usually have little or no previous knowledge of the concepts of limit, differentiation and integration; they usually meet these concepts for the first time in calculus I. The syllabus for calculus I given to the teachers of UFRJ consists of the following topics:

- . Limits: definition, theorems on limits, one-sided limits, limits at the infinity and infinite limits, asymptotes;
- . Continuity: definition, theorems on continuity;
- . The derivative: the tangent line, definition of derivative, the relation between differentiation and continuity;
- . Applications of the derivative: related rates, maximum and minimum values of a function, Rolle's theorem and the mean value theorem, L'Hôpital's rule, the first derivative test, the second derivative test, concavity and point of inflection, sketch of graphs;
- . The definite integral: definition (Riemann sum), properties of the definite integral, the mean value theorem for integrals, the Fundamental Theorem of Calculus, improper integrals;
- . Applications of the definite integral: area, volumes of solids of revolution;
- . Inverse functions: the inverse function theorem, inverse trigonometric functions and their derivatives, the logarithmic and the exponential functions,
- . Techniques of integration: integration by parts, the substitution formula, partial fractions.

The textbooks used varies, although a popular one is Leithold's book (Leithold, 1994), an American book translated into Portuguese. The curriculum of calculus I is exactly the same for all scientific-technological areas, however the assessments are not the same.

It should be said that although the students will learn the definitions, and some of the proofs will be given by the teacher in class, this course has no similarity with a course in

analysis, in which much more emphasis is given to theoretical aspects. Only mathematics students will later follow a course in analysis.

### **3.4 Data collection**

Data for the study came from several sources derived from its different phases. The study consisted of three phases : a questionnaire, interviews based on the answers given to the questionnaire and interviews with students engaged in specially designed computer activities.

The questionnaire was divided into two parts The first part consisted of questions evaluating students' understanding of the FTC, continuity, differentiation and integration. The second part consisted of two questions aiming to explore their conceptions of the processes of proving.

The interviews following analysis of the questionnaire responses were planned as a way to explore more deeply some of the arguments the students had used to solve the questions. They were organised in a semi-structured way, some of the questions intended for all the interviewees and some devised on the basis of particular answers to the questionnaire.

The computer activities were used in the last phase to explore in more detail how students visualise some of the concepts related to the FTC, as well as the FTC itself. There was no intention to evaluate the software or to use the activities as a teaching intervention. The computer was used in this research as another window to examine students' understanding of the FTC (see Noss and Hoyles, 1996).

It is important to bear in mind that the main focus of this research was the questionnaire, with the interviews and the interviews around the computer designed to enrich and help to explain some of the data obtained from the questionnaire.

A pilot study was held in November and December 1994 and the main study in November and December 1995 and January 1996. By the time the questionnaire was administered the students had already learnt the FTC but had not finished all the exams bearing on their calculus course. The interviews were undertaken during the exams or after they had finished them.

### **3.5 The pilot study**

The pilot study was undertaken with one computer science class and two engineering classes. Its aim was to try out the instruments and design a methodology of analysis to be used in the main study. First, a questionnaire was given to 72 students, with one hour to complete it.

The questionnaire was marked and twelve of the students were interviewed. These students were chosen either because their answers represented different kinds of arguments or because their answers had arguments which deserved to be followed more closely. The interviews lasted between 30 minutes and one hour.

The interviews around the computer were very informal at this stage. The aim was to help in the design of computer activities for the main study and to make it clear which criteria should be taken into account when choosing computer software to work with students in this research. Five of the twelve students took part in this last phase. Two of them worked with Graphic Calculus (Tall, Blockland and Kok, 1990) and the other three with Logo. The interviews lasted between one and a half and two hours each.

All the phases of the pilot study were conducted by the researcher. The analysis of the data obtained in the pilot study was very important for suggesting modifications to be incorporated into the main study.

### **3.6 The main study**

The main study was conducted with two mathematics classes, one computer science class and three engineering classes. The questionnaire was distributed to 148 students in all, who had one and a half hours to answer it.

Following the application of the questionnaire, 26 students were interviewed. The main strategies used to solve the questions and the answers which needed to be investigated more closely were all represented in this group. Seventeen of these interviews were chosen for detailed analysis.

Nine of the 17 students were selected to work with the computer, each was interviewed individually. The interviews were task based and lasted about four hours. The activities were designed incorporating the use of the Graphic Calculus software. The interviews of seven out of the nine students were analysed in detail.

The main study questionnaire was administered by two researchers in mathematics education. The interviews were all conducted by the researcher. The data was analysed using both, quantitative and qualitative methods, depending on the nature and on the purpose of this analysis.

## **CHAPTER 4**

### **THE PILOT STUDY**

#### **4.1 Introduction**

The pilot study which will be the subject of this chapter took place at the Federal University of Rio de Janeiro (UFRJ) in November and December 1994. It was administered to three classes of first-year undergraduate students: one computer science class and two engineering classes (chemical and industrial engineering). The pilot study was basically divided into three phases: administration of the questionnaire, follow-up interviews based on responses to the questionnaire and computer task-based interviews. Each of these phases will be described next.

#### **4.2 The questionnaire**

The pilot study questionnaire was important not only for examining up to which point the questions achieved their aims but also for deciding the optimal setting in which to administer it. In this section, all the important issues which emerged from the pilot study and subsequently influenced the design of the main study will be discussed.

##### **4.2.1 The sample and the setting**

Altogether, 72 students answered the questionnaire: 29 students from the computer science class, 32 from the industrial engineering class and 11 students from the chemistry engineering class.

The calculus classes last 1:50 min each and the time planned for the questionnaire was 1 hour. Three different situations were tried. In the computer science class the questionnaire was given in the first hour of the class, in the industrial engineering class it was given in the last sixty minutes and in the chemistry engineering class it was given at a different time which the teacher had previously asked the students to come at.

Comparing these situations, those during class time were definitely the most successful in making sure that sufficient students took part. It should be said that none of the students were obliged to participate in this research.

#### 4.2.2 Structure and aims

The pilot version of the questionnaire is given in Appendix 1. The questionnaire consisted of two parts. The first part contained five questions and aimed to evaluate if the students knew the FTC, understood the concepts related to it and knew how to apply them. Bearing that in mind, some questions were designed related more directly to the FTC, aiming to verify if students knew that  $F'(x) = f(x)$  and could apply this result both in a purely analytical context and in a graphical context. Other questions were designed to explore the following points: understanding the concept of continuity, understanding that the derivative and the integral functions are functions as well, and understanding how the graph of the integral function is built up. These are points in which students meet difficulties, as previous research indicates (see Chap. 2, sections 2.4 and 2.5). It was decided that the first question should be of a type students are used to meeting in their textbook, to give them confidence. Thus the decision was taken to choose a question in which they had to evaluate integrals.

All the questions were open-ended with the exception of question 4, which contained a multiple choice part. In this question, instead of students drawing graphs, they were to choose one correct graph for each of the items of the question and then justify their choices. Multiple choice was opted to guarantee that even if students did not know how to sketch the graph of the integral function  $F(x)$  precisely, they could by visual inspection and using the association of integral with area or derivative with slope of the tangent line, verify which graph was the correct.

After finishing the first part they were given the second part, with two questions that aimed to probe their beliefs about proof in the context of the FTC. This part was influenced by research on proof conducted with students (see section 2.7). Particularly the work of Chazan (1993), helped to make clear, by the time the study was planned, what the aims of each of the questions should be (stated above).

The aims of each question are as follows:

### Part One

Q.1 - To give confidence to the students and make sure they know how to apply the corollary of the FTC to evaluate integrals, and in item 1b to see also if they use a graphical strategy to solve it.

Q.2 - To find out if they apply the theorem directly, even in a situation where they can easily find the elementary primitive and to discover if they understand the role of the variables in the functions  $g(x)$  and  $g'(x)$ .

Q.3 - To find out if they know what a continuous function is.

Q.4 - To find out if they have a good graphical understanding of the integral function  $F(x)$  and of the derivative function  $F'(x)$ .

Q.5 - To find out if they know how to state the FTC (or its corollary) or, if not, what ideas come to their minds when they are asked to do state it.

### Part Two

To find out if they hold one, both or none of the following beliefs in connection with proving: evidence is proof (question 1) and proof is simply evidence (question 2).

#### **4.2.3 Analysis of the questions**

In order to analyse the questionnaire, categories representing patterns of answers were identified for each of the questions. Some of the categories had been chosen previously, based on mathematical strategies that were well known to be used in answering these types of questions. Others were identified after examining the results of the questionnaire: they represented either interesting arguments or very frequently kinds of answers. Each of the questions was analysed separately, the results will be described below.



## First part: students' knowledge of the FTC and related concepts

### Question 1

In the first question the students had to evaluate the integrals: (1a),  $\int_0^2 (3x^2 - 4x + 1)dx$  and (1b),  $\int_{-1}^1 |x|dx$ . The answers to this question were divided into three groups: right (R), wrong (W) or those in which was added a constant (+ C) to the correct answer. The right answers to question 1b were divided into two subcategories: those solved analytically ( $R_a$ ) and those solved graphically ( $R_g$ ).

The answer was classified as right in question 1a even if the method of solution was correct but the final answer was wrong owing to a slip in arithmetic. Only correct answers, with no mistakes, were coded as R in question 1b. The reason why 1b was evaluated differently from 1a was that it was easier to check visually what the final answer to 1b should be.

Thus, the categories set up for this question were:

R: Right  $\begin{cases} R_a: \text{solved analytically} \\ R_g: \text{solved graphically} \end{cases}$

C: A constant (+ C) was added to the correct answer

W: Wrong

X: Not answered

The results found were:

**TABLE 4.1** Distribution of categories of responses to question 1- part I (% *italics*)

Question 1a		Question 1b		
Category	Total (n=72)	Category	Subcategory	Total (n=72)
R	64 88.9		$R_a$	19
C	3 4.2	R		26.4
W	4 5.5		$R_g$	2
X	1 1.4			2.8
		C		1 1.4
		W		43 59.7
		X		7 9.7

The result of question 1a was very good, with 88.9 % of the students answering it correctly. However, for question 1b, only 29.2 % gave the correct answer.

The only unexpected point which appeared in the analysis of (1a) was the addition of the constant ( $C$ ) to the correct answer by 3 students. These students added  $C$  to the right (1 student) or wrong (2 students) answers in (1b).

Most of the students who answered the question incorrectly gave 0 as the result. A very interesting point was that some students wrote  $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$  in the space reserved for solving the question, but did not use this information. It was interpreted that they had probably learnt the definition of a modular function in secondary school but did not know how to apply it in a new situation.

One of the aims of question 1b was to evaluate if the students associated the concept of integral with the geometric concept of area. Analysis of the answers shows that only 2.8 % made use of this association. This result reveals that either they did not know this association or, if they knew it, were not accustomed to making use of it to solve an integral.

## Question 2

In question 2, the students were given  $g(x) = \int_2^x (2t - 1)dt$  and had to find  $g(2)$ ,  $g'(x)$  and  $g'(2)$  in items (a), (b) and (c) respectively. When preparing this question a doubt arose in connection with what kind of function to choose to be integrated. It could be a function whose elementary primitive could easily be found, such as  $f(t) = 2t - 1$ , or a function such as  $f(t) = e^{-t^2}$  for which the students could not find any elementary primitive. The decision to select the first one was taken considering that it would show to what extent the FTC was a tool which would come to students' minds to solve the question directly.

This question was very difficult to analyse and needed to be reformulated for the main study. The reason for this difficulty was that question 2 should reveal whether the students would need to solve the integral or not in order to answer (2a) and (2b).

However, some of the students did not make clear which strategy they used in question 2b. This was because some of them solved the integral in (2a) and it was impossible to determine whether they used this result in (2b) or not. There was also the case of the students who solved the integral beside the general statement of the question instead of underneath one of the items, so possibly used this result for both items. The criteria taken for classifying the answers was the following: if the student had not solved the integral in the space provided for question 2b, it was considered that the question had been solved directly (i.e. that the FTC had been applied); and if the student had solved it beside the general statement of the question, it was considered he had first solved the integral.

The categories used to classify the responses to this question were the following:

R: Right

W: Wrong

X: Not answered

In questions 2a, some students answered 0 directly without solving the integral, while others solved the integral first. In question 2b, some students applied the FTC, while others first solved the integral and so found the derivative of it. Thus for questions 2a and 2b category R was divided into 2 subcategories:

Right  $\begin{cases} R_d: \text{directly, without solving the integral} \\ R_p: \text{the elementary primitive was found} \end{cases}$

In question 2b, some wrong answers deserved to be separated into subcategories. They were:  $2t-1$  or  $2t-1+C$  (mixing up the variables) and  $(2x-1)dx$  (keeping the differential element in the correct answer). The wrong answers in (2b) were divided into:

W: Wrong  $\begin{cases} W_v: \text{mixing up variables} \\ W_d: \text{misunderstanding of the differential element} \\ W_o: \text{other mistakes} \end{cases}$

The results of question 2 according to the categories described above were:

**TABLE 4.2** Distribution of categories of responses to question 2 - part I (% italics)

Question 2a			
<i>Category</i>	<i>Subcategory</i>	<i>Total (n=72)</i>	
R	R <sub>d</sub>	30	
		41.7	64
	R <sub>p</sub>	34	88.9
		47.2	
W			7
			9.7
X			1
			1.4

Question 2b			
<i>Category</i>	<i>Subcategory</i>	<i>Total (n=72)</i>	
R	R <sub>d</sub>	15	
		20.8	37
	R <sub>p</sub>	22	51.4
		30.6	
W	W <sub>v</sub>	11	
		15.3	
	W <sub>d</sub>	2	29
		2.8	40.3
	W <sub>o</sub>	16	
		22.2	
X			6
			8.3

Question 2c	
<i>Category</i>	<i>Total (n=72)</i>
R	45
	62.5
W	21
	29.2
X	6
	8.3

The results show that most of the students solved question 2 correctly (88.9% in 2a, 51.4% in 2b and 62.5% in 2c). Question 2a was the one with the best results: only 7 students gave an incorrect solution. From the analysis of (2a) on the basis of the questionnaire only, it seems that students could recognise  $g(x)$  as a function, at least in

respect of replacing  $x$  by 2 in  $g(x)$ . It was not clear, however, if they associated the integral with the image of a null area or if they memorised the fact that when the upper limit of the integral is equal to the lower limit, the value of the integral is 0.

The analysis of question 2b had to be done keeping in mind possible errors in classification (as mentioned above). As Table 4.2 shows, a significant mistake which appeared was to write  $2t-1$  or  $2t-1+ C$  as answers (11 students did so), which shows that the students had problems in understanding the role of variables in the FTC. Another mistake made by 2 students which was significant, if not for the number of those who made it but for what it might mean, was to answer  $(2x-1) dx$ . These 2 students did not seem to understand the meaning of the differential element.

Question 2c did not really raise many interesting points for discussion apart from the fact that 5 students, who had their answers classified as  $W_t$  in (2b), replaced  $t$  by 2 in  $2t-1$ . They thus obtained the correct answer, but did not realise the mistake in changing variables they had previously made. These students had their answers classified as R.

### Question 3

In question 3a the students had to define a continuous function; in (3b) sketch the graph of a discontinuous function; and in (3c) justify why the graph they had drawn was that of a discontinuous function.

The definition of a continuous function taken when devising this question was: a function is continuous if it is continuous at every point  $a$  in its domain ( $\lim_{x \rightarrow a} f(x) = f(a)$ ).

Therefore, examining students' answers to this question, it was possible to see that many students gave examples in (3b) of graphs of functions as  $f(x) = \frac{1}{x}$ , which are not defined at one point (in this case 0). According to Leithold (1994), one of the textbooks used,  $f(x) = \frac{1}{x}$  is discontinuous at 0.

Another example given in (3b) was a graph of a function with a small hole. This idea probably comes also from their textbook, which states that it is sufficient that  $f$  is not

defined at a point for  $f$  to be discontinuous at this point (see the first condition in the three-condition definition, section 2.5).

The analysis of this question led to an investigation of other textbooks, the result of which is reported in Chapter 2 (section 2.5). Owing to these problems, the answers to question 3 were not categorised. The question had to be reformulated and another approach taken to investigate students' understanding of a continuous function in the main study.

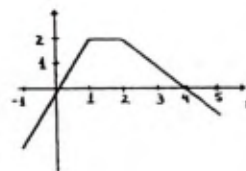
#### Question 4

In question 4, given the graph of a function  $f(x)$ , the students were required to recognise the graphs of  $F(x) = \int_{-1}^x f(t)dt$  (4a) and of  $F'(x)$  (4b). Figure 4.1 shows question 4.

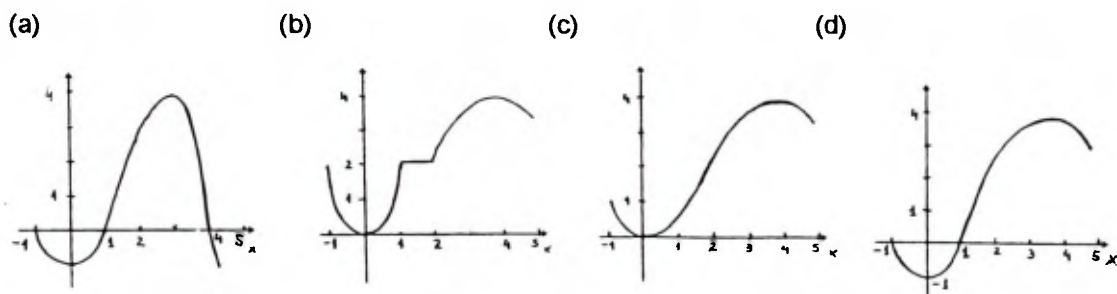
**FIGURE 4.1** Question 4

4) The graph at the right side is the graph of function

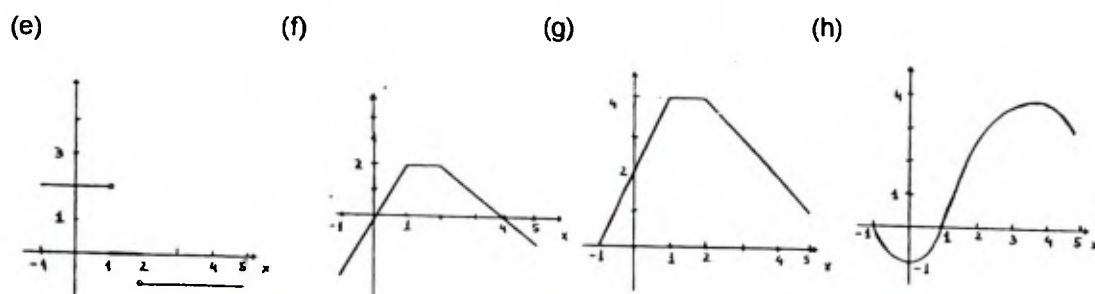
$f(x)$  defined in the interval  $[-1, 5]$ .



a) Which of the graphs (a), (b), (c), (d) could be the graph of  $F(x) = \int_{-1}^x f(t)dt$ ?.....Justify your choice.



b) Which of the graphs (e), (f), (g), (h) could be the graph of  $F'(x)$ ?...Justify your choice.



The table below displays the number of students who chose each item for question 4a. One student answered “none” instead of choosing one of the items.

**TABLE 4.3** Distribution of students’ choices in question 4a (*% in italics*)

<i>Choice</i>	<i>Total (n=72)</i>
a	10 <i>13.9</i>
b	10 <i>13.9</i>
c	22 <i>30.5</i>
d	18 <i>25.0</i>
none	1 <i>1.4</i>
no answer	11 <i>15.3</i>

Table 4.3 shows that the results were very widely distributed, only 25.0 % of the students choosing (d), the correct choice. The justifications for the choices were of various kinds. Some students considered  $f = F'$  and either examined the sign of  $f$  to decide where  $F$  is increasing or decreasing, or analysed the intervals where  $f$  is increasing or decreasing to decide about the concavity in the graph of  $F$  (category D: derivative). Others found the analytic expressions of the three lines in the graph of  $f$  and integrated them in order to find  $F$ , thinking of  $F$  as the primitive of  $f$  (category E: expression). Some analysed the area under the graph of  $f$  to find the graph of  $F$  (category A: area). The remaining kind of answers categorised were from the students who only wrote that  $F(-1) = 0$  in the justification (category S: single point). The categories can be summarised as follows:

D:  $f$  was seen as the derivative of  $F$

E: The analytic expressions of the lines in the graph of  $f$  were found in order to find  $F$

A: Analysed the area under the graph of  $f$

S:  $F(-1) = 0$  was the only aspect considered.

O: Others

X: Not answered

Table 4.4 shows for the choices of (c) and (d) the absolute number and the percentage of answers per strategy (according to the categories described above).

**TABLE 4.4** Distribution of categories of justifications for the choices of (c) and (d) in question 4a  
(% in *italics*)

<i>c/d</i>	<i>Category</i>	<i>Total (n = 72)</i>
<i>c</i>	D	5 <i>6.9</i>
	E	8 <i>11.1</i>
	A	1 <i>1.4</i>
	O	5 <i>6.9</i>
	X	3 <i>4.2</i>
	<i>Total</i>	22 <i>30.5</i>
<i>d</i>	D	3 <i>4.2</i>
	E	2 <i>2.8</i>
	A	2 <i>2.8</i>
	S	2 <i>2.8</i>
	O	6 <i>8.3</i>
	X	3 <i>4.2</i>
	<i>Total</i>	18 <i>25.0</i>

Table 4.4 shows that the most frequent strategy used was to consider  $F$  as the primitive of  $f$  and try to find the analytic expressions of the lines of  $f$  (category E). In fact, altogether 13 students had their answers classified as E, 10 chose (c) or (d) and the others 3 made other choices. It should be said that many mistakes were made in this process of finding  $F$ . On the other hand not many answers came into category A, which shows that few of the students had in mind the association of definite integral with area. In fact, only 3 students made this association.

Category S comprises the answers of the two students who only wrote that  $F(-1) = 0$ . Although these students made the correct choice, they probably took other aspects into consideration as well, if not, they would have two choices: (a) or (d).



Many students had their answers classified as O (19 in all). Among the justifications classified as O there are some which clearly show false conceptions of integration. Very representative of this is the one which shows a conception of integration as having to add 1 to the degree of the function (which works in a polynomial function) and uses this rule for a general expression of  $f(x)$ . One student wrote in the space reserved for his justification:

$$F(x) = \left[ \frac{f^2(t)}{2} \right]_{-1}^x = \left[ \frac{f^2(x)}{2} - \frac{f^2(-1)}{2} \right]$$

$$F(-1) = 0$$

$$F(4) = \left[ \frac{f^2(4)}{2} - \frac{f^2(-1)}{2} \right]$$

$$F(4) > 0$$

Moving to the analysis of question 4b, the results were better. Table 4.5 shows the distribution of the students' choices:

**TABLE 4.5** Distribution of students' choices in question 4b (% in *italics*)

<i>Choice</i>	<i>Total (n=72)</i>
<b>e</b>	23 <i>32.0</i>
<b>f</b>	33 <i>45.8</i>
<b>g</b>	4 <i>5.6</i>
<b>h</b>	5 <i>6.9</i>
no answer	7 <i>9.7</i>

As Table 4.5 shows, 45.8% of the students marked the correct item (f). The second most popular choice was (e): 32.0 % of the students chose it. Of those who marked (e), 18 made it clear in the justification that the reason why they did so was that they found the derivative of  $f(x)$  instead of  $F(x)$ . In the interview, one of the students who chose (e) said that had not even paid attention to the small or capital letter. The same might be the case with other students.

Of the students who marked (f), only one explicitly wrote that his choice was due to the FTC. There were students who wrote that  $F'(x) = f(x)$  and some that the derivative of the integral of a function is the function itself or something similar. Since they were still

making reference to the FTC, these sorts of answers were put into the same category (category F: FTC).

Another sort of relevant justification that appeared was to analyse the graph chosen in (4a) in order to see which graph in (4b) would correspond to its derivative (category P: procedural). To summarise, the different categories set up for the justifications of this question were:

F: Used the FTC

P: Procedural reasoning: used (4a)

O: Others

X: Not answered

Table 4.6 shows for item (f), the absolute number and percentage of answers per strategy (according to the categories described above).

**TABLE 4.6** Distribution of categories of justifications for the choice of (f) in question 4b (% in *italics*)

<i>Category</i>	<i>Total (n = 72)</i>
F	20 <i>27.8</i>
P	5 <i>6.9</i>
O	6 <i>8.3</i>
X	2 <i>2.8</i>
<i>Total</i>	33 <i>45.8</i>

Table 4.6 shows that most of the students who marked (f), 20 in all, did so it by the correct and most direct reasoning (category F). However, it still does not seem much, given that the question was very simple if students knew the FTC.

Question 4 was useful for analysing some of the aspects of visualisation involved in the FTC: recognising graphically the relation between the functions  $f(x)$  and  $F(x)$  stated by the FTC and understanding how the graph of  $F(x)$  is built up. The justifications the students gave were important for judging whether they had chosen the correct item for

the correct reason and for analysing how they had established the relation between  $f(x)$  and  $F(x)$ .

### Question 5

In question 5 the students had to state the FTC (or its corollary). The answers to this question were very difficult to classify. Confusion over the FTC was such that first a rough classification was made. The only point analysed was whether the students had mentioned that  $f$  had to be continuous and  $F' = f$  (or expressed this same idea using words like primitive or antiderivative). Only one student mentioned both, but he did not state the FTC correctly, one of his problem being that he was confused over variables in the function  $F(x)$ , as his answer shows:

“Given  $f(x)$ , continuous, there is  $F(x)$  such that  $F'(x) = f(x)$  and  $F(x) = \int_a^b f(x)$  is the area of the graph of  $F(x)$  between the points  $a$  and  $b$ .”

Considering the number of students who only mentioned that  $F'(x) = f(x)$  (or expressed this idea), the result was still very poor: 16 students mentioned it.

One student, despite mentioning that  $f$  had to be continuous, gave an answer that had nothing to do with the FTC, it is as follows:

“To every continuous function in an interval  $[a,b]$  there is a number  $c$  between  $a, b$  such that  $f(c) = \frac{f(a)-f(b)}{b-a}$ .”

He probably thought the FTC was the mean value theorem and tried to state it, but the statement he gave was wrong.

After this first classification a second one was tried. This time what was mainly taken into consideration was whether the students had mentioned that  $F'(x) = f(x)$ . If they had mentioned it, their answers were classified in category A: antiderivative. If they had not, they were classified in category N: not mentioned  $F' = f$ . There was also the case of students who, instead of stating the general statement, gave a particular example in which the FTC was applied. These answers were put into a separate category (category E: example).

After this process of analysis the categories developed were as follows:

A: Mentioned  $F' = f$  or expressed this idea in words as antiderivative

N: Not mentioned that  $F' = f$

E: Gave an example

X: Not answered

Table 4.7 shows the results according to these categories:

**TABLE 4.7** Distribution of categories of responses to question 5 (*% in italics*)

<i>Category</i>	<i>Total (n = 72)</i>
A	16 <i>22.2</i>
N	29 <i>40.3</i>
E	3 <i>4.2</i>
X	24 <i>33.3</i>

Table 4.7 confirms that few students (only 22.2 %) showed they knew that the FTC states that  $F' = f$ . The percentage of students who gave no answer, 33.3%, is also significant.

### Students' conceptions of proof: second part

#### Question 1

In question 1 students were put into a situation where they had to decide whether, on the basis of only three examples, they could state the FTC. The statement of it follows:

**FIGURE 4.2** Question 1 - part II

1) It is true that:

a) if  $f(x) = \int_0^x 2t dt$ , then  $f'(x) = 2x$ ;

b) if  $h(x) = \int_{-\frac{\pi}{2}}^x \sin t dt$ , then  $h'(x) = \sin x$ ;

c) if  $m(y) = \int_{-1}^y 3 dt$ , then  $m'(y) = 3$ .

Considering only the data above, is it possible to affirm that for any continuous function,

$$\text{if } g(x) = \int_a^x f(t) dt, \text{ then } g'(x) = f(x) ?$$

Considering the aims of this question, the students' answers could be basically divided into two types: those in which the need for a proof was affirmed and those which clearly reflected the belief that evidence is proof. However, there were other kinds of answers which revealed that the students concerned had not understood the aim of the question: some made a reference to the FTC or tried to justify the results and some thought it was a trick question. The categories which emerged from these criteria will now be described.

The first category was formed of the justifications of students who gave the correct answer to the question, saying "no" (category P: proving). In contrast to this, the second category was formed of the students who seemed to hold the belief that evidence is proof (category E: empirical). They answered "yes" because of the examples given and some even added one more example in their answers. In this category there were the answers of two students who considered  $f$  in  $g(x) = \int_a^x f(t)dt$  to be one more example. One of them wrote the three examples again and  $g(x)$  as one more case and the other student wrote that:

"In the examples above the FTC is presented. The same theorem works in the function presented in the statement [ $g(x) = \int_a^x f(t)dt$ ] because  $f(x)$  is a continuous function and  $x$  belongs to the interval  $[a,x]$  where the function is continuous."

There was also in this category the answer of one student who thought that the three examples illustrated different kinds of functions, and so covered all the possibilities necessary for making a generalisation; that was clearly an empirical view of mathematics.

The following three categories comprise the answers of students who probably did not understand the aim of the question. Some students did not pay attention to the word "only" and answered "yes" because they knew it was true by the FTC (or as some expressed it: because differentiation and integration are inverse processes). These answers were classified in category F: FTC. Others also answered "yes" and tried incorrectly to prove the theorem or give some justification why  $g' = f$  (category J: justification).

There were some students who thought that the statement was not correct. A very common objection to the statement was that the interval in which  $f$  was defined should be closed. Probably these students read the statement of the FTC in question 2 of the

same part (in which it was written that  $f$  was defined in a closed interval) and since nothing was said about the interval in question 1, they thought this was missing. Other students said that  $x$  had to be different from  $a$ . Such answers were classified in category S: statement.

Analysis of this question showed that it needed to be reformulated. The first problem arose from the expectation that the students would write “yes” or “no” before giving any justification. However, this did not necessarily happen. In the case of some of the students (8 in all) it was necessary to interpret from the justification given whether they meant “yes” or “no”. In the case of one student it was impossible to say which answer he had chosen. What he wrote was inconclusive, as can be seen from the following:

“By the Fundamental Theorem of Calculus, I believe that this is valid, but I am not sure if the data above are enough to guarantee the truth of the data above.”

Probably he meant ‘by the FTC’ instead of the second ‘the data above’. Although he knew it was true by the FTC, he was uncertain whether he could state the theorem only by giving the examples.

The second problem with the question was that its purpose was not clear to most of the students, as categories F, J and S show.

The different categories can be summarised as follows:

P: Proving as necessary to generalise

E: Empirical view

F: By the FTC or making references to the ideas contained in it

S: Statement is not correct

J: Justified why  $g' = f$  (incorrectly)

O: Others

X: Not answered

The table below displays the results of this question according to the “yes” or “no” answer, bearing in mind that some of the “yes” or “no” were not explicitly written.

**TABLE 4.8** Distribution of categories of responses to question 1- part II (% in italics)

<i>no / yes</i>	<i>Category</i>	<i>Total (n = 72)</i>
no	P	2
		2.8
	S	16
		22.2
	O	1
		1.4
	X	1
		1.4
<i>Total</i>		20
		27.8
yes	E	11
		15.3
	F	24
		33.3
	J	5
		6.9
	O	6
		8.3
	X	2
		2.8
<i>Total</i>		48
		66.6
inconclusive	O	1
		1.4
<i>Total</i>		1
		1.4
no answer	X	3
		4.2
<i>Total</i>		3
		4.2

Table 4.8 shows that the largest percentages (33.3% and 22.2%) correspond respectively to the students who said “yes”, referring to the FTC, and “no” because the statement was not correct. Even considering that these students perhaps did not understand the purpose of the question, they probably also did not have any conviction about the necessity of generalising through proof in mathematics.

## Question 2

In question 2, the students were asked to give a counter-example for the FTC (if they thought it was possible to give one), although the word “counter-example” did not appear in the question. The translated statement of it follows:

---

**FIGURE 4.3** Questions 2 - part II

---

2) The Fundamental Theorem of Calculus can be stated in the following way:

“Let  $f$  be a continuous function on  $[a,b]$  and  $x$  be any number in  $[a,b]$ . If  $f$  is the function defined by  $F(x) = \int_a^x f(t)dt$ , so  $F'(x) = f(x)$ .”

Your proof can be found in several calculus textbooks, as for example Leithold's book.

Can you give an example of a continuous function  $f$  which does not verify the Fundamental Theorem of Calculus?

---

Considering the aim of this question, two main categories were set up: one comprising the students who believed that proofs generalise and that it would therefore be impossible to give a counter-example, and the other comprising the students who thought that there could be counter-examples to theorems. Apart from these two, two more came up: one from the students who did not affirm whether a counter-example exists or not, but only said they had no knowledge of such a function, and one from the students who tried to give a proof for the FTC. In addition, just as in the case of the other questions, there were some answers which could not be classified in any of the categories and so were grouped separately. A description of each of the categories is given below.

Category G (general) comprises the answers given by the students who said “no” because they knew that the FTC was true or that differentiation and integration are inverse processes. Although they did not make any explicit reference to the fact that the FTC was proved, they knew there were no exceptions to the FTC. In contrast, category CE (counter-example), comprises the answers given by the students who believed there are counter-examples of the FTC and thought they were giving one.

Apart from these two kinds of justifications, there were two other significant justifications. The first corresponded to the answers in which students gave an inconclusive argument, saying that they had no knowledge of a function which



contradicted the FTC, category I (inconclusive). The second corresponded to the answers in which students tried to give a proof for the FTC. In every case, this proof reflected many misunderstandings of the theory, category J (justifications). One example of such an answer is the following:

$$\begin{aligned} \text{" Because if } f(x) \text{ is continuous, } F(x) &= \left[ \frac{f^2(t)}{2} \right]_a^x = \frac{f(x)^2}{2} \\ F'(x) &= \frac{2f(x)}{2} \\ F'(x) &= f(x) \text{"} \end{aligned}$$

As can be seen, apart from other mistakes, this student thinks of integration as having to add 1 to the degree of the function.

The various categories are thus the following,

G: Proofs generalise

I: Inconclusive: no knowledge of any examples of such a function

CE: Counter-examples for the FTC exist

J: Justified why  $g' = f$  (incorrectly)

O: Others

X: Not answered

The results are shown in the table which follows (remembering that some of the “yes” or “no” were not explicitly written).

**TABLE 4.9** Distribution of categories of responses to question 2 - part II (% in italics)

<i>no / yes</i>	<i>Category</i>	<i>Total (n = 72)</i>
no	G	23
		<i>31.9</i>
	I	3
		<i>4.2</i>
	J	6
		<i>8.3</i>
	O	1
		<i>1.4</i>
	X	6
		<i>8.3</i>
<i>Total</i>		39
		<i>54.2</i>
yes	CE	19
		<i>26.4</i>
<i>Total</i>		19
		<i>26.4</i>
no answer	X	14
		<i>19.4</i>
<i>Total</i>		14
		<i>19.4</i>

Table 4.9 shows that slightly more than half of the students, 39 students (54.2%) chose “no”, the majority of them (23 students) for the correct reason, categorised as G. However, it is still surprising that 19 students (26.4%) thought they were giving a counter-example. Of the counter-examples given, 3 were a modular function, 3 “a continuous function in an open interval” (as the students wrote), among others of various kinds.

The analysis of the questionnaire shows that this question seems to have achieved its aim. Apparently the students understood the statement of the question well and answered it according to their beliefs. There was, however, the problem of some students who did not answer “yes” or “no”. In question 2 the number of these was greater than in question 1: 17 students gave the justification only.

### **4.3 The interviews based on responses to the questionnaire**

Follow-up interviews were undertaken one or two weeks after the administration of the questionnaire to probe responses to the questionnaire in more detail. The sample was chosen on the basis of the criteria described below.

#### **4.3.1 The sample**

The main criterion used to select the students to take part in the interviews was to choose those whose answers were representative of the different patterns of responses. There should, for example, be students who solved some questions by using graphical approaches as well as students who solved them by procedural approaches. In addition, any student who wanted to be interviewed was allowed to participate. Twelve students came in all, 4 females and 8 males. As the teacher in the industrial engineering class encouraged his students to take part, this was the class with most representatives, 7 students: Gabriela, Julieta, Nair, Daniel, Ernesto, Fábio and Marcos. There were only 3 students from the computer science class: Joana, Igor and Ronaldo, and 2 of the chemistry engineering class: Arthur and Márcio.

#### **4.3.2 The schedule**

The interviews lasted between 30 minutes and 1 hour. They were organised in a semi-structured form. Some questions were asked to all the interviewees and some differed on the basis of particular answers given by them in the questionnaire. Prior to each student interview the student questionnaire was analysed and interesting points marked to be followed up in the interview.

The first two questions were general, asking the students what the first image was that came to their minds when they thought about the derivative and the integral of a function. The next set of questions was formulated taking into account the answers given in the questionnaire. The students had to explain why they had answered the questions in that way, sometimes giving a graphical interpretation to them. Also, if they had not tried to solve a question, they were asked to try and do so and then an attempt was made to discover if they knew the tools they should use to solve it, although they

were not able to apply them. In connection with question 4 they had to find the values of  $F(-1)$ ,  $F(0)$ ,  $F'(-1)$  and  $F'(0)$ .

In part II, if the students answers clearly showed they did not realise the point of the questions (evaluating their view of the role of proving in mathematics), the situations put forward in the questions were rephrased in order to help them understand them better. Next, they were asked if they were convinced of the truth of the FTC, how they had convinced themselves and how they would convince someone else (questions based on an approach suggested by Mason, Burton & Stacey (1982)).

They also answered some questions about their habits of study, whether they studied using the book or their class notes and if they tried to understand the proofs while studying. Afterwards, looking at the proof of the FTC in their book or lecture notes, they answered questions related to the proof, for example where the hypothesis of continuity is used. They were further asked if they understood the graphical meaning of the FTC and if they knew the mean value theorem for integrals.

All the students who came to the interviews did so willingly or at least had agreed to come; there was no pressure and no payment was offered. Some of them only answered the questions posed, adding nothing more, but others volunteered to talk more about the course, about themselves or about their view of their future work.

The idea of adding a set of common questions to the interviews was to enrich students' explanations about particular answers with broader queries related to the proof of the FTC itself, to their views of proving and to their study habits.

#### **4.3.3 Analysis of the interviews**

The interviews were tape-recorded and full transcriptions were made of all of them. They were read bearing in mind that they should complement the results found in the questionnaire. Three points emerged from the analysis of the questionnaire and the examination of the interviews which were considered especially interesting from the point of view of this research, namely, visualisation, variables and mathematical proof. All the quotations related to each of these points were highlighted.

For the purpose of the description of the pilot study, which is not intended to be detailed, this chapter presents the analysis under the following three headings: visualisation, variables and mathematical proof.

### Visualisation

One of the purposes of this research is to identify the role of the visual imagery in students' understanding of the concepts of continuity, differentiation and integration. Responses to questions 1b, 2a and 4 in the first part of the questionnaire and some of the answers in the interview helped to clarify this point. Responses to question 3, related to continuity, raised some very interesting points as well, but need to be modified in order to achieve its aim.

The result of question 1b shows that most of the students did not solve it correctly. In the interview the students were asked to sketch the graph of  $f(x) = |x|$  and say what the graphical meaning of  $\int_1^1 |x| dx$  was. Seven of these students associated it with area, although none of them had explicitly solved the question graphically in the questionnaire. It was clear that these students knew theoretically the association of integration with area but did not know how to make use of it. There was also the case of one student who said that the teacher had never taught the modular function before the questionnaire. Modular function is one of the topics of the secondary school curriculum, so what this student probably meant was that the teacher never gave an example of an integral of a modular function.

The analysis of questions 2a and 4 show that students find it difficult to visualise the function  $F(x) = \int_a^x f(t) dt$ . When most of them see an integral their first reaction is to solve it. The interview confirms this result. When they were asked what  $F(-1)$ ,  $F(0)$ ,  $F'(-1)$  and  $F'(-0)$  would be, with  $F$  being defined as in question 4, some of them did not even look at the graphs they had marked in order to try to find the values. Instead, they engaged in complicated calculations, trying to substitute values in the integral function.

At the beginning of the interview the students were asked what first image came to their minds associated with the derivative and the integral of a function. Six of them said the tangent line in connection with the derivative but only three said area in connection with

the integral. One of the students gave a very interesting answer, showing no visual graphical image linked with integration and confirming the misconception already noticed in the analysis of question 4 of requiring 1 to be added to the degree of the function. The following passage of his interview reveals this:

Researcher: What is the first image that comes to your mind when one speaks about an integral?

Joana: You will increase the value of the function, because when you integrate you always add 1 to the exponent.

At the end of the interview the students were asked if they had some graphical image associated with the FTC. Only one student, Arthur said that he had previously thought about that, five answered “no”, three answered area under the curve, one said she had never seen the graph of an integral and one that the FTC was related to separating the integral into two integrals. Although Arthur tried to associate the two images, area and slope of the tangent line, he said he found this very difficult because he did not understand how to relate these two concepts at a point, indicating that he had difficulty in understanding the limit concept.

If the students are not used to visualising and applying the visual images connected with differentiation and integration, it will be even more difficult for them to understand the graphical meaning behind the FTC.

### Variables

While marking the questionnaire it was possible to identify two main problems students have with variables: recognising the role of  $x$  and  $t$  in the function  $F(x) = \int_a^x f(t)dt$  and understanding why it is necessary to change the variables when applying the FTC in the function  $F$  defined above, such that  $F'(x) = f(x)$ . In fact the second problem can be viewed as a consequence of the first one.

It was very significant that in question 2b, 11 students answered  $2t-1$  and two students answered  $(2x-1) dx$ . The first answer clearly resulted from difficulties with variables and the interview showed that the second one also did so, at least in the case of one of the students, Ernesto. He explained that he had answered in that way because he had to change the variables; probably that was his conception of the FTC. In his own words:

Ernesto: I can consider that the integral is the inverse of the derivative. So, if I am integrating to find the derivative afterwards, it is as if I were going and coming back, I am at the same point, but there will be just a change of variables.

The interview also showed that other kinds of reasoning connected with question 2 and related to difficulties with variables may also have taken place in students' minds. This was the case of one student interviewed, Ronaldo, whose answer to (2a) was the correct value, 0, being classified as  $R_4$ . He explained in the interview that he would substitute 2 for  $t$  in  $f(t) = 2t-1$  twice, subtracting one value from the other (as if  $f(t) = 2t-1$  were the primitive of  $g(x)$ ).

The other part of the interview in which the question of variables was explored was when the students were asked what the values of  $F(0)$  and  $F(-1)$  would be, with  $F$  being defined as in question 4. They were not sure where to substitute 0 and -1 to find  $F(0)$  and  $F(-1)$ . Ronaldo, for example, would replace  $t$  by 0 in order to find  $F(0)$ , as he said:

Researcher: Do you know what the value of  $F(0)$  is?

Ronaldo:  $F(0)$  is 0.

Researcher: Why is it 0? Where did you substitute the 0?

Ronaldo: In  $t$ .

Researcher: Then, is  $F(0)$  the integral from -1 to  $x$  of  $f(0)$ ?

Ronaldo: I think so.

One of the obstacles the students found in evaluating  $F(0)$  was that the function to be integrated was not explicitly defined in  $F$  in question 4 as it was in question 2. Two students mentioned this point in the interview. Nair was one of them and she said that:

Nair: Do you know what is complicated for me? This  $f(t)$  here. If I had, let us suppose that I had a function written here ... I am confused.

In fact, problems with variables are closely related to problems with the concept of a function.

### Mathematical Proof

The initial motivation for investigating how students regard proving in mathematics was the proof of the FTC itself: did it help to explain the theorem in the undergraduate class? The proof of the FTC, at least proof 1 (see section 2.2.2), is very illuminating for the



algebraic and geometric aspects of the theorem and the key point in this proof is to use the fact that  $f$  has to be continuous.

In section 2.2.2 it was mentioned that in order to understand proof 2, the students needed to understand the mean value theorem for integrals (MVTI). In the interview the students were asked about this theorem. Three of them did not even remember having been taught it and none of the others stated it correctly.

Some of the interviewees brought their lecture notes to the interviews, which made it possible to examine how their teachers had proved the FTC. The computer science class teacher followed a structure similar to proof 2 (using the MVTI) and the engineering class teacher a structure similar to proof 1. No information was obtained about how the chemistry engineering class teacher proved it. In none of the lecture notes was it written in the proof itself where the condition that  $f$  has to be continuous was used, but perhaps the teacher said this in class while proving without writing it on the blackboard.

In the interviews the students were asked some questions about the proof itself and about how they regarded proving. Seven of them were asked if the FTC was proved in class. Four answered “yes” and the 3 others said either “no” or that they did not remember. When asked if they had read the proof, it was not clear if they had not read it at all or had read it only in their lecture notes. When they were asked where the condition that  $f$  has to be continuous was used, some of them could indicate where it was stated in their textbooks, but none of them could explain where it was used.

The interviews showed that one of the reasons they had not studied the proofs was that they thought they would not need them in the exams. When they were asked if they had studied the proofs and if the proofs had helped them to understand the theory, there were several interesting answers which show they think they did not need to study the theory. Two of these are given here by way of example:

Igor: Proof ... when I am studying for a test for instance, I don't like to examine that. A test... we can say, the teacher is going to ask us practical things, for example to evaluate .... That is why I don't like to study proofs when I am studying for a test.

Ronaldo: In algebra I learnt a lot of proofs, but in calculus... Perhaps because the only things that come in the test... there isn't any theory, just practical things.



These statements suggest that the theoretical aspects of calculus are neglected in favour of what the students consider practical ones, as for example evaluating derivatives and integrals.

The questions in the second part should have helped to classify the students according to their beliefs about generalising in mathematics. However, as already mentioned, question 1 needed to be reformulated. A problem which the interview revealed was that the students were not used to this sort of question, which made it more difficult to discover what they really believed. Arthur expressed this very clearly, saying in connection with question 1:

Arthur: In this second part I found it was difficult, this part, for us to see if it is possible to affirm or not....Because it is not explored as an exercise, so it is very difficult. We come to the test without expecting this kind of exercise.

Although question 2 seemed to be more clearly understood by the students, one point which emerged from the interview was that some of the students might have thought that a positive answer was associated with it. Two students made this point, one of them saying:

Marcos: When you asked me this ... if you know an example of a continuous function that does not verify the FTC. When you asked me this I was convinced that there must be a function for which the FTC doesn't work. So I tried to find this function.

In the interview, in order to discover students' beliefs in a different way from the questionnaire, the students were asked if they were convinced of the truth of the FTC, how they convinced themselves and how they would convince somebody else. Three different groups came out: the students who believed that proving was necessary to convince, those who believed it was necessary to test with several examples, and those who took a conformist position and thought that if the teacher taught the theorem and if it was written in their textbooks, then they did not have to think about it. This last attitude is illustrated by the following quotations:

Marcos: Am I convinced that it is true? Are you asking me if I believe that what the teacher teaches is true?

Nair: I can't even say whether this is true or not because I didn't analyse it, I didn't stop to reason about it, if this is true. This was shown to me as true.

Julieta: I would say ... this is the FTC, isn't it? Did you understand it? This is the FTC, if you have this you would do this. Take it as true, nobody is going to question you about whether it is right or wrong, they will ask you to apply it.

The quotations given in this section show that some of problems linked with students view of proving in mathematics are closely connected with the way they are taught and with what they are supposed to learn.

#### **4.4 The computer- based interviews**

The computer-task based interviews were very informal. The aim of doing them was to help in the design of the activities around the computer and to make it clearer which criteria should be taken into account when choosing a computer program to work with the students in this research. The students interviewed in this phase worked either with the programming language Logo or with the software Graphic Calculus (Tall, Blockland and Kok, 1990). These interviews will be described briefly in this section.

##### **4.4.1 The sample**

Five students were interviewed in this phase. There was no criterion of selection. The twelve students who had taken part in the interviews about the questionnaire were asked if they wanted to take part in this last phase of the study and were available to do so. Five accepted, all of the industrial engineering class. Three of them: Marcos, Julieta and Nair worked with Logo. The other two: Daniel and Fábio worked with Graphic Calculus.

##### **4.4.2 The schedule**

A brief description follows of how the students used Logo and Graphic Calculus to develop the activities proposed. It should be kept in mind that, as the activities were informal there was no set of written activities that were to be followed systematically. The general idea was to work with several functions over different intervals, aiming to explore the students' understanding of the integral function and of the FTC.

##### **The Logo-based activities**

The activities explored with Logo were conducted in a session of about one and a half hours each. The students worked with procedures taken from Lewis (1990). It was not

expected that they would do any programming, since they had not worked with Logo before and the time was very short. Pascal was the language these students had learnt in the first semester at the university. Nevertheless, during the interviews, using the Edit command, they could see how the procedures were written. This made it possible to ask questions in order to see if they understood why the procedure was written in that way, paying attention to the mathematical ideas used.

The students worked mainly with two procedures: areagraph and slopeareagraph. Areagraph takes a function rule, an initial and final value of  $x$  for the domain of the function and the number of intervals into which the domain established should be divided. Areagraph outputs an approximate value for the area between the graph of the function and the axes  $x$  in the domain. It also plots the graph of the area function. Areagraph uses a trapezoidal approximation.

Slopeareagraph plots the graph of the derivative function of the area function. The value of the derivative in an interval is determined by the value of the coefficient of the secant line built which joins the end points in this interval. Like areagraph, slopeareagraph takes in a function rule, an initial and final value of  $x$  for the domain of the function and the number of intervals into which the domain should be divided. As a consequence of the FTC, when calling up the slopeareagraph of a continuous function, what should appear on the screen is the graph (approximate) of the function whose rule was given as the output.

The general design of the interviews was the following. First the students were asked to draw on the screen the graph of the area function of any constant function they could choose. They were to compare the different results produced when changing the intervals. This was done in order to explore why, depending on the interval chosen, the graph of  $F(x)$  could assume different values at the same point  $x$ . They were often asked to predict the graph that would appear on the screen as a result of changing the intervals. After working with constant functions they were asked to write the area function  $F(x)$  as an integral function (they should answer  $F(x) = \int_a^x f(t)dt$ ).

Next, the students were asked to work with some linear functions and also with the modular function  $f(x) = |x|$ . Following that, they had to draw the graph of the derivative

function of the area function of the functions they had just worked on. Their knowledge of the FTC was evaluated from that point. The proof was examined and questions were asked to find out if they could relate aspects of the proof with what they saw on the screen.

### Graphic Calculus-based activities

Graphic Calculus is a software first designed and implemented by David Tall for the BBC computer and later programmed and developed by Piet Van Blockland for IBM compatible computers. The students working with Graphic Calculus are offered a Menu to choose which program they want to work with. In this study they worked with two programs, Gradient: Introducing Differentiation and Area: Introducing Integration .

The Gradient program has essentially two options: (1) drawing chords through a selected point and (2) gradient curve. In the first option an  $x$ -coordinate is chosen to be kept fixed while the other point will be away at an initially default distance of 1 (which can be changed). This point will get closer and closer and a table is shown relating  $\Delta x$  with the respective slope of the chord which joins the two points. The second option shows graphically how the gradient curve is plotted, drawing the chords through  $x$  (varying over the domain) and  $x+h$  ( $h$  chosen by the student).

The program Area has two options: (1) area picture and (2) plot area function. The area picture outputs the graph of the function and demonstrates graphically how the area approximation can be evaluated by using one of the rules: left, middle, random, trapezium or Simpson. Plot Area Function plays the same role as Areagraph in Logo, with the difference that, with each step (interval) it advances, it shows the value of the area up to that point, and not only the final value of the area. For both of these options the starting and ending points have to be chosen, as well as the number of intervals into which the domain is to be divided.

The general design was the same, but the activities varied a bit, since the program offered different options to use. The students first worked with constant functions. They used area picture and plot area function. They chose the number of steps and the kind of approximation they would work on. They were asked to guess the formula of the primitive function and to check if it was correct. They used for this an option named

“input primitive” provided by the program and tried over different intervals. They were also asked to write  $F(x)$  as an integral function, using one of the  $f(x)$  they had dealt with.

They repeated the same activities using linear functions. Following that, they were asked to go to “Gradient” to plot the gradient function of the primitives. At this point they were asked questions more directly related to the FTC. At the end they were asked to draw on the screen the graph of  $f(x) = \sin x$ , from  $x = 1$  to  $1.001$  and  $y = -2$  to  $2$ , as suggested in the manual of Graphic Calculus (p. 21). With these scales the graph is stretched horizontally and appears on the screen as the graph of a constant function. This activity, using this image, was done in order to explore students’ understanding of :

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

#### 4.4.3 Analysis of the interviews

The interviews were tape-recorded and practically complete transcriptions were made of all of them (points only concerning explanations about how to use the software were not transcribed). They were examined bearing in mind what they could add to the points that stood out in the analysis of the first interviews: visualisation, variables and mathematical proof. These same headings will be used to describe the results of this analysis.

##### Visualisation

Analysis of the questionnaire and of the first interviews showed that it was difficult for the students to visualise the function  $F(x) = \int_a^x f(t)dt$ . In the interviews around the computer, one of the aspects examined, in order to see how they coped with this function, was whether they could predict the formula of the graph of  $F(x)$  that would appear on the screen. The interviews with both: Logo and Graphic Calculus showed that if  $F(x) = ax + b$ , it was easy for them to find the  $a$ , but not the  $b$ . This was because they tended to use formulas of antidifferentiation, which help only to find the  $a$ .

It was not easy for them to see why, when the interval of the function changed, the graph of  $F(x)$  would change. Daniel thought the  $b$  in  $F(x) = ax+b$  would change because as the

domain increased (taking a large domain), the area above the graph would increase as well.

Marcos was the only student interviewed who proved able to visualise the process of constructing the area function. At the beginning of the interview he found it difficult to guess the formula of the primitive, but he progressively got better. When reading in the editor how the procedure areagraph was written, he associated  $\frac{A(x+h) - A(x)}{h}$  with rate of change.

However, he was the exception. All the other interviewees kept thinking in terms of antidifferentiation formulas, and it did not help even when they examined how the procedures were written (in the case of Logo) or saw the area being evaluated at each step (in the case of Graphic Calculus).

### Variables

In the interviews with Marcos, Julieta, Daniel and Fábio, each student, after working with some functions, was asked to write  $F(x)$  as an integral function, using one of the functions she or he had worked with. None of them did so correctly. Marcos wrote  $F(x) = \int_a^b f(t)dt$ ; Daniel wrote  $F(x) = \int_0^3 3x dx$ ; Fábio  $F(x) = \int_{-3}^3 5 dx$  and Julieta wrote  $F(x) = \int_0^2 2$ . Marcos got his first attempt wrong, but when he was told that  $x$  would vary over the domain, he changed it and wrote it correctly.

With Nair a different approach was tried. After the first example she was told that the area function was defined as  $F(x) = \int_a^x f(t)dt$ . During the interview she was asked to relate the function defined in this way with the graphs she was looking at on the screen. Even after doing that she wrote  $F(x) = \int_{-1}^1 x^2 dx$ . There is no doubt that understanding the role of the variables in the function  $F(x)$  was very difficult for the students.

### Mathematical Proof

Although it was planned to put more questions related to the proof of the FTC during the interviews around the computer, the fact was that this aspect was not much explored,

since the students came up against many difficulties even in simpler exercises and interpretations. None of them explained why  $f$  had to be continuous in the FTC. Three of them explained that  $f$  had to be continuous, because if not,  $F(x)$  would not exist. This is obviously not true, since  $f$  may be integrable even if it is not continuous. However, they said that because they were thinking of functions with vertical asymptotes, in which case the area function would not be defined in an interval containing the asymptote (the work with the computer helped them to see that the value of the area would increase tremendously).

With some help the two students who worked with Graphic Calculus were able to understand the graph of  $f(x) = \sin x$  stretched, but were not able to use this information in order to make connections with the proof in general.

## 4.5 Preparing the main study

The pilot study suggested that some modifications had to be made to the questionnaire. It also helped in the preparation of the activities, making clear what structure these interviews should have. The interviews based on the questionnaire did not suffer any significant change. The modifications incorporated and the rationale for them will be described below.

### 4.5.1 The questionnaire

The main study questionnaire (Appendix 2) was similar to the pilot study questionnaire (Appendix 1). Questions 1b and 5 (corresponding to 6 in the main study) were kept with no modifications. In question 1a the polynomial function to be integrated was made even easier in order to prevent the students from making algebraic mistakes. In part I, question 2, the item asking the students to evaluate  $g(2)$  was removed. This was done in order to eliminate the problem of not being able to decide how they solved  $g'(2)$ . Question 3 was completely modified in order to avoid the problem of different definitions cited in section 4.2.3. The way found to avoid it was to ask the students to classify only functions defined in  $\mathbf{R}$ . Question 3 of the main study was based on a question in Tall and Vinner (1981). One more item was included in the main version of question 4, however since a mistake was made when typing it, it was not considered in the analysis. In



question 4 of the main study the graphs were grouped together (it was not found necessary to draw two separate sets of graphs) and one graph of item b in the pilot study was taken out, graph (g), the least chosen in this question.

One more question was added: question 5 in the main study. This was done in order to discover whether the students knew how to apply the FTC in a situation in which they had no other choice (different from question 2). It was also desired to evaluate how they found  $h(0)$ , since question 2a of the pilot study was removed.

In the second part, the following sentence was added before the wordings of the question: "The first part was devoted to testing the learning of certain mathematical concepts; in this second part we want to know some of your ideas about mathematics". It was supposed that this would make it clear to the students that there was a significant change in the aims of the first and second parts. In addition, question 1 was changed to avoid the answer "by the FTC". The idea of this question was to put the students into an experimental situation to see whether they believed that examples were a sufficient test. By giving only one example instead of the three in the pilot questionnaire and creating a hypothetical situation to put the student in the role of a researcher, the intention was to give the question stronger impact. Question 2 was not changed in any significant way, except that "in an interval  $[a,b]$ " was added to the question whether they could find a function which did not satisfy the theorem, in order to avoid objections concerning the interval (for example saying "yes if it was open").

#### **4.5.2 The interviews around the computer**

Three sets of activities were planned to be given to the students one by one. They were planned bearing in mind the results of the analysis of pilot interviews around the computer. The main points which these interviews raised were: difficulty in representing  $F(x) = \int_a^x f(t)dt$ , difficulty in understanding the visual connection between  $f(x)$  and  $F(x)$ ; and not using the graphical images for the questions posed. The activities planned for the main study aimed at investigating these aspects more systematically and also at examining the visual aspects linked to the proof of the FTC itself.



In the first set, the students were to familiarise themselves with the software and solve activities concerning differentiation and integration. In these activities their images of differentiation and integration connected with tangent line and area would be explored. Their view of the functions  $f'(x)$  and  $F(x)$  would also be investigated.

In the second set the focus was on the function  $F(x)$ . Investigation centred on whether the students would be able to sketch with paper and pencil the graph of  $F(x)$  given the analytical formula of  $f(x)$ . There was great interest in discovering whether they would use the same method as Graphic Calculus does in building up the graph of  $F(x)$ . A further activity was planned (solving equations) to explore their understanding of the role of variables in  $F(x)$ .

The third set of activities was planned to explore explicitly the students' image of the FTC and its corollary, and also to see how they coped with the more formal aspects of the FTC related to its proof.

Having planned the activities, the second decision concerned whether to choose Logo or Graphic Calculus to work with the students. The decision went in favour of Graphic Calculus, taking into account the fact that the students would have little time to familiarise themselves with the software and Graphic Calculus is more user-friendly and certainly provided all the tools needed to undertake the activities. Of course, as Logo is a programming language, it would have been possible to write all the desired procedures and work with it. However, the time that would be spent was not felt to be justified, given the aims of the research. It should also be mentioned that Graphic Calculus was designed by a researcher in mathematics education and reflects a concern of the author with the formal aspects of calculus. The main limitation of the program is that it cannot be extended and apparently its development stopped in 1990. This is, of course, an important aspect when one speaks about computer programs. However, the choice of the software has to be guided by the purpose for which it is intended and Graphic Calculus was well suited to the purposes of the task-based interviews.

All the activities were planned to be done on one day, but the interviews were to take longer than in the pilot study. They were, in fact, planned to take about four hours each.

The activities themselves are set out in Chap. 7 (section 7.3) as well as in the Appendix 3. The decision was taken to include them in Chap. 7 (together with their aims) in order to help the reader to follow the analysis there.

## **CHAPTER 5**

### **THE QUESTIONNAIRE**

#### **5.1 Introduction**

As described in Chapter 4, the questionnaire was divided into two parts. The aim of the first part was to evaluate students' knowledge of the FTC and other concepts related to it such as: continuity, differentiation and integration. The second part aimed at identifying students' attitudes towards proof. The two parts were given to the students separately. They started with the first part, and were given the second part after completion of the first.

The questionnaire was administered in November 1995 by two researchers in mathematics education from UFRJ. They were not the teachers of any of these classes. The questionnaire was administered during class time and the students were allowed one and a half hours if they wanted it. None of the students were obliged to take part in this research.

#### **5.2 The sample**

The main study was conducted with two mathematics classes, one computer science class and three engineering classes. The idea of including mathematics students in the main study was to see if, in relation to part II, they paid more attention to the processes of proving in mathematics and would get better results. The questionnaire was distributed to 148 students, 111 males and 37 females in all. Of these, 46 were studying mathematics, 33 computer science and 69 engineering. The students in the three classes of engineering were from different areas: industrial (11), civil (25) and mechanical (33). The reason why few students from the industrial engineering class answered the questionnaire was that on the day it was arranged to apply it to this class unexpected circumstances made it difficult to get to the university.

The students in the engineering classes were in the same position as the computer science students: they were mostly doing calculus I for the first time (having entered the university in August), but some had failed before and were doing it again. However, the two mathematics classes were very different from the other groups and also very different from each other.

One of these classes is held during the day, like most of the courses at UFRJ. All the students in this class were repeating calculus I. In fact, no calculus I classes are offered to new mathematics' students in the second semester. They all do a common basic course in two years and then follow one of the next specialist courses: BSc, teacher training, statistics or actuarial studies. However, most of these students would actually have preferred to pass the entrance examination to study computer science.

The other mathematics class is held during the evenings. This class only prepares the students to be mathematics teachers. As the students in evening classes are considered weaker, the calculus I course is given in two semesters instead of one. The students in this class of the sample were in their second semester of calculus.

By the time the questionnaire was administered, the students had already learnt the FTC but had not yet finished all their examinations. Checking their final results in January showed that of the 148 students who answered the questionnaire, 78.4% passed in the calculus course. By classes, the percentage was 70.8% in the daytime mathematics class, 54.5% in the evening mathematics class, 100% in the computer science class; 81.8% in the mechanical engineering class; 64% in the civil engineering class and 100% in the industrial engineering class.

However, the examinations were not uniform for all the classes, an exception being made for the three engineering classes, so some of the above results need to be considered with care, because they do not necessarily reflect the competence of students in calculus.

Special mention should be made of the results of the computer science class (all the students of the sample passed) and the mathematics class held during the evenings, of whom 45.5 % failed. The teacher who was teaching calculus for the computer science class is considered not to be very demanding in examinations. The mathematics class held during the evenings had had problems with the teacher who taught calculus in the first

semester; the teacher of the second semester explained to me that this was what caused the high level of failure.

In the course of this study the classes will be abbreviated as follows: computer science, cs; mathematics (daytime class), dm; mathematics (evening class), em; industrial engineering, ie; civil engineering, ce; and mechanical engineering, me.

### **5.3 Methodology of Analysis**

In order to analyse the questionnaire, two procedures were followed: one to characterise the different kinds of answers and arguments used in each question and the second to describe the performance of the students in the questionnaire as a whole.

The first procedure was the same as that adopted in the pilot study: identifying categories representing patterns of answers. Some of the pilot study categories were kept, others were changed, and some new ones which had not been devised for the pilot were created (for questions 3 and 5 of the main study). There was a concern to group some of the categories of the pilot study, selecting the central ideas behind the answers and not considering small differences which did not matter for the purposes of the research.

The second procedure was marking the questions and giving each student a final score for the first and second parts of the questionnaire. In the first part of the questionnaire, the scores ranged from 0 to 10 and in the second part from 0 to 4. The previous analysis helped to provide a basis for marking each question.

Once the scores were obtained, they not only showed the performance of each student but also suggested differences between the groups of students following the different courses. These results therefore received a statistical treatment, which will be presented next.

## 5.4 Comparing the classes

In order to compare the performance of the classes in relation to the results obtained in the questionnaire, the two parts were examined separately, with different statistical tests used for each part.

### 5.4.1 Analysis of the results of part I

Given the students came from six different classes, the first step was to ascertain if the six classes have the same mean. To do this, an analysis of variance (ANOVA) model was used.

Table 5.1 shows the pertinent statistics used to construct the ANOVA table (see Appendix 4).

**TABLE 5.1** Sample Data to Part I

<i>Statistic</i>	<i>Class</i>						<i>Total</i>
	<i>em</i>	<i>dm</i>	<i>c</i>	<i>ie</i>	<i>me</i>	<i>ce</i>	
Sample Size	22	24	33	11	33	25	148
Sample Mean	3.68	4.88	5.34	6.8	6.08	5.23	5.28
Standard Deviation	1.72	1.81	1.57	1.89	2.17	2.22	2.07

The results of ANOVA showed that the means of the six classes differ,  $F = 5.85$ ,  $p < 0.001$ . The next step was to examine in what way the classes differed: whether each one was different, or whether some of the classes were equal to one another. In order to do that, the classes were compared in pairs by constructing the confidence intervals between the difference of the means, following an approach presented in Neter, Wasserman and Kutner (1990), see table in Appendix 4.

The conclusion of these comparisons was that the evening mathematics class (em) performed significantly worse than every other class. The daytime mathematics class (dm), the computer science class (cs), and the civil engineering class (ce) all performed similarly. The industrial engineering class (ie) and the mechanical engineering class (me), both performed better than the middle group. For the purposes of this research therefore, the sample was divided into three groups: group 1 (ie and me); group 2 (dm, cs and ce) and group 3 (em). There are 44 students in group 1, 82 in group 2, and 22 in group 3.

### 5.4.2 Analysis of the results of part II

The results of the second part were poor; with most of the students not answering any of the questions correctly and therefore scoring zero. It should be added that if they did not justify their answers to these questions, they were given 0, since it was very easy just to answer yes or no with no justification at all.

The relative frequency of students who obtained 0, 1, 2, 3 or 4 in each of the classes was as follows:

**TABLE 5.2** Frequency in percentage of each score per class.

Score	Class						Total
	em	dm	cs	ce	me	ie	
0	72.7	66.7	42.4	52.0	39.4	72.7	54.1
1	13.6	12.5	15.2	20.0	27.3	9.1	17.6
2	13.6	12.5	24.2	24.0	24.2	9.1	19.6
3	0	4.2	12.1	0	6.1	9.1	5.4
4	0	4.2	6.1	4.0	3.0	0	3.4

Since the frequencies of the last two categories (3 and 4) were very low, they were grouped in only one category so that the Chi- square test to contingency tables could be applied.

So, grouping (3) and (4) the table of absolute frequency observed was:

**TABLE 5.3** Absolute frequency of each score per class.

Score	Class						Total
	em	dm	c	ie	me	ce	
0	16	16	14	8	13	13	80
1	3	3	5	1	9	5	26
2	3	3	8	1	8	6	29
3/4	0	2	6	1	3	1	13

The results of the Chi –square test indicated that the classes did not differ significantly as to the second part of the questionnaire,  $\chi^2 = 16.47$ ,  $df = 15$ . The  $p$ -value of this test is approximately 0.36.

### 5.4.3 Conclusion

The quantitative analysis suggested that it was possible to arrange the classes into three groups according to responses to the questions in Part I, and this grouping will be used when analysing those questions. When analysing the questions in Part II, the division into groups did not add any relevant information. However, particularly in this part of the questionnaire, it was interesting to see how the several classes answered the questions, by examining the different categories, and paying closer attention to the answers given by the students of the mathematics classes. For this reason the data in the tables will be given by classes.

## 5.5 Aims of the questions

The main version of the questionnaire is in Appendix 2. The aims of questions 1, 2, 3, 4 and 6 (corresponding to question 5 of the pilot study) of part I and of both questions of part II are the same as stated in Chapter 4, section 4.2.2. Question 5 of the main study aimed to find out if students know to apply the FTC in a situation in which they had no other choice (unlike question 2), and also to discover how they found  $h(0)$ . This question was deliberately put a long way from question 2.

## 5.6 Results per question: part I

For each of the questions in the questionnaire the following will be presented: the results of the scores, the categories used and the distribution of the answers according to the categories in the different groups. Each of the questions will be discussed separately, with the exception of questions 2 and 5, whose aims complement each other more closely. The presentation of the categories, analysis and results of each of the questions will be given in greater detail than in the pilot study.

### 5.6.1 Evaluating integrals: question 1

Question 1 asked the students to evaluate two integrals: (1a),  $\int_0^2 (3x^2 + 1)dx$  and (1b),  $\int_{-1}^1 |x|dx$ . The students' achievements in (1a) and (1b) were very different. An



analysis was made of the main mistakes in an attempt to explain some of the misconceptions students hold in connection with the process of integration.

#### *Grading the question*

The total score for this question was 2, i.e. 1 for each part. Only completely right answers received the total score. Otherwise, the score ranged from 0 to 2, depending on the mistake made

The table below displays the means for each of the parts (a and b) by groups and also for the whole sample.

**TABLE 5.4** Means and standard deviations found for question 1 - part I

Questions	Groups						Total	
	1		2		3			
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1a	0.96	0.10	0.91	0.14	0.70	0.21	0.89	0.15
1b	0.48	0.46	0.28	0.42	0.14	0.32	0.32	0.43

All groups performed well in question 1a, but the results for question 1b were considerably weaker. There was a significant difference between the overall averages (0.89 as against 0.32). Even the best group did not achieve a good average (0.48) in question 1b. The overall standard deviation was great (0.43). Note that question 1b would have been easier than question 1a if the students had remembered to use the graph of the modular function to evaluate the area under it.

#### *Bases for the categories*

The answers to this question were basically classified in the same way as in the pilot : as right (R) or wrong (W). The difference was that the answers in which a constant (+ C) was added to the correct answer were classified as wrong as well and not put in a separate category. The right answers to question 1b were divided into: those solved analytically ( $R_a$ ) and those solved graphically ( $R_g$ ).

The following codes will be used to present the results below. Remember that the right answers are divided into R<sub>a</sub> and R<sub>g</sub> only in question 1b.

R: Right  $\begin{cases} R_a: \text{solved analytically} \\ R_g: \text{solved graphically} \end{cases}$

W: Wrong

X: Not answered

### Discussion of the results

The number and percentage of students by groups who answered question 1 fall into the various categories as follows:

**TABLE 5.5** Distribution of categories of responses to question 1 - part I (% italics)

Question 1a					
Category		Group			Total (n=148)
		1 (n=44)	2 (n=82)	3 (n=22)	
R		41 <i>93.2</i>	72 <i>87.8</i>	14 <i>63.6</i>	127 <i>85.8</i>
W		3 <i>6.8</i>	8 <i>9.8</i>	6 <i>27.3</i>	17 <i>11.5</i>
X		0	2 <i>2.4</i>	2 <i>9.1</i>	4 <i>2.7</i>

Question 1b					
Category	Subcategory	Group			Total (n=148)
		1 (n=44)	2 (n=82)	3 (n=22)	
R	R <sub>a</sub>	14 <i>31.8</i>	17 <i>20.7</i>	2 <i>9.1</i>	33 <i>22.3</i>
	R <sub>g</sub>	1 <i>2.3</i>	3 <i>3.7</i>	0	4 <i>2.7</i>
W		27 <i>61.4</i>	52 <i>63.4</i>	12 <i>54.5</i>	91 <i>61.5</i>
X		2 <i>4.5</i>	10 <i>12.2</i>	8 <i>36.4</i>	20 <i>13.5</i>

As can be seen from the results in Table 5.5, question 1a achieved its aim of helping to give the students more confidence: 85.8% of the students got the right answer. The

poorest result was obtained by group 3 (only 63.6% correct answers), particularly if it is considered that this kind of question is common in most textbooks. The results of question 1b, on the other hand (25.0% of correct answers for all the students), were very poor in all the groups, group 3 in particular. Only 9.1 % of the students in group 3 gave the right solution to this question; 36.4% of them did not even try to solve the question.

The quickest way to solve 1b was by doing it graphically, but only 2.7 % of the students (4 students) evaluated the integral by finding the area below the graph of  $f(x) = |x|$ . Three of these were from the computer science class, which is in group 2, and all three were repeating the calculus course. Perhaps some influence had been exerted by the previous teacher.

Some mistakes deserve to be mentioned in question 1: substituting the upper and lower limits of integration direct into the function inside the integral; leaving the final answer in an algebraic form; adding a constant to the numerical answer; writing  $\frac{x^2}{2}$  or  $\left| \frac{x^2}{2} \right|$  (or, similarly,  $\frac{|x|^2}{2}$ ) as the primitives of  $|x|$ ; and separating the integral into  $\int_{-1}^1 x dx$  and  $\int_{-1}^1 x dx$ .

Although not many students made the first three mistakes listed above, they are mentioned because they revealed a complete misunderstanding of what a definite integral is. Three students (2 from group 1 and 1 from group 2) found  $(3.4+1) - 1 = 12$  as the final answer to (1a) and 0 to (1b). Three other students (1 from group 1 and 3 from group 2) left the final answer in an algebraic form (for example  $x^3 + x + C$  and  $\frac{x^2}{2}$ ). Five others (4 from group 2 and 1 from group 1) gave answers such as  $10+C$  and  $0+C$ .

On the other hand, the number of students who made the remaining three mistakes listed, which only correspond to question 1b, is significant. The following table shows how these mistakes are distributed among the different groups.

**TABLE 5.6** Distribution by groups of the main mistakes made in question 1b

Mistakes	Groups			Total (n=148)
	1 (n= 44)	2 (n=82)	3 (n=22)	
$\frac{x^2}{2}$ as the primitive of $ x $	3 6.8	15 18.3	2 9.1	20 13.5
$\left \frac{x^2}{2}\right $ (or $\frac{ x ^2}{2}$ ) as the primitive of $ x $	11 25.0	14 17.1	1 4.5	26 17.6
Separating the integral into $\int_{-1}^1 x dx$ and $\int_{-1}^1 x dx$ .	1 2.3	8 9.7	2 9.1	11 7.4
Total	15 34.1	37 45.1	5 22.7	57 38.5

Examining the overall result, the two most frequent mistakes were to consider  $\left|\frac{x^2}{2}\right|$  or

$\frac{|x|^2}{2}$  as the primitive for  $|x|$ . Comparing the groups, it was seen to be the least representative mistake only in group 3. In fact, the smallest total percentage of students who made any of the three mistakes listed above (22.7%) was found in group 3. However, 36.4% of the students in this group did not even try to give any answer to the question.

Although  $\frac{|x|^2}{2}$  is equal to  $\left|\frac{x^2}{2}\right|$ , the answer from one student indicates that perhaps the

line of reasoning of the students who wrote  $\frac{|x|^2}{2}$  was different from the ones who

answered  $\left|\frac{x^2}{2}\right|$ . Attempting to explain why he regarded  $\frac{|x|^2}{2}$  as the primitive for  $|x|$ , this

student wrote :  $|x| = u$ ,  $\frac{u^{1+1}}{2} = \frac{u^2}{2}$ . So, replacing  $u$  with  $|x|$ , he got  $\frac{|x|^2}{2}$ . By the other

hand the students who wrote  $\left| \frac{x^2}{2} \right|$ , apparently just added the sign of the modular function at the end of having integrating  $f(x) = x$ .

Most of the students who considered  $\frac{x^2}{2}$  to be the primitive for  $|x|$  did not justify their answer. It seems that they just ignored the sign of the modular function. However, two of them considered  $|x| = \sqrt{x^2}$  and followed their reasoning by writing that  $(x^2)^{1/2} = x$ , these students concluded that they could treat  $|x|$  as just  $x$ . Besides considering  $|x|$  to be  $\sqrt{x^2}$ , another student solved the integral by substitution, writing that  $x^2 = u$ ,  $dx = \frac{du}{2x}$  and solving the integral  $\int_{-1}^1 \sqrt{u} \cdot \frac{du}{2\sqrt{u}} = \frac{1}{2} u = \frac{x^2}{2}$ .

Eight of the students who separated the integral into  $\int_{-1}^1 x dx$  and  $\int_{-1}^1 x dx$ , gave two numbers as the final answers (for example: 0 and 0 or -1/2 and 1/2), and four of them added these two results to produce one single answer. The students who separated the integral in this way had probably learnt in the secondary school that  $|x|$  is equal to  $x$  if  $x \geq 0$  and to  $-x$  if  $x < 0$ , but this information had no graphical or even analytical meaning for them. They only memorised a part of it and it appears that they did not know how to apply it in a different situation.

### 5.6.2 Understanding continuity: question 3

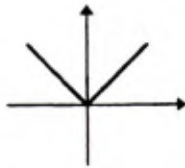
In question 3 the students were given four functions, which they had to classify as continuous or discontinuous, justifying their answers. Asking them to justify their answers was important to see if they just used graphical arguments with no link to the definition or if they at least tried to make use of the definition of continuity.

---

**FIGURE 5.1** Question 3

---

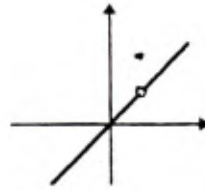
Classify the following functions as continuous or discontinuous. Give a brief justification for your answer.



$$f_1(x) = |x|$$

continuous ☐ discontinuous ☐

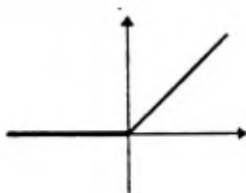
Justification:



$$f_2(x) = \begin{cases} x & (x \neq 1) \\ 2 & (x = 1) \end{cases}$$

continuous ☐ discontinuous ☐

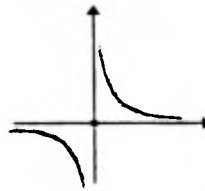
Justification:



$$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x > 0) \end{cases}$$

continuous ☐ discontinuous ☐

Justification:



$$f_4(x) = \begin{cases} \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

continuous ☐ discontinuous ☐

Justification:

---

### *Grading the question*

This question was graded twice, the first time when marking the whole questionnaire. The total value attributed to it was 2 points, 0.5 points for each correct choice. This time what was taken into account when marking it was if they could recognise visually what a continuous or discontinuous function is irrespective of the justifications they gave.

The second grading was done when analysing the question alone. After classifying the justifications it seemed important to attribute a grade in which was reflected the relation between the justifications and the answers. This result was not used in the general grading on which the statistical analysis was based.

In this new grading the total value of the question was 6. For each correct answer (continuous or discontinuous) the students got 0.5 points, as before. If their answer was correct the justification was examined. Justifications in which they used the definition correctly scored 1 point. Justifications in which they used partial aspects of the definition or showed that they had not well understood the meaning of all the

mathematical ideas contained in it scored from 0.3 to 0.5 points. Graphical justifications could get from 0 to 0.5 points, depending on how vague they were; arguments such as being able or not to take the pencil off the paper or whether the function gives a jump at one of its points got 0.5, while an argument such as there is no hole in the graph or the function follows a certain pattern would get 0. Any other kind of justification got 0 points.

The following table displays the means obtained in each of the grading systems.

**TABLE 5.7** Means and standard deviations found for question 3

Grading	Groups						Total	Value of the question	
	1		2		3				
	Mean	S.D.	Mean	S.D.	Mean	S.D.			
Answers only	1.82	0.46	1.71	0.54	1.68	0.39	1.74	0.52	2.0
Answers plus justifications	2.95	1.35	2.82	1.41	2.32	1.39	2.78	1.39	6.0

In the first line of Table 5.7 the overall mean is 1.74 and the standard deviation is 0.52. Given that the total value was 2.0, the results were good. In the second line the overall mean is 2.78 and the standard deviation is 1.39. Given that the total value was 6.0, the results were poor. This shows that most of the students could recognise visually whether the functions in question 3 were continuous or not but did not know how to justify their answers properly.

#### *Bases for the categories*

The main point that was taken into account when analysing the justifications in question 3 was if the students had tried to use the definition or if they had used an argument based only on a graphical interpretation. However, another sort of justification appeared in a fair number of answers, using arguments related to the definition of other concepts than continuity. So there were four categories, some of them with subcategories. There are also the justifications that could not be classified in any of these and which were grouped separately. The categories used were as follows:

- Category D (Used the Definition)

When students used the definition of continuity their justifications were classified as D.

Four different big groups appeared in D, which were coded as D<sub>c</sub>, D<sub>p</sub>, D<sub>l</sub> and D<sub>w</sub>.

- D<sub>c</sub> (Correct use of the Definition)

If they wrote the definition correctly, their answers were classified as D<sub>c</sub>. Examples for  $f_1$  and  $f_3$  are:

“(i)  $f(a)$  exists;

(ii)  $\lim_{x \rightarrow a} f(x)$  exists; and

(iii)  $\lim_{x \rightarrow a} f(x) = f(a), \forall a \in \mathbf{R}.$ ”

and

“ $\lim_{x \rightarrow a} f(x) = f(a), \forall a \in \mathbf{R}.$ ”

For  $f_2$  and for  $f_4$  they had to show that it was not true that  $\lim_{x \rightarrow a} f(x) = f(a), \forall a \in \mathbf{R}.$

One example of answer to  $f_2$  was:

“ $\lim_{x \rightarrow 1} f_2(x) \neq f(1)$ ”

And the follows are examples for  $f_4$ :

“ $\nexists \lim_{x \rightarrow 0} f_4(x)$ ”

and

“ $f_4(x) \neq \lim_{x \rightarrow 0^+} f_4(x) \neq \lim_{x \rightarrow 0^-} f_4(x), \nexists \lim_{x \rightarrow 0} f_4(x)$ ”

These are only examples. There might be some mistakes even in the justifications classified as D<sub>c</sub>. Some students for example, did not write that  $\lim_{x \rightarrow a} f(x) = f(a)$  was

valid for  $\forall a \in \mathbf{R}$ . However, since they wrote  $a$  (or any other letter) and did not choose a specific point as 0, it seems that, in their minds, they generalised for all the points over the domain.



- D<sub>p</sub> (Partial use of the Definition)

Taking only parts of the definition or a partial aspect of it was very common in justifying to  $f_1$  and  $f_3$  as continuous. This happened in two ways:

(1) considering a particular point ( $x = 0$ ) and justifying the function to be continuous at that point, for example:  $f(0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$ , or

(2) the final argument being that the limits of the function at all the points (or only at 0) both on the right and on the left were equal, for example, writing that:

$$“f(a) \text{ exists for every } a \in \mathbb{R} \text{ and } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)”$$

or only that

$$“\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)”$$

Although in (1) and (2) they only used a partial aspect of the definition, the reasons why they did so are probably very different. In (1) they seem to have had a good notion of what a continuous function is. They analysed the function,  $f_1$  or  $f_3$ , to determine whether it was continuous at the point at which the graph had a corner, but probably either forgot to generalise the result for all the points of  $\mathbb{R}$  or thought this procedure was not necessary. In (2) they really had in mind only one aspect of the definition, that is, to check if the limit of the function at each point (or only one point) existed.

. D<sub>l</sub> (Infinite as an existing limit)

In justifying why the function  $f_4$  was not continuous they were expected to say that the  $\lim_{x \rightarrow 0} f_4(x)$  did not exist, but some of them did not write that; they treated the limit at 0 as

if it existed. They had the definition of continuity in mind when they gave answers as:

$$“\lim_{x \rightarrow 0^-} f_4(x) \neq \lim_{x \rightarrow 0^+} f_4(x)”;$$

$$“\lim_{x \rightarrow 0^-} f_4(x) \neq f(0) \text{ and } f(0)=0”;$$

$$“\lim_{x \rightarrow 0^+} f_4(x) = +\infty \text{ and } \lim_{x \rightarrow 0^-} f_4(x) = -\infty”$$

or

$$“\lim_{x \rightarrow 0} f_4(x) = \infty”$$

In the first example the right- and left- hand limits are being treated as if each of them existed. Probably the students who answered in this way thought since they are different the  $\lim_{x \rightarrow 0} f_4(x)$  does not exist. In the second example the  $\lim_{x \rightarrow 0} f_4(x)$  itself is considered to exist. In the third one it not clear whether the students thought that these limits existed or not. The same can be said about the fourth example (some textbooks use this sort of language, in which  $\infty$  can mean  $+\infty$  and  $-\infty$ ). If this test were applied again, the function  $f_4$  should be replaced with  $f(x) = \frac{1}{x^2}$  if  $x \neq 0$  and 0 if  $x = 0$ . At least this would clarify how the students who gave answers as exemplified by the third example thought.

Although students who answered only that  $\lim_{x \rightarrow 0} f_4(x)$  does not exist had their answers classified as  $D_c$ , perhaps some of them thought in the same way as the students who had their answers classified as  $D_\infty$ .

#### - $D_w$ ( Wrong evaluation of the limit)

There were students who used the definition but gave the wrong answer because they evaluated the limit at certain points incorrectly. They wrote  $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$  in the case of  $f_1$  or  $f_3$  and  $\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x)$ , in the case of  $f_2$ . In  $f_2$ , some of them considered that  $\lim_{x \rightarrow 1^-} f(x) = 1$  and  $\lim_{x \rightarrow 1^+} f(x) = 2$ .

#### - Category M ( Mixing up arguments)

Sometimes students mixed up the definition of continuity with the definition of other concepts, using arguments related to them, very frequently showing that none of these concepts were well understood. Answers of this kind were classified as M. There are two groups in this category, coded as  $M_f$  and  $M_d$ .

#### - $M_f$ ( Mixing up arguments related to the concept of Function)

When defining in class what a function is, two points are emphasised: (1) to every  $x$  in the domain there corresponds a value of  $y$ , such that  $f(x) = y$  and (2) the  $y$  is unique.

The first of these conditions was used to classify the functions as continuous or discontinuous. This is the case of the following examples:

$$“\forall x \exists y \in \mathbf{R}”$$

and

“The function is not defined at a point” (argument for classifying  $f_2$  or  $f_4$  as discontinuous).

These arguments are related to the first condition in the definition of continuity of a function at a point in the three - condition definition:  $f(x)$  exists (see section 2.5). It may be that the students only took this condition to classify the function as continuous or not, but it is more likely that they got mixed up with the definition of a function.

-  $M_d$  (Mixing up arguments related to the concept of a Derivative of a function)

In  $M_d$  justifications are classified in which the students used arguments related to the concept of differentiability, such as:

“There is a derivative over the whole of its domain” (*for any of the functions*);

“There is no derivative at the origin” (*only for  $f_1$  and  $f_3$* );

“It is not differentiable in  $I$ ” (*only for  $f_2$* ).

- Category G (Graphical arguments)

Using graphical arguments for determining whether the function was continuous or not was a much used procedure, which was categorised as G. Sometimes it was clear that the argument was purely a graphical one as for example when students wrote:

“The function gives a jump at one of its points.”

But there are other examples in which it was not clear whether they were reasoning graphically or analysing the formula of the function. One example is:

“The function obeys a certain pattern.”

Category G brings together justifications in which a graphical reason was presented and those in which there is the idea that the function follows a certain pattern or rule. Other examples of justifications classified as G are:

“It is possible to draw the graph of the function without taking the pencil off the paper.”;

“It is not possible to draw the graph of the function without taking the pencil off the paper.”;

“There is no break in the graph of the function”;

“All the values of  $x$  and  $y$  are on the same curve”;

“The graph can be divided into two parts”;

“The function has two non-connected branches”;

“For certain values the function requires a defined  $y$ ; it does not follow the rule.”

- Category O (Others)

Justifications which were not classified in any of the categories above were classified as O. Some examples are:

$$“\exists f_1(x) \forall x \in \mathbb{R}, \lim_{x \rightarrow -\infty} f_1(x) = +\infty = \lim_{x \rightarrow +\infty} f_1(x)”;$$

“ $f_2(x) = 1$  is an inflexion point”;

“The function  $f_4$  has vertical and horizontal limits, not being continuous at  $x = 0$ .”

A short description of each category is given below, summarising the classification according to the types of arguments described above:

D: Used the definition  $\left\{ \begin{array}{l} D_c \text{ (correct use)} \\ D_p \text{ (partial use)} \\ D_i \text{ (infinite as an existing limit)} \\ D_w \text{ (wrong evaluation of the limit)} \end{array} \right.$

M : Mixed up continuity  $\left\{ \begin{array}{l} M_r \text{ (with the concept of a function)} \\ M_d \text{ (with the concept of a derivative of a function)} \end{array} \right.$

G : Used graphical arguments

O : Others

It should be noted that although all the justifications were classified in only one category, some of them included more than one argument. In this case an attempt was made to feel which argument was predominant.

### *Discussion of the results*

The tables below were obtained according to the answers chosen and the categories described above. Students could answer continuous (c) or not continuous (nc). However, two students marked both options: continuous and not continuous (c and nc)

for functions  $f_2$  and  $f_4$  and justified why they chose both at the same time. Some students did not mark any of the options (no answer) although some of them wrote arguments in the space reserved for justification. Their justifications were also categorised.

**TABLE 5.8** Distribution of categories of responses to function  $f_1$  in question 3 (% in *italics*)

<i>c / nc</i>	<i>Category</i>	<i>Subcategory</i>	<i>Group</i>			<i>Total</i> ( <i>n</i> = 148)
			1 ( <i>n</i> = 44)	2 ( <i>n</i> = 82)	3 ( <i>n</i> = 22)	
<b>c</b>	<b>D</b>	<b>D<sub>c</sub></b>	6 <i>13.6</i>	9 <i>11.0</i>	2 <i>9.1</i>	17 <i>11.5</i>
		<b>D<sub>p</sub></b>	8 <i>18.2</i>	18 <i>22.0</i>	2 <i>9.1</i>	28 <i>18.9</i>
	<b>G</b>		12 <i>27.3</i>	12 <i>14.6</i>	3 <i>13.6</i>	27 <i>18.2</i>
	<b>M</b>	<b>M<sub>f</sub></b>	7 <i>16.0</i>	13 <i>15.9</i>	5 <i>22.7</i>	25 <i>16.9</i>
		<b>M<sub>d</sub></b>	2 <i>4.5</i>	1 <i>1.2</i>	1 <i>4.6</i>	4 <i>2.7</i>
	<b>O</b>		1 <i>2.3</i>	7 <i>8.5</i>	1 <i>4.6</i>	9 <i>6.1</i>
	<b>X</b>		4 <i>9.1</i>	7 <i>8.5</i>	7 <i>31.8</i>	18 <i>12.2</i>
	<b>Total</b>		40 <i>90.9</i>	67 <i>81.7</i>	21 <i>95.5</i>	128 <i>86.5</i>
<b>nc</b>	<b>D</b>	<b>D<sub>w</sub></b>	1 <i>2.3</i>	3 <i>3.7</i>	-	4 <i>2.7</i>
	<b>G</b>		-	1 <i>1.2</i>	-	1 <i>0.7</i>
	<b>M</b>	<b>M<sub>d</sub></b>	1 <i>2.3</i>	7 <i>8.5</i>	-	8 <i>5.4</i>
	<b>O</b>		1 <i>2.3</i>	3 <i>3.7</i>	1 <i>4.6</i>	5 <i>3.4</i>
	<b>X</b>		1 <i>2.3</i>	1 <i>1.2</i>	-	2 <i>1.3</i>
	<b>Total</b>		4 <i>9.1</i>	15 <i>18.3</i>	1 <i>4.6</i>	20 <i>13.5</i>

Table 5.8 reveals that 86.5% of the students gave the correct choice for the function  $f_1$ . Group 3 did particularly well, 95.5% of the students marking  $f_1$  as being continuous. However, examination of the justifications for the choice of  $f_1$  as continuous gives another view of overall results. Although 30.4% tried to use the definition, only 11.5% did so correctly, the other 18.9% used only parts of it. In addition, 16.9% of students mixed up the concept of a function with that of a continuous function. Group 3, despite their high percentage of correct answers, also had the largest percentage of students who did not give any justification at all (31.8%) and who mixed up the concepts (27.3%).

Considering the data referring to the students who marked  $f_1$  as being discontinuous, the reason which led most of them to classify  $f_1$  wrongly was mixing up the differentiability with the continuity of a function .

**TABLE 5.9** Distribution of categories of responses to function  $f_2$  in question 3 (% in italics)

<i>c / nc</i>	<i>Category</i>	<i>Subcategory</i>	<i>Group</i>			<i>Total</i> ( <i>n</i> = 148)		
			1 ( <i>n</i> = 44)	2 ( <i>n</i> = 82)	3 ( <i>n</i> = 22)			
<b>nc</b>	<b>D</b>	<b>D<sub>c</sub></b>	10 22.7	25 30.5	3 13.6	38 25.7	50 33.8	
		<b>D<sub>w</sub></b>	4 9.1	6 7.3	2 9.1	12 8.1		
	<b>G</b>		14 31.8	15 18.3	4 18.2		33 22.3	
		<b>M</b>	<b>M<sub>f</sub></b>	3 6.8	14 17.1	2 9.1	19 12.8	23 15.5
	<b>M<sub>d</sub></b>		2 4.5	1 1.2	1 4.5	4 2.7		
	<b>O</b>		5 11.4	8 9.8	2 9.1		15 10.1	
	<b>X</b>		2 4.5	5 6.1	5 22.7		12 8.1	
		<b>Total</b>		40 90.9	74 90.3	19 86.4		133 89.9
	<b>c</b>	<b>G</b>		1 2.3	-	-		1 0.7
			<b>M</b>	<b>M<sub>f</sub></b>	-	2 2.4	1 4.5	3 2.0
<b>M<sub>d</sub></b>		1 2.3		4 4.9	-	5 3.4		
<b>O</b>			-	-	-		-	
<b>X</b>			2 4.5	2 2.4	1 4.5		5 3.4	
		<b>Total</b>		4 9.1	8 9.7	2 9.1		14 9.5
<b>c/nc</b>	<b>G</b>		-	-	1 4.5		1 0.7	
<b>Total</b>			-	-	1 4.5		1 0.7	

Table 5.9 shows that 89.9% of the students marked the correct choice when classifying function  $f_2$ . It is clear in this case not only that the greatest percentage of these students tried to use the definition (33.8%), but also that most of them did so correctly (25.7%). Graphical reasons were popular (22.3%), probably because of the fact of the function “having a hole”, or “giving a jump” as some students stated. The mixing up of concepts repeated the pattern of  $f_1$ : M<sub>f</sub> was more common with students who made the right

choice (12.8%), while  $M_d$  was more common with students who chose wrongly (3.4%). Group 2 was the one which most mixed up concepts (25.6% in all).

The group which showed the largest percentage of students who did not give any justification (27.2% in all) was group 3, it was also the group with the lowest percentage of correct answers (86.4%). It was a student from group 3 who answered that  $f_2$  was continuous and discontinuous, writing twice: “There is a discontinuity, a kind of jump in this function”, as he thought that this argument was valid both for continuous and discontinuous functions.

**TABLE 5.10** Distribution of categories of responses to function  $f_3$  in question 3 (% in *italics*)

<i>c / nc</i>	<i>Category</i>	<i>Subcategory</i>	<i>Group</i>			<i>Total</i> ( <i>n</i> = 148)
			1 ( <i>n</i> = 44)	2 ( <i>n</i> = 82)	3 ( <i>n</i> = 22)	
<b>c</b>	<b>D</b>	<b>D<sub>c</sub></b>	5 <i>11.4</i>	9 <i>11.0</i>	2 <i>9.1</i>	16 <i>10.8</i>
		<b>D<sub>p</sub></b>	6 <i>13.6</i>	17 <i>20.7</i>	2 <i>9.1</i>	25 <i>16.9</i>
		<b>D<sub>w</sub></b>	-	-	1 <i>4.5</i>	1 <i>0.7</i>
	<b>G</b>		12 <i>27.3</i>	14 <i>17.1</i>	2 <i>9.1</i>	28 <i>18.9</i>
	<b>M</b>	<b>M<sub>f</sub></b>	7 <i>15.9</i>	17 <i>20.7</i>	2 <i>9.1</i>	26 <i>17.6</i>
		<b>M<sub>d</sub></b>	3 <i>6.8</i>	2 <i>2.4</i>	-	5 <i>3.4</i>
	<b>O</b>		2 <i>4.5</i>	3 <i>3.7</i>	1 <i>4.5</i>	6 <i>4.1</i>
	<b>X</b>		5 <i>11.4</i>	9 <i>11.0</i>	9 <i>40.9</i>	23 <i>15.5</i>
	<b>Total</b>		40 <i>90.9</i>	71 <i>86.6</i>	19 <i>86.4</i>	130 <i>87.8</i>
<b>nc</b>	<b>D</b>	<b>D<sub>w</sub></b>	2 <i>4.5</i>	2 <i>2.4</i>	-	4 <i>2.7</i>
	<b>M</b>	<b>M<sub>d</sub></b>	1 <i>2.3</i>	4 <i>4.9</i>	-	5 <i>3.4</i>
	<b>O</b>		-	3 <i>3.7</i>	-	3 <i>2.0</i>
	<b>X</b>		1 <i>2.3</i>	-	1 <i>4.5</i>	2 <i>1.4</i>
	<b>Total</b>		4 <i>9.1</i>	9 <i>11.0</i>	1 <i>4.5</i>	14 <i>9.5</i>
<b>no answer</b>	<b>G</b>		-	-	1 <i>4.5</i>	1 <i>0.7</i>
	<b>X</b>		-	2 <i>2.4</i>	1 <i>4.5</i>	3 <i>2.0</i>
	<b>Total</b>		-	2 <i>2.4</i>	2 <i>9.1</i>	4 <i>2.7</i>

As Table 5.10 demonstrates, 87.8% of the students marked the correct choice for  $f_3$ , as against 9.5% who classified  $f_3$  as discontinuous and 2.7% who gave no answer. Although 28.4% of these students tried to use the definition, only 10.8% did so correctly. It is noticeable that 20.9% of the students who marked  $f_3$  as continuous mixed up concepts, most of these results being classified as  $M_f$  (17.6%). The mixing up of continuity with differentiability was, as in the case of the previous functions, the category with the largest percentage of students who made the wrong choice for  $f_3$  (3.4%). Group 2 was the one which most mixed up concepts ( 28.0 % in all). In  $f_3$ , even more noticeable is the percentage of students in group 3 who gave no justification at all (45.4%).



**TABLE 5.11** Distribution of categories of responses to function  $f_4$  in question 3 (% in *italics*)

<i>c / nc</i>	<i>Category</i>	<i>Subcategory</i>	<i>Group</i>			<i>Total</i> ( <i>n</i> = 148)		
			1 ( <i>n</i> = 44)	2 ( <i>n</i> = 82)	3 ( <i>n</i> = 22)			
<i>nc</i>	<i>D</i>	<i>D<sub>c</sub></i>	2 4.5	9 11.0	-	11 7.4	57	
		<i>D<sub>l</sub></i>	15 34.1	25 30.5	6 27.3	46 31.1	38.5	
	<i>G</i>		8 18.2	9 11.0	2 9.1		19 12.8	
		<i>M</i>	<i>M<sub>f</sub></i>	4 9.1	10 12.2	-	14 9.5	17
	<i>M<sub>d</sub></i>		2 4.5	1 1.2	-	3 2.0	11.5	
	<i>O</i>		3 6.8	6 7.3	1 4.5		10 6.8	
	<i>X</i>		6 13.6	7 8.5	7 31.8		20 13.5	
	<i>Total</i>		40 90.9	67 81.7	16 72.7		123 83.1	
	<i>c</i>	<i>G</i>		1 2.3	3 3.7	1 4.5		5 3.4
			<i>M</i>	<i>M<sub>f</sub></i>	-	3 3.7	1 4.5	4 2.7
<i>M<sub>d</sub></i>		1 2.3		5 6.1	1 4.5	7 4.7	7.4	
<i>O</i>			1 2.3	1 1.2	-		2 1.4	
		<i>X</i>		1 2.3	2 2.4	1 4.5		4 2.7
<i>Total</i>				4 9.1	14 17.1	4 18.2		22 14.9
<i>c/nc</i>		<i>O</i>		-	1 1.2	-		1 0.7
<i>Total</i>				-	1 1.2	-		1 0.7
<i>no answer</i>		<i>M</i>	<i>M<sub>f</sub></i>	-	-	1 4.5		1 0.7
			<i>X</i>		-	1 4.5		1 0.7
	<i>Total</i>		-	-	2 9.1		2 1.4	

Table 5.11 shows that 83.1% of the students marked the correct choice for  $f_4$ . This was the function in which the largest percentage of the students who got it right used the definition in their justifications (38.5%), although most of them did not state that the function is discontinuous because the limit does not even exist at 0 (31.1%). However, this was not a great mistake considering that it is very frequently treated in this way in

class and in some textbooks. Few of the students who chose  $f_4$  as discontinuous mixed up the concepts (11.5%). As with the other functions, mixing up continuity with differentiability was what led to most wrong choices (4.7 %).

Group 3 was the one with the largest percentage of absence of justifications (35.8%). In group 2 there was one student who chose both, continuous and discontinuous, for  $f_4$ . He wrote, "It depends on the point of view. If we analyse the function in each of the intervals, it is continuous. If we analyse it as a whole, we can affirm that it is discontinuous". This justification of his is very interesting as regards discussion of the definition of continuity ( see section 2.5).

### *Comparing the overall results*

Function  $f_2$  was the one which most of the students classified correctly (89.9%). It was also the one in which a large number of the students justified their answers by using the definition correctly (25.7 %) . This was perhaps because it was simpler in this case than in any other function, and also because it is a typical example of a discontinuous function used to teach students that the right- and left-hand limits are not always equal to the value of the function at one point.

Using the definition (category D) was predominant in the case of the justifications for the choices for all the functions. However, for  $f_1$  and  $f_3$  a significant percentage used only partial aspects of the definition, subcategory  $D_p$ , 18.9% and 16.9% respectively. Justifications classified as  $D_p$ , in which only the right- and left-hand limits were checked show that comparing these limits was seen by the students as such a strong argument that many of them thought that it is enough.

Graphical arguments were very common ( 18.9 % for  $f_1$ , 23.0 % for  $f_2$ , 18.9 % for  $f_3$  and 16.2 % for  $f_4$ ). The most common was that: it is possible (or not) to draw the graph of the function without taking the pencil off the paper; 35 (23.6%) of the justifications were written in this way.

The kind of argument which led most of the students to mark the incorrect answer for each of the functions was  $M_d$ , using arguments related to differentiability to classify the functions (5.4% for  $f_1$ , 2.0% for  $f_2$ , 3.4% for  $f_3$  and 4.7% for  $f_4$ ). It should be noted that

since none of the functions was differentiable at all the points of the domain, they could not use the fact that all differentiable functions are continuous to distinguish the continuous functions.

Furthermore, using arguments related to the concept of a function was another significant mistake, when justifying correct answers in some cases. If the students had used the argument  $\forall x \exists y \in \mathbb{R}$  consistently, they would have classified all the functions in question 3 as continuous (since all of them are defined in  $\mathbb{R}$ ).

A comparison of the performance of the different groups shows that if only the choice of continuous or discontinuous is considered, group 1, except for  $f_1$ , got the best results (90.9% for  $f_2$ ,  $f_3$  and  $f_4$ ); group 2 was the intermediate group; and group 3 was the weakest. However, the difference in performance is not very great.

As regards the justifications given, the groups which most used the definition correctly in all their justifications were groups 1 and 2: group 1 for  $f_1$  (13.6%) and  $f_3$  (11.4%), and group 2 for  $f_2$  (30.5%) and  $f_4$  (11.0%). Group 1 was also the one as well which used most graphical arguments for all the functions and used them in such a way as to get the best results in terms of marking the correct choice.

However, the most noticeable fact that derives from comparing the groups is the high percentage of students who did not give any answer at all in group 3 (31.8% for  $f_1$ , 27.3% for  $f_2$ , 45.5 % for  $f_3$  and 36.4% for  $f_4$ ). Even if these students indicated the correct choice, they could not justify why.

Finally, it should be mentioned that some of the students who tried to justify their answers by using the definition, obviously do not know how to use mathematical language. Mistakes in the logical structure of the sentences and in the use of symbols such as  $\forall$  and  $\exists$  were common. The students also had some difficulty in understanding the meaning of such sentences as:  $\lim_{x \rightarrow a} f(x) = f(a), \forall a \in \mathbb{R}$ . In this particular case, some

of the students added that  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$ .

### 5.6.3 Finding the derivative of the integral function: questions 2 and 5

In both questions, 2 and 5, students had first to find the derivative of a function in the form  $\int_a^x f(t)dt$  in order to find the value of the function at a given point. This was the reason why these questions have been grouped in the same section, comparison is going to be made of the tables showing the categories. Additionally, in question 5 they had to find the value of  $\int_a^a f(t)dt$ . Despite these similarities, the wording of these two questions was different, as we can see below:

---

**FIGURE 5.2** Questions 2 and 5

---

2) If  $g(x) = \int_2^x (2t - 1)dt$ , determine:

a)  $g'(x)$

b)  $g'(2)$

5) Let  $h(x) = 5 + \int_0^x e^{-t^2} dt$ . Find the point  $P = (a,b)$  of the graph of  $h$  such that  $h'(a)$  is equal to 1.

---

Another difference is that in question (2a) there were two options for solving it: applying the FTC directly or finding first the elementary primitive and then finding the derivative of it, while in question 5 they had no other option than applying the FTC.

The scoring of these two questions was in fact undertaken separately, with no comparison between them. Equivalent mistakes did not necessarily receive the same value. Question 5 was considered to be the more difficult, receiving, however the same value as question 2, since it was decided not to award a single answer more than 1.5 point, in order not to affect very much the total score of the student who did not answer one single question.

## Grading the questions

### . Question 2

The function  $f(x) = 2x-1$  is a polynomial function which is very easy to integrate. When students gave the correct answer in question 2a with no mistakes, they scored 1 point, whichever method they used, by integrating  $f(x) = 2x-1$  first or by using the FTC. If they made a mistake in integrating, for example answering  $-2x + 1$ , they scored 0.4. If they changed the variables, answering  $g(x) = 2t-1$  they scored 0.2. Completely wrong answers, such as writing a number as 0 for the final answer, scored 0.

In question 2b, if they obtained 3 as the final answer, they could score 0.5 or 0.3, depending on whether they answered  $2x-1$  or  $2t-1$  in question 2a. For any other answer they scored 0.

The following shows the distribution of the means according to the different groups:

**TABLE 5.12** Means and standard deviations found for question 2 - part I

Questions	Groups						Total	
	1		2		3			
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
2a	0.84	0.34	0.72	0.42	0.37	0.42	0.70	0.42
2b	0.38	0.20	0.36	0.22	0.20	0.23	0.34	0.23

Except for group 3, the results of questions 2a and 2b were reasonably good, the means for the overall group being respectively 0.70 and 0.34. Group 3 in both questions did not achieve even half of the mean: in question 2a it was 0.37 with a very significant standard deviation, 0.42, while in question 2b the mean was 0.2 and the standard deviation 0.34.

### . Question 5

The total value of this question was 1.5. If the students found  $h'(x)$  correctly, but were unable to find the point P correctly, they scored 1. If they changed the variables, answering  $h'(x) = e^{-t^2}$ , 0.2 was deducted from 1 or 1.5.

**TABLE 5.13** Means and standard deviations found for question 5

<i>Groups</i>						<i>Total</i>	
1		2		3			
<i>Mean</i>	<i>S.D.</i>	<i>Mean</i>	<i>S.D.</i>	<i>Mean</i>	<i>S.D.</i>	<i>Mean</i>	<i>S.D.</i>
0.83	0.67	0.38	0.57	0.13	0.35	0.48	0.62

Table 5.13 reveals that in question 5 the students of group 1 were much better than groups 1 and 2, while the mean of group 1 was 0.83, from groups 2 and 3 were 0.38 and 0.13. The data were not on the whole concentrated around the mean (s.d. equal to 0.62), particularly regarding groups 2 and 3 (s.d. equal to 0.57 and 0.35 respectively). In order to explain the high standard deviation, it is important to state that the number of students scoring 0 was 90, while 28 scored 1.5. Of these 28 students who got 1.5, 1 was from the daytime mathematics class, 3 from the computer sciences, 6 from the industrial engineering, 12 from the mechanical engineering and 6 from the civil engineering. It is clear, therefore, that it was mainly the engineering classes that scored 1.5.

#### *Bases for the categories*

##### *. Question 2*

The answers to question 2 were classified as right (R) or wrong (W). The right and wrong answers to question 2a were divided into two subcategories each: those solved by first finding the elementary primitive ( $R_p$ ) and those solved by applying the FTC directly ( $R_d$ ). The students only got  $R_p$  or  $R_d$  if the answer was  $2x-1$ . Small mistakes were considered when grading but not in the categories. If the question was approached by solving the integral first and a mistake was made in this process which caused the final answer to be wrong, the answer was classified as wrong, since the mistake could have been noticed afterwards if the student had an idea of the FTC.

The wrong answers were divided into those where variables were changed obtaining  $2t-1$  or  $2t-1+C$  ( $W_v$ ) and those with any other kind of mistake, for example:  $dt (2t-1) = 2$

and  $-2x+1$  ( $W_o$ ). Although in  $2t-1+C$  not only were changed variables, but the constant  $C$  was also added to the final answer, changing of the variables was the mistake which more deserved to be pointed out. As can be noticed the categories were kept the same as in the pilot with exception of one category, the one comprising students who answered  $(2x-1)dx$ . One student made this mistake, his answer was classified as  $W_o$ .

In question 2b there were no subcategories to R and W. Even students whose answers were classified as  $W_v$  in question 2a had their answers classified as right in 2b if they obtained 3 at the end.

The codes listed below will be used. It should be borne that the right and wrong answers are divided into two subcategories each only in question 2a..

R: Right  $\begin{cases} R_d: \text{directly by applying the FTC} \\ R_p: \text{the elementary primitive was found} \end{cases}$

W: Wrong  $\begin{cases} W_v: \text{mixing up variables} \\ W_o: \text{other mistakes} \end{cases}$

X: Not answered

#### . Question 5

The answers to question 5 were first divided only into right (R) and wrong (W) and the table drawn up on the basis of this division (see Table 5.17) gives a view of the question as a whole. However, in order to compare it with question 2, it should be remembered that question 5, although not divided into sub-items, can be thought of as being made up of three steps: (1) finding  $h'(x)$ , (2) finding the value of  $a$  (for which it was necessary to express  $h'(a)$  correctly), and (3) finding  $h(a)$ . The first two steps are similar to questions 2a and 2b respectively. Therefore, in order to compare questions 2 and 5 two further tables were drawn up .

In one of them only step (1), finding  $h'(x)$ , was considered. The answers were also divided into right and wrong, but the wrong answers were divided into three subcategories: those where variables were changed obtaining  $h'(x) = e^{-t^2}$  ( $W_v$ ), those where students tried to solve the integral by finding the elementary primitive ( $W_p$ ) and those with any other kind of mistake ( $W_o$ ), for example writing that  $h'(x) = 5 + e^{-x^2}$ .

In the other table only step (2), finding the value of  $a$ , was evaluated . The answers were divided into right and wrong. Even students whose answers were classified as  $W_v$  in step (1) had their answers classified as right in step (2) if they found the correct value of  $a$ .

In order to facilitate the description from here on, the steps numbered (1) and (2) will be termed 5a and 5b. The following codes will be used to present the results of question 5, remembering that only in the table with the results of (5a) the wrong answers were divided into three categories:

R: Right

W: Wrong  $\begin{cases} W_p: \text{tried to find the elementary primitive} \\ W_v: \text{mixing up variables} \\ W_o: \text{other mistakes} \end{cases}$

X: Not answered

### *Discussion of the results*

In order to discuss the results of the two questions, the tables for the overall students will be presented in the following order: first the results of (2a) and (5a); second, a comparison of the results of (2b) and (5b); and third, the results of question 5 alone.



**TABLE 5.14** Distribution of categories of responses to questions 2a and 5a (% *italics*)

Question 2a						
Category	Subcategory	Group			Total (n=148)	
		1 (n=44)	2 (n=82)	3 (n=22)		
R	R <sub>d</sub>	29 65.9	12 14.6	-	41 27.7	102 68.9
	R <sub>p</sub>	7 15.9	47 57.3	7 31.8	61 41.2	
W	W <sub>v</sub>	3 6.8	3 3.7	4 18.2	10 6.8	37 25.0
	W <sub>o</sub>	5 11.4	16 19.5	6 27.3	27 18.2	
X	-	-	4 4.9	5 22.7	9 6.1	

Question 5a						
Category	Subcategory	Group			Total (n=148)	
		1 (n=44)	2 (n=82)	3 (n=22)		
R	W <sub>p</sub>	24 54.5	21 25.6	-	45 30.4	63 42.6
		8 18.2	26 31.7	2 9.1	36 24.3	
W	W <sub>v</sub>	2 4.5	5 6.1	3 13.6	11 7.4	16 10.8
	W <sub>o</sub>	6 13.6	8 9.8	3 13.6	16 10.8	
X		4 9.1	22 26.8	14 63.6	40 27.0	

Examining Table 5.14, one point that stands out is that 68.9 % of the students were able to find the derivative of  $g(x)$ , while only 32.4 % managed to find the derivative of  $h(x)$ . If they had known how to apply the FTC correctly, these two questions would have had the same degree of difficulty. But most of them (41.2%) found the derivative of  $g(x)$  by first solving the integral, which is impossible to do in question 5.

While group 1 stood out significantly, compared with the other groups, by virtue of its

percentage of right answers (81.8% in question 2a and 54.5% in question 5), group 3 was noticeable for its percentage of questions left unanswered ( 22.7% for question 2a and 63.6% for question 5) and for its wrong solutions ( 45.5% for question 2a and 36.6% in question 5).

Comparing the mistakes common to these two questions, writing  $t$  for  $x$  in the final answer was the most conspicuous (6.8% in question 2a and 7.4% in question 5). In question 5 the most common mistake was in trying to solve the integral (24.3%). Most of the students did that either by using methods of integration by using the substitution formula or by treating  $e^{-x^2}$  as if it was not a composition of functions and therefore writing  $h(x) = e^{-x^2}$ .

Since the results of group 1 really stand out from the others and group 1 is made up of two classes, industrial engineering (ie) and mechanical engineering (me), the results of these classes should be examined separately:

**TABLE 5.15** Distribution of categories of responses for questions 2a and 5a in group 1 (% italics)

Question 2a			Question 5a		
Category	Class		Category	Class	
	ie (n = 11)	me (n = 33)		ie (n = 11)	me (n = 33)
R <sub>d</sub>	4 <i>36.4</i>	25 <i>75.7</i>	R	6 <i>54.5</i>	18 <i>54.5</i>
R <sub>p</sub>	4 <i>36.4</i>	3 <i>9.1</i>	W <sub>p</sub>	4 <i>36.4</i>	4 <i>12.1</i>
W <sub>v</sub>	-	3 <i>9.1</i>	W <sub>v</sub>	1 <i>9.1</i>	1 <i>3.0</i>
W <sub>o</sub>	3 <i>27.3</i>	2 <i>6.1</i>	W <sub>o</sub>	-	6 <i>18.2</i>
X	-	-	X	-	4 <i>12.1</i>

Table 5.15 shows that the highest percentage of students who solved question 2a directly came from the mechanical engineering class, a very different result compared to the other classes. However, not all the students who solved question 2a by applying the FTC were able to apply it in question 5. Of the 25 involved, 7 failed to do. The fact that the teacher of this class also taught one of the classes of the pilot study may have caused some interference.

The tables below show the results of questions 2b and 5b. However, it should be borne in mind that not all the students solved the previous item correctly and that one is linked to the other.

**TABLE 5.16** Distribution of categories of responses to question 2b and 5b (*% in italics*)

Question 2b				
Category	Group			Total (n=148)
	1 (n= 44)	2 (n=82)	3 (n=22)	
R	34 77.3	60 73.2	10 45.5	104 70.3
W	9 20.4	17 20.7	7 31.8	33 22.3
X	1 2.3	5 6.1	5 22.7	11 7.4

Question 5b				
Category	Group			Total (n=148)
	1 (n=44)	2 (n=82)	3 (n=22)	
R	25 56.8	21 25.6	3 13.6	49 33.1
W	10 22.7	23 28.1	3 13.6	36 24.3
X	9 20.5	38 46.3	16 72.7	63 42.6

Table 5.16 shows that most of the students who had their answers classified as R and W, in question 2a (75.7%) and 5a (37.8%), got questions 2b and 5b right : 70.3% and 33.1% respectively.

Examination of the answers classified as wrong shows that 51.5% of them in the case of question 2a and 58.3% in the case of question 5 were consistent with the answers given to 2a and 5a.

An interesting point about question 2b was the number of students who found 0 as the result of  $g'(2)$ : 9 students in all. It seems that they have an attraction for this number and forced the result to be 0. Arguments of various kinds were used to obtain 0 at the end and some of the students just wrote  $g'(2) = 0$ . As for question 5b, 4 students did not

know how to find  $a$  such that  $e^{-a^2}=1$ . Two of them stopped at this point: one said that such an  $a$  did not exist and one that  $a = 1$ .

Besides finding  $a$  such that  $h'(a) = 1$ , students had to find  $h(0)$  in question 5, to see if they associated  $h(0)$  with the image of a null area. However, as they were not asked to find  $h(0)$  directly, many intermediate steps may have interfered in the final answer. It was difficult to discover how they found  $h(0)$ , since they had not done well in the first two steps. So it seemed wiser to examine the result of question 5 as a whole, using the following table.

**TABLE 5.17** Distribution of categories of responses to question 5 (*% in italics*)

Question 5				
Category	Group			Total ( <i>n = 148</i> )
	1 ( <i>n = 44</i> )	2 ( <i>n = 82</i> )	3 ( <i>n = 22</i> )	
R	18 <i>40.9</i>	11 <i>13.4</i>	-	29 <i>19.6</i>
W	22 <i>50.0</i>	49 <i>59.8</i>	8 <i>36.4</i>	79 <i>53.4</i>
X	4 <i>9.1</i>	22 <i>26.8</i>	14 <i>63.6</i>	40 <i>27.0</i>

As we can see, even group 1 did not get even 50% of correct answers in question 5. The results of the two other groups were even worse: only 13.4% of correct answers in group 2 and no correct answer in group 3. There was also a large percentage of students who did not give any answer at all in group 3 (63.6%). Incidentally, the percentage of students who did not give any answer to question 5 (27.0 %) seems to indicate that they perhaps found it very complicated.

The results of these questions suggests that most of the students do not even know how to apply the FTC and in trying to apply it become confused and make such mistakes as changing the variables. The FTC is probably regarded as a rule which not all of them could memorise well. This section can be aptly closed with the very illustrative answer of one of the students who wrote in his questionnaire:

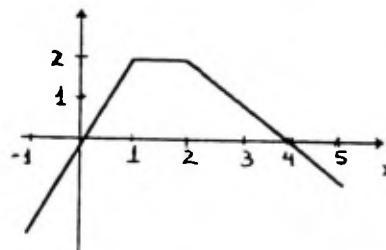
“By the first fundamental rule of calculus,  $g'(x) = (2x-1) dx$ .”

### 5.6.4 Visualising the functions $F(x)$ and $F'(x)$ : question 4

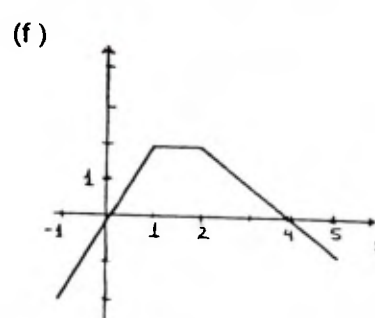
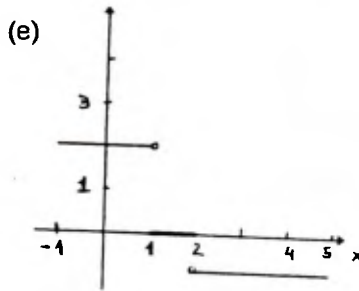
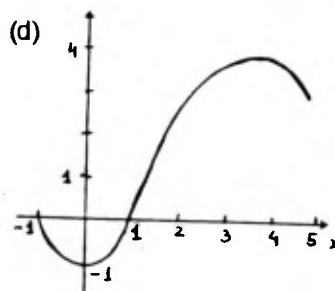
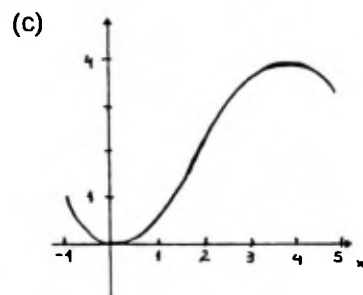
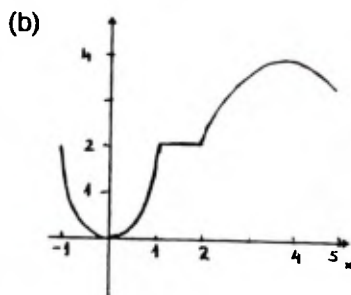
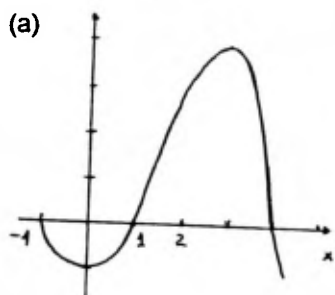
Question 4 was important for analysing some aspects of visualisation involved in the function  $F(x) = \int_a^x f(x)dx$  and in the FTC. Given the graph of a function  $f(x)$ , the students were asked to recognise the graphs of  $F(x) = \int_{-1}^x f(t)dt$  and  $F'(x)$ .

FIG. 5.3 Question 4

4) The graph at the right side is the graph of function  $f(x)$  defined in the interval  $[-1, 5]$ .



Consider the graphs (a), (b), (c), (d), (e), (f) below.



a) Which of the graphs (a), (b), (c), (d), (e), (f) could be the graph of  $F(x) = \int_{-1}^x f(t)dt$  ?

Drawn a circle around the letter corresponding to the right answer. Justify your choice

a   b   c   d   e   f

b) Considering the function  $F(x)$  defined in the item above, which of the graphs (a), (b), (c), (d), (e), (f) could be the graph of  $F'(x)$  ? Drawn a circle around the letter corresponding to the right answer. Justify your choice

a   b   c   d   e   f

### *Grading the question*

The total score for this question was 2; 1 for part (4a) and 1 for part (4b). Each correct answer, (d) for (4a) and (f) for (4b) scored 0.5 points. If the answer was right the justification was examined. For a completely correct justification students would score 0.5, otherwise the score could vary from 0 to 0.4.

The means obtained per groups and the overall results are:

**TABLE 5.18** Means and standard deviations found for question 4

Questions	Groups						Total	
	1		2		3			
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
4a	0.3	0.41	0.16	0.30	0.1	0.25	0.19	0.34
4b	0.55	0.45	0.53	0.44	0.23	0.36	0.49	0.45

Considering that the value of each of these questions was 1, Table 5.18 shows that the results are bad, for both of the questions. Although the result for question 4b (0.49) was better than for question 4a (0.19), this question was much easier than the other one. Students only had to remember that  $F'(x) = f(x)$  and to be able to visualise the meaning of this expression. Of course, the fact that the easier question came after the more difficult one has to be considered. Some students might have thought that one was a pre-requisite of the other. Even group 1, the best one, did not achieve good results. Group 3 was consistently worse than the other ones. There was also a lack of homogeneity in the scores. In question 4a particularly, the standard deviation was high (0.34).

### *Students' choices in question 4a*

Table 5.19 displays the number of students in the various groups who chose each item for question 4a. One student chose two answers, (b) and (f) instead of only one.

**TABLE 5.19** Distribution of students' choices in question 4a (% in *italics*)

Choice	Group			Total (n=148)
	1 (n = 44)	2 (n =82)	3 (n =22)	
a	1	7	1	9
	2.3	8.5	4.6	6.1
b	5	13	5	23
	11.4	15.9	22.7	15.5
c	8	17	2	27
	18.2	20.7	9.1	18.2
d	16	20	3	39
	36.4	24.4	13.6	26.4
e	2	4	1	7
	4.5	4.9	4.6	4.7
f	2	7	1	10
	4.5	8.5	4.6	6.8
b and f	1	—	—	1
	2.3	—	—	0.7
no answer	9	14	9	32
	20.4	17.1	40.9	21.6

Table 5.19 indicates that the highest percentage of students marked the correct answer. However, this represents only 26.4% of them. The other two commonest answers were (c) and (b), chosen by 18.2% and 15.5% of students respectively. Group 3 was the only group in which the highest percentage of students did not choose (d) but made some other choice, (b) in this case (22.7%). The percentage of students who did not give any answer at all in this group is also high, standing at 40.9%. The reasons why they chose each answer were of various kinds and not always right.

#### *Bases for the categories*

In order to see what strategies the students used in making their choices, attention was paid to how they made the connection between  $f(x)$  and  $F(x)$  and whether they analysed the area under the graph of  $f(x)$ . With these ends in view, four categories resulted from examining the students' questionnaires and considering the pilot categories. Two of them contain respectively the justifications of the students who considered  $f$  as the derivative of  $F$  and  $F$  as the primitive of  $f$ . The third contains the justifications of students who analysed the area under the graph of  $f$ , and the fourth contains justifications in which



none of these arguments were found. Two changes were made to the pilot categories. The first was that as none of the students justified their choice by answering only that  $F(-1) = 0$ , the category which represented this answer did not appear in the main study. The other change was that the pilot study contained a category E, comprising the justifications of students who found the analytic expressions in the graph of  $f$  in order to find  $F$ . These answers were included in the main study in category P ( $F$  was considered to be the primitive of  $f$ ). However, P is broader than E. There follow some examples taken from the questionnaires representing each of the categories.

- Category D ( $f$  as the derivative of  $F$ )

This category contains the justifications of students who used the result that  $f(x)$  is the derivative of  $F(x)$ . In fact, they used the FTC directly in their reasoning to solve the question.

In most of these cases the students examined the sign of  $f$  in order to analyse when  $F$  is decreasing or increasing. A very illustrative example is:

“ $F'(x) = f(x)$ , so  $f(x)$  is the graph of the values of the tangents to the graph  $F(x)$ .  
In item d: from -1 to 0 the tangent is negative and increasing,  
from 0 to 1 the tangent is positive and increasing,  
from 1 to 2 the tangent is constant and positive  
from 2 to 4 the tangent is positive and decreasing  
in 4 the tangent is null and  
from 4 to 5 the tangent is negative and decreasing.”

- Category P ( $F$  as the Primitive of  $f$ )

This category comprises the answers of students who constructed their arguments thinking of  $F(x)$  as the primitive of  $f(x)$ .

Some students tried to find an analytic expression for  $F(x)$ . They were supposed to find:

$$F(x) = \begin{cases} x^2 - 1, & \text{for } -1 \leq x \leq 1 \\ 2x - 2, & \text{for } 1 < x < 2 \\ -\frac{x^2}{2} + 4x - 4, & \text{for } 2 \leq x \leq 5 \end{cases} \quad ”$$



but most of them found only two of the three expressions, while others did find the three expressions, but with mistakes.

Others gave answers which simply focused on the degree of the  $F$  being one bigger than that of  $f$ , as in the following examples:

“From -1 to 1  $f(x)$  is a linear function of the first degree, so its integral will be of the second degree; from 1 to 2  $f(x)$  is linear of 0 degree, so its integral will be of the first degree; from 2 to  $\infty$   $f(x)$  is linear of the first degree, so its integral will be of the second degree.”

and others used numbers in the upper limits of integration of each of the integral functions corresponding to each interval, obtaining at the end numbers and not functions.

$$\text{“} \int_{-1}^1 2x dx = 0; \int_1^2 2 dx = 2; \int_2^5 (-x + 4) dx = \frac{3}{2} \text{”}$$

#### - Category A (Area)

In category A, the students analysed the area under the graph of  $f$  to decide which graph should correspond to  $F$ . One example in this category is:

“From -1 to 1, the area increases non-uniformly and is negative; from 1 to 2 the area increases uniformly, from 2 to 4 the area increases non-uniformly, from 4 to 6 diminishes, even though it is positive.”

There is only one example in this category in which it is not completely clear that the student reasoned using area. He wrote in the questionnaire:

“The graph must pass through for (-1,0) because  $\int_{-1}^1 f(t) dt = 0$  and it cannot be zero at the point  $x = 4$ .”

For this student to decide that  $F(4) \neq 0$  directly, he must have thought in terms of the area under the graph of  $f$ . This is why this answer was classified as A. This strategy was the most straightforward way to approach this question.

#### - Category O (Others)

Category O contains only the answers which do not fit into any of the other categories mentioned. Some of them reflect the fact that students are applying a rule for integration

of polynomial functions (the degree of the function increases of 1 after integration) to a general expression of  $f(x)$ . Three students answered in the way shown below:

$$“F(x) = \left[ \frac{f(t)^2}{2} \right]_{-1}^x”$$

Other answers reflect students' confusion are over what it is the relation between  $f(x)$  and  $F(x)$  is, as for example:

$$“F(x) = \int_{-1}^x f(x) dx = f'(x)”$$

In fact, there are all sort of examples in O and it is not easy make sense of some of them.

Summarising, the different categories can be listed as follows:

D:  $f$  was seen as the derivative of  $F$

P:  $F$  was seen as the primitive of  $f$

A: Analysed the area under the graph of  $f$

O: Others

X: Not answered

### Discussion of the results

The tables below present the justifications for (b), (c) and (d), which were the three items most commonly chosen:

**TABLE 5.20** Distribution of categories of justifications for the choices of (b), (c) and (d) in question 4a (% in *italics*)

<i>b/c/d</i>	Category	Group			Total ( <i>n</i> = 148)
		1 ( <i>n</i> = 44)	2 ( <i>n</i> = 82)	3 ( <i>n</i> = 22)	
<i>b</i>	D	–	1 <i>1.2</i>	–	1 <i>0.7</i>
	P	3 <i>6.8</i>	7 <i>8.5</i>	3 <i>13.6</i>	13 <i>8.8</i>
	A	–	–	–	–
	O	2 <i>4.5</i>	3 <i>3.7</i>	2 <i>9.1</i>	7 <i>4.7</i>
	X	–	2 <i>2.4</i>	–	2 <i>1.4</i>
	<i>Total</i>	5 <i>11.3</i>	13 <i>15.8</i>	5 <i>22.7</i>	23 <i>15.6</i>
<i>c</i>	D	1 <i>2.3</i>	2 <i>2.4</i>	–	3 <i>2.0</i>
	P	3 <i>6.8</i>	12 <i>14.6</i>	2 <i>9.1</i>	17 <i>11.5</i>
	A	–	–	–	–
	O	3 <i>6.8</i>	1 <i>1.2</i>	–	4 <i>2.7</i>
	X	1 <i>2.3</i>	2 <i>2.4</i>	–	3 <i>2.0</i>
	<i>Total</i>	8 <i>18.2</i>	17 <i>20.6</i>	2 <i>9.1</i>	27 <i>18.2</i>
<i>d</i>	D	4 <i>9.1</i>	1 <i>1.2</i>	–	5 <i>3.4</i>
	P	4 <i>9.1</i>	11 <i>13.4</i>	2 <i>9.1</i>	17 <i>11.5</i>
	A	3 <i>6.8</i>	1 <i>1.2</i>	–	4 <i>2.7</i>
	O	3 <i>6.8</i>	5 <i>6.1</i>	–	8 <i>5.4</i>
	X	2 <i>4.6</i>	2 <i>2.4</i>	1 <i>4.5</i>	5 <i>3.4</i>
	<i>Total</i>	16 <i>36.4</i>	20 <i>24.4</i>	3 <i>13.6</i>	39 <i>26.4</i>

Analysing Table 5.20 above, it can be seen that the most frequent strategy used was P. Few students had their answers classified as D and even fewer classified as A. Group 1 was the only one with a significant percentage of answers classified as D or A. In fact, it was the only one in which there were students who used the strategy A (analysing the

answers (b), (c) and (d)). In group 3 none of the students who made these three choices had their justifications classified as A or D.

No tables are given for the choices of (a), (e) or (f), not only because few students chose these answers (26 students in all), but also because 15 had their justifications classified as O and 6 did not give any justification at all. The justifications of 4 students were classified as P, and the remaining one was classified as A, although he chose (a) as his right answer. From the 15 justifications classified as O, 6 of them reflect the fact that the students mixed up what the FTC states. Four of these students chose f because they thought that  $F(x) = f(x)$  and two chose (e) because they thought that  $F'(x) = f'(x)$ .

### *Students' choices in question 4b*

The following table displays the number of students in the various groups who chose each item for question 4b.

**TABLE 5.21** Distribution of students' choices in question 4b (% in *italics*)

Choice	Group			Total ( <i>n</i> =148)
	1 ( <i>n</i> = 44)	2 ( <i>n</i> =82)	3 ( <i>n</i> =22)	
a	–	1 <i>1.2</i>	–	1 <i>0.7</i>
b	3 <i>6.8</i>	2 <i>2.4</i>	–	5 <i>3.4</i>
c	2 <i>4.5</i>	5 <i>6.1</i>	3 <i>13.6</i>	10 <i>6.8</i>
d	–	2 <i>2.4</i>	1 <i>4.6</i>	3 <i>2.0</i>
e	2 <i>4.5</i>	6 <i>7.3</i>	2 <i>9.1</i>	10 <i>6.8</i>
f	28 <i>63.6</i>	51 <i>62.2</i>	7 <i>31.8</i>	86 <i>58.1</i>
no answer	9 <i>20.5</i>	15 <i>18.3</i>	9 <i>40.9</i>	33 <i>22.3</i>

Table 5.21 shows that 58.1% of the students got the right answer, (f). The percentage of students who chose other items is not significant compared with this one. However, 22.3% of the students did not give any answer at all.

No significant difference was found comparing group 1 (63.6% of correct answers) with group 2, in which this percentage was 62.2%. Group 3 was the worst, only 31.8% of the students answering (f). Group 3 was also the one in which the highest percentage of students (40.9%) did not choose any of the graphs, however, for all the groups this percentage was also significant, even for the best group, 20.5%.

### *Bases for the categories*

The categories used in the pilot for this question were kept the same as in the main study. The criterion for classifying the justifications to question 4b being to see if students used the FTC or not. The justifications were divided into those that used the FTC, those that used the result of (4a) and those that did not use any of these arguments. Here are some examples of each of the categories.

#### **- Category F (Used the FTC)**

This category groups together the students who knew how to apply the result of the FTC visually. They simply answered:

“Because  $F'(x) = f(x)$ .”

Or used words trying to express this relation, for example:

“The derivative of the integral of a function is the function itself.”

#### **- Category P (Procedural)**

This category comprises the justifications of the students who used the answer they gave to question 4a in order to find the answer to question 4b. Some students analysed the graph they obtained in (4a) in order to see what graph would correspond to the derivative of it. One example was the student who chose (b) for question 4a and (f) for question 4b and wrote:

‘Because in the interval  $x = -1$  to  $x = 1$  the graph is a parabola and the derivative of it is a straight line, from  $x = 1$  to  $x = 2$  it is a constant, so it turns into a constant and for  $x > 2$  it is a parabola, finding the derivative of it, it becomes a straight line.’

One can see from the context that when this student says a straight line, he means a non horizontal line.

There is also the case of others who solved question 4a in a completely wrong way but used the result obtained to solve question 4b. One student, for example, who had chosen

(a) in question 4a because  $F(x) = \left[ \frac{f(t)^2}{2} \right]_{-1}^x$  and so  $F(x) = \frac{(f(x))^2 - 4}{2}$ , found in

question (4b) that:

$$“F'(x) \text{ was equal to } \frac{1}{2} 2f(x) = f(x) \Rightarrow f(x) = F'(x)”$$

#### - Category O (Others)

All the justifications which are not classified as F or P are in category O. In O there are justifications like the ones below (the first was a justification for the choice of (b) and the second for the choice of (e)):

“The function assumes 0 in  $-1 < x < 1$ ”

and

“This is the correct derivative of the function  $F(x) = \int_{-1}^x f(t)dt$ , there is no other that puts this affirmation in doubt.”

Summarising, the classifications for question (4b) were:

F: Used the FTC

P: Procedural reasoning: used (4a)

O: Others

X: Not answered

#### *Discussion of the results*

Table 5.22 displays the number of students who chose (f) according to the categories described above. Only the table for (f) will be presented since (f) is the only answer with a significant percentage of students who chose it.

**TABLE 5.22** Distribution of categories of justifications for the choice of (f) in question 4b (% in *italics*)

Category	Group			Total ( <i>n</i> = 148)
	1 ( <i>n</i> = 44)	2 ( <i>n</i> = 82)	3 ( <i>n</i> = 22)	
F	20 <i>45.4</i>	37 <i>45.1</i>	3 <i>13.6</i>	60 <i>40.5</i>
P	4 <i>9.1</i>	9 <i>11.0</i>	3 <i>13.6</i>	16 <i>10.8</i>
O	4 <i>9.1</i>	3 <i>3.7</i>	1 <i>4.6</i>	8 <i>5.4</i>
X	—	2 <i>2.4</i>	—	2 <i>1.4</i>
Total	28 <i>63.6</i>	51 <i>62.2</i>	7 <i>31.8</i>	86 <i>58.1</i>

Table 5.22 reveals that 60 of the 86 students who chose (f) did so for the correct reason, using arguments classified as F, while 16 students reasoned in a more procedural way, analysing the answer given to (4a). Group 3 was the only one in which the percentage of students whose justifications were classified as P (13.6%) is significant. In fact, it is equal to the percentage of justifications which were classified as F. In groups 1 and 2 the difference between the frequencies in categories F and P is greater.

Of the 60 students who chose (f) and had their answers classified as F, all except 7 scored 1 point in this question. These 7 received 0.8. Two because for them  $F'(x) = f(x) + C$  ( $C$  is a constant) and 5 because they wrote that  $F'(x) = f(t)$ . Again in this question the problems with the constant and variables appeared.

The results of question 4 suggest that if it was difficult for the students to apply the FTC in a graphical context, it was even more difficult for them to understand visually what an integral function is.

### 5.6.5 Stating the FTC: question 6

The students were asked in this question simply to state the Fundamental Theorem of Calculus. The aim was to see what ideas came to their mind when they were asked that.

Attention was paid to two main points in their answers: if they wrote that  $f$  has to be continuous and if they wrote that  $F'(x) = f(x)$ , which would mean that they had some grasp of the foundations of what the theorem is about.

### *Grading question 6*

The value of this question was 1 point. If the students' answers had nothing to do with the FTC or had clearly been copied from a classmate (a problem which will be discussed below), they got 0 points. If they answered that  $\int_a^b f(x)dx = F(b) - F(a)$ , not writing the relation between  $f$  and  $F$ , the score was 0.2 points. All the other answers in which the students tried to establish a relation between the integral and the derivative, but which contained mistakes or were incomplete, scored between 0.3 and 0.9 points.

The means and standard deviations obtained were:

**TABLE 5.23** Means and standard deviations found for question 6

Question	Groups						Total	
	1		2		3			
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
6	0.17	0.26	0.14	0.24	0.12	0.27	0.14	0.25

The overall average was very low (0.14). The same can be said comparing the different groups. This low average and the high standard deviation can be explained by the fact that 103 students got 0 points in this question. None of the students scored 1. Only one student got 0.9. He mentioned both, that  $f$  has to be continuous and  $F'(x) = f(x)$ , but made one mistake. The next highest score was 0.8 and only one student achieved it. His answer would have been correct if he had not changed the condition of  $f$  from having to be continuous to being differentiable.

### *Bases for the categories*

In the case of this question, it was decided to make a much more detailed analysis of the answers than in the pilot study, in which the criterion used to classify the answers was to see if students stated that  $F' = f$ . In the main study an analysis was made to see whether they stated any association between derivative and integral (which included  $F' = f$ ) and, if they did, how. Three categories were set up with this aim in mind. One more unexpected



category appeared: that of the students who copied the FTC from their classmates. A description of the categories follows with examples for each of them.

- Category A (Associated derivative and integral)

In all the answers in this category students stated an association between derivative and integral. There are two groups in this category, abbreviated as  $A_e$  and  $A_i$ .

-  $A_e$  (Explicit Association:  $F'(x) = f(x)$ )

All the answers in this category mention that  $F'(x) = f(x)$ . Two examples of students' answers are:

$$F(x) = \int_a^x f(t) dt,$$

$$F(x) = f(t) \big|_a^x \Rightarrow F(x) = G(x) - G(a)$$

$$F'(x) = G'(x)$$

$$G'(x) = f(x) \Rightarrow F'(x) = f(x)$$

and

$$\int_a^b f(x) dx = G(b) - G(a) \Rightarrow f(x) = G'(x)."$$

As we can see from these examples, not only did these students fail to mention that  $f$  has to be continuous, but also made several mistakes in trying to prove the theorem, showing that they had not understood the main concepts taught in calculus.

In this category there are also the answers of the two students who were the only ones to mention that  $f$  has to be continuous. They would have had their answers classified as correct if they had known how to enunciate the theorem correctly, having clear in their minds the hypothesis and the thesis of the theorem. One of the students wrote:

"Let  $f(x)$  be a continuous function in the interval  $[a,b]$  and  $g(x)$  a primitive of  $f(x)$ , so we define:  $F(x) = \int_a^b f(x) dx = g(b) - g(a)$  and also  $F'(x) = f(x) = g'(x)$ ."

This student would have enunciated the corollary of the FTC right if he had not defined  $F(x)$  wrong, owing probably to his not understanding the role of the variable  $x$ . The other point is that he used  $g'(x) = f(x)$  as the hypothesis (saying that  $g(x)$  is the primitive of  $f(x)$ ) and the thesis of the theorem.

The handwriting of the other student was not clear. His answer is quoted below, and in brackets is a version of what he might have written at the end:

“Let  $f$  be a continuous function in the interval  $[a,b]$ ,  $\forall x \in [a,b]$ , and let  $F$  be a function  $F(x) = \int_a^x f(t)dt$ , we have that  $F'(x)$  will be equal to  $F'(x) = f(x)dx$  (or  $f'(x) = F(x) dx$ ).”

Probably the first version is the right one. Questions 2a, 4b and 5 were examined in this student's questionnaire. In these three questions he knew how to apply the FTC and made the same mistake of including the differential element.

#### - $A_I$ (Inadequate Association)

It was not mentioned in  $A_I$  that  $F'(x) = f(x)$ . The students tried to establish an association between integral and derivative, but either they only characterised the indefinite integral or they made many mistakes in describing this association. Some examples are:

$$\text{“ } \frac{d}{dx} \int f(x)dx = f(x) \text{”};$$

$$\text{“Let } f(x) = \int_a^b f(x)dx, (\int_a^b f(x)dx)' = f(b) \text{”}$$

and

$$\text{“ } \int_a^b f(x)dx = F'(b) - F'(a) \text{”}$$

#### - Category F (Formula not characterised)

The following answer occurs in F:  $\int_a^b f(x)dx = F(b) - F(a)$ . Probably the students had in mind that  $F'(x) = f(x)$  when writing that  $\int_a^b f(x)dx = F(b) - F(a)$ , but perhaps they only wrote it down without reflecting about it.

#### - Category C (Copied from classmates)

In this category are the answers copied from classmates who were already solving the second part of the questionnaire. Unfortunately, some students, in spite of having been told that they were doing the test for purposes of research, still insisted on cheating and

copying the FTC . This was detected in the case of eight students in the computer science class.

- Category B (Bits of calculus)

In this category are classified the answers of students who could not express anything about the FTC or any comprehensible relation between derivative and integral. They wrote random bits of calculus. The follows are examples:

$$\text{“} \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x) - f(x)}{\Delta x} \text{”},$$

“Every differentiable function is continuous”;

“The integral of a function can be seen as the sum of the areas of small intervals of the function”;

and

“If we know how to evaluate the tangent of a graph at a certain point, we will know how to evaluate the area of the graph”.

Summarising the description of the categories, we have:

A: Associated derivative and integral  $\begin{cases} A_e: \text{explicit association: } F'(x)=f(x) \\ A_i: \text{inadequate association} \end{cases}$

F: Formula no characterised

C: Copied from classmates

B: Bits of calculus

X: Not answered

## Discussion of the results

The following table displays the results of the analysis of this question:

**TABLE 5.24** Distribution of categories of responses to question 6 (% *italics*)

Category	Subcategory	Group			Total ( <i>n</i> =148)
		1 ( <i>n</i> =44)	2 ( <i>n</i> =82)	3 ( <i>n</i> =22)	
A	A <sub>e</sub>	7 <i>15.9</i>	9 <i>11.0</i>	1 <i>4.6</i>	17 <i>11.5</i>
	A <sub>i</sub>	5 <i>11.4</i>	9 <i>11.0</i>	3 <i>13.6</i>	17 <i>11.5</i>
F		5 <i>11.4</i>	6 <i>7.3</i>	—	11 <i>7.4</i>
C		—	8 <i>9.8</i>	—	8 <i>5.4</i>
B		8 <i>18.2</i>	22 <i>26.8</i>	5 <i>22.7</i>	35 <i>23.7</i>
X		19 <i>43.2</i>	28 <i>34.1</i>	13 <i>59.1</i>	60 <i>40.5</i>

As can be seen from Table 5.24 there was a very large percentage of students who did not even write anything in question 6 (40.5%) or if they did, wrote things that had nothing to do with the FTC (23.7%). It is now clear from the table why 103 students (69.6 %) scored 0 in this question. This is the result of adding together the number of students who had their answers classified as C, B or X (8+35+60).

Comparison of the three groups did not contribute much new information. Adding the percentages of categories C, B and X, which reflect the fact that students had no idea of what the FTC is, the results for each group are: 61.4% for group 1, 70.7% for group 2 and 81.8% for group 3. None of the groups did well; the worst was group 3.

Question 6 proved to be very useful in helping to analyse how students stated theorems. Some of them, instead of stating a theorem, tried to give a proof for what they wanted to state. These proofs revealed several misconceptions: writing the hypothesis as the thesis, not defining the arguments (the functions) they used in the “proof”, or using the thesis as the hypothesis as well. See the examples shown in category A<sub>e</sub>.

An interesting point that also appears is the fact that some students missed the main point of the FTC, that is to establish that a continuous function  $f$  always has a primitive:

$F(x) = \int_a^x f(t)dt$ . They only stated what the indefinite integral is (answers coded as  $A_1$ ).

In category  $A_1$  some answers are also included in which it is clear that the students know that there is a relation between integral and derivative, but made several mistakes when attempting to express this relation. They tried to mix the results of the FTC and its corollary and did not really know what a definite integral was.

An analysis of the answers in category B could by itself be the object of a separate study. There were some students who confused a theorem and a definition, thinking that the FTC established that the definite integral is the area of the graph. Others tried to state the mean value theorem and others even wrote expressions which showed that they mixed up several concepts learnt in calculus.

Another interesting point that arose from analysis of this question was to examine in more detail the questionnaires of the eight students who had copied answers from their classmates. Instead of these questionnaires being annulled, they were used to find out if this cheating had influenced the solution of the other questions, particularly questions 4b and 5. Three students did not even answer 4b; one chose item f but explained in his own words that differentiation and integration are inverse processes. Four, however, may have been influenced, since they wrote  $F'(x) = f(x)$  or used the words “by the FTC” as justification for having chosen f. Even if some students were influenced, it was only in question 4b; none of them could find the derivative of  $h$  in question 5. That in itself is a very interesting fact.

## 5.7 Results per questions: part II

This part aimed to identify students’ beliefs about the processes of proving in mathematics. In order to help them see that there was a difference between the aims of part II and the aims of part I, the following was written at the beginning of part II:

“The first part was devoted to the learning of certain mathematical concepts; in this second part we want to know some of your ideas about mathematics.”

Whether these words produced the desired effect or not will be seen from the analysis which follows. The presentation of the analysis and results of the two questions will be set down basically following the same model as the questions of part I. One difference is that there will be no division into groups. The Chi-square did not show any significant

difference in the performance of the classes considering part II as a whole. However, particularly in this part of the questionnaire, it was interesting to examine how the different classes answered the questions, by examining the different categories, and paying a closer attention to the students of the mathematics classes. For this reason the data in the tables will be given by classes.

The other difference in relation to the analysis of part I is that instead of showing the mean and standard deviation of each of the questions, the frequency of each score will be given, since for each of the questions there were only three possible scores: 0, 1 or 2, and most of the students in fact got 0.

### 5.7.1 Generalising in mathematics: question 1

In the first question the students were asked if they believed that, working from examples alone, they could generalise a result in mathematics that they had been (hypothetically) the first to find. This situation was put for them in the following words:

---

**FIG. 5.4** Questions 1: part II

---

1) You are studying mathematics and noticed that:

$$\text{if } F(x) = \int_0^x 2t dt \text{ then } F'(x) = 2x.$$

Suppose that you are the first person in the world to observe this result. On the basis of this example alone, could you publish the following in a book:

“ Let  $f$  be a continuous function on the closed interval  $[a,b]$  and  $x$  any number in this interval.

If  $F$  is the function defined as  $F(x) = \int_a^x f(t)dt$  , then  $F'(x) = f(x)$ .” ?

Yes ☐ No ☐

---

#### *Grading the question*

The value of this question was 2. If the students wrote that it was necessary to provide a proof, they scored 2. If they gave the idea that this example was not enough but did not make clear the need to supply a proof, they scored 1; otherwise they scored 0. The table below displays the frequency of scores by classes.

**TABLE 5.25** Frequency of scores found for question 1 - part II

Score	Class						Total
	em	dm	cs	ce	me	ie	
0	19 86.4	20 83.3	25 75.8	20 80.0	22 66.7	9 81.8	115 77.7
1	3 13.6	2 8.3	5 15.1	4 16.0	9 27.3	2 18.2	25 16.9
2	0	2 8.3	3 9.1	1 6.0	2 6.1	0	8 5.4

Table 5.25 shows that only 5.4% of the students answered this question correctly, none of them belonging to the classes which were considered the worst in part I, the em class (evening mathematics) or the best, the ie class (industrial engineering). Adding the frequencies of the students who got 1 and 2, the class which performed best was the mechanical engineering class: 33.4% scored 1 or 2 in this question. However, none of the classes did well. The mean of this question was very low, 0.28.

#### *Bases for the categories*

The categories used were almost the same as in the pilot study, with two differences: one pilot category was excluded from the main study, while one other was included. This category comprised the inconclusive answers, in which, although students made it clear that one example was not enough, they did not affirm the necessity of a proof. The category excluded was the one in which students tried to give a proof for the FTC, their answers being grouped together with those in which students referred to the FTC.

#### **- Category P (Proving as necessary to generalise)**

Consonant with the aims of this question, this was the only category with the correct answers. It comprises those of the students who believed that in order to make a generalisation in mathematics a formal proof was necessary. One example is:

“With one example we cannot prove anything (...); to prove something, it is necessary to use a general case.”

#### **- Category I (Inconclusive: insufficient data to state a theorem)**

Category I comprises the answers in which students expressed the idea that there were insufficient data to state a theorem. However, these answers do not guarantee that the

students believed that it is necessary to generalise through a formal proof or that they thought that it was sufficient to test with further examples. There are also cases in which they mentioned the need of a mathematical explanation, but it is not clear what kind of explanation they had in mind. Their answers were inconclusive. Two examples follow.

“A theorem cannot be based on only one example.”

“I would have to test with other examples to really be sure that the above hypothesis is true. After verifying that, I would go on to a mathematical explanation, thus removing any doubt as to the hypothesis.”

#### - Category E (Empirical view)

Category E contains the answers of students who made it clear that they believed that examples are enough to guarantee the truth of a theorem. Some tried to explain why the example was true, by using the following argument:  $\int_0^x 2t dt = t^2 \Big|_0^x$ . Others offered different examples, a popular one being to substitute 1 for 0 at the lower limit of the integral. Others simply stated it was true on the basis of the example given in the question:

“It is right; since I was the first to have verified this, my law is true until someone proves the contrary.”

#### - Category F (By the FTC or making reference to the ideas contained in it)

Category F contains the answers of students who were not able to put themselves into the situation of having no previous knowledge of the theorem. For them it was difficult to imagine that they were the first to have met the example or to have to answer the question on the basis of only one example. Most of them just cited the FTC, while others tried to give an explanation for the theorem. Here are some examples of this sort:

“This is the Fundamental Theorem of Calculus.”

“ $F(x)$  is an anti-derivative of  $f(x)$ , if  $F(x) = \int_a^x f(t) dt$ , finding the derivative we have  $F'(x) = f(x)$ .”

This category also includes the case of some students who misread the statement of the FTC, but believed what they had misread. Although the statement misread was not the



FTC, they thought it was. This situation of misreading will be discussed later in the discussion of the results.

#### - Category S (Statement is not correct)

Category S contains the answers of students who doubted the statement of the FTC. They showed that either by trying to find a mistake in the statement or by saying that the statement was not complete. This category contains answers by students who wrote, for example, (1) that there should be a restriction with regard to the  $a$  in  $\int_a^x f(t)dt$ ; or (2) that is not enough for the function to be continuous; or (3) that it is not true that  $F' = f$ , or (4) that there are exceptions to the theorem. Although this case (4) seems to deserve to be fitted into a special category of its own, it does, in the last analysis, reflect the belief that the statement is not complete, which means, in other words, the statement is not correct.

“It will only be true if  $a = 0$ .”

“No, because not all the continuous functions are differentiable, so this fact cannot be generalised.”

“There are cases in which this does not happen, as in the example below:

$$\frac{d}{dt} \int_0^{x^2} 2t \, dt = 2t \, dt = 2x^2 (2x) = 4x^3$$

$$(t = x^2 \, dt = 2x)''$$

#### - Category O (Others)

Category O contains answers of all sorts. Some in which no mathematical argument was used, such as:

“Probably because even before we learn the subjects, the theorems are thrown at us in such a way as to press us to memorise them and not to reflect. The time is very short and there is no time to make analyses of this type.”

Some with a serious mathematical mistake, for example:

$$F(x) = \int_a^x f(t)dt = F(x) - F(a)$$

$$F(x) = F'(x) = [f(t)]^x \rightarrow F'(x) = f(x)''$$

And others which made sense in mathematics but did not belong to any of the categories quoted . This is the case of one student who first solved  $F(x) = \int_0^x 2t dt = t^2 \Big|_0^x$  and found  $F'(x)$  ; then did the same to  $\int_1^x 2t dt$  , and finally to  $\int_a^x 2t dt$  , writing that this was the “general example”. This student certainly thought that to prove the theorem meant to generalise it for  $a$ .

Summarising the description of the categories, we have:

P: Proving as necessary to generalise

I: Inconclusive: insufficient data to state a theorem

E: Empirical view

F: By the FTC or making references to the ideas contained in it

S: Statement is not correct

O: Others

X: Not answered

## Discussion of the results

The table below displays the results of this question according to the yes or no answer given.

**TABLE 5.26** Distribution of categories of responses to question 1- part II (% in *italics*)

no / yes	Category	Class						Total (n = 148)
		<i>em</i> (n=22)	<i>dm</i> (n=24)	<i>cs</i> (n=33)	<i>ce</i> (n=25)	<i>me</i> (n=33)	<i>ie</i> (n= 11)	
no	P	–	1 <i>4.2</i>	2 <i>6.1</i>	1 <i>4.0</i>	2 <i>6.1</i>	–	6 <i>4.1</i>
	I	3 <i>13.6</i>	2 <i>8.3</i>	5 <i>15.1</i>	4 <i>16.0</i>	9 <i>27.3</i>	2 <i>18.2</i>	25 <i>16.9</i>
	S	2 <i>9.1</i>	5 <i>20.8</i>	1 <i>3.0</i>	3 <i>12.0</i>	4 <i>12.1</i>	5 <i>45.4</i>	20 <i>13.5</i>
	O	1 <i>4.6</i>	1 <i>4.2</i>	–	1 <i>4.0</i>	–	–	3 <i>2.0</i>
	X	1 <i>4.6</i>	–	–	–	1 <i>3.0</i>	–	2 <i>1.3</i>
<i>Total</i>		7 <i>31.8</i>	9 <i>37.5</i>	8 <i>24.2</i>	9 <i>36.0</i>	16 <i>48.5</i>	7 <i>63.6</i>	56 <i>37.8</i>
yes	P	–	1 <i>4.2</i>	1 <i>3.0</i>	–	–	–	2 <i>1.3</i>
	E	–	–	6 <i>18.2</i>	2 <i>8.0</i>	2 <i>6.1</i>	2 <i>18.2</i>	12 <i>8.1</i>
	F	8 <i>36.4</i>	9 <i>37.5</i>	13 <i>39.4</i>	7 <i>28.0</i>	5 <i>15.1</i>	2 <i>18.2</i>	44 <i>29.7</i>
	O	2 <i>9.1</i>	4 <i>16.7</i>	4 <i>12.1</i>	7 <i>28.0</i>	7 <i>21.2</i>	–	24 <i>16.2</i>
	X	5 <i>22.7</i>	1 <i>4.2</i>	1 <i>3.0</i>	–	2 <i>6.1</i>	–	9 <i>6.1</i>
<i>Total</i>		15 <i>68.2</i>	15 <i>62.5</i>	25 <i>75.7</i>	16 <i>64.0</i>	16 <i>48.5</i>	4 <i>36.4</i>	91 <i>61.5</i>
no answer	X	–	–	–	–	1 <i>3.0</i>	–	1 <i>0.7</i>
<i>Total</i>		–	–	–	–	1 <i>3.0</i>	–	1 <i>0.7</i>

Table 5.26 reveals that only 8 students (5.4%) answered this question correctly, by saying that it was necessary to produce a proof (category P). Although two of these students answered “yes”, they made it clear that the answer would only be “yes” if a proof were given. Twenty-five students (16.9%) gave inconclusive answers (category I) and twelve (8.1%) did not answer the question. Most of the students answered this question incorrectly, 103 in all (69.5%), the number obtained by adding the results of categories S, E, F, and O.

Comparing the classes, it is clear why the mechanical engineering class was the one with the best scores. It was the class with the highest percentage of students (27.3%) who answered “no”, giving answers classified as I (insufficient data to state a theorem), which scored 1 point. The industrial engineering class was the one with the highest percentage of “no” as the answer, 63.6%, but most of these students had their answers classified as S. Neither of the mathematics classes did well: only one student from the daytime class said it was necessary to prove the theorem. Most of the students in both classes (68.2% of the em class and 62.5% of the dm class) answered “yes”.

Category E should be the one in which the students who believed that examples are enough to guarantee the truth of a theorem would fit. However, probably some of them had their answers classified as I. On the other hand, category P should be the one comprising the students who believe that in order to make a generalisation in mathematics a formal proof is necessary. But again, probably some of them had their answers classified as I.

One interesting point in this question is that some students doubted the statement of the FTC. Probably, when asked if they could state the theorem, it never entered their heads that the problem could lie in the method for generalising in mathematics; they thought there might be a mistake in the statement of the theorem. Their answers were classified as S, and represented 13.5 % of the students. It is incredible that even though the theorem was stated in the second question with the remark that its proof could be found in several books on calculus, these students still insisted on finding mistakes in the statement.

Unfortunately, the statement was misread by some students. The theorem was stated between quotation marks. Some students understood  $f(x)$  at the end as the second derivative of  $f$ . This was certainly the case of 5 students, but perhaps more than this number had the same problem. This could have been an obstacle to the analysis but, in fact, analysing the justifications the students wrote proved to be very interesting. One student answered “no”, explaining that the derivative of the integral of a function is the function itself. Her answer was classified as S since she tried to find a mistake in the statement and in fact her reading was influenced by this attempt. The students who

answered “yes”, had their answers classified as F and tried to explain why  $F'(x) = f''(x)$ . One student even gave a proof, writing QED at the end. One example is:

“ $F(x)$  is a function whose derivative is  $f(t)$ , that is,  $f(t)$  is the derivative of a function, so the derivative of  $F(x)$  will be  $f(t)$ , which itself is a derivative, and is therefore the second derivative.”

Compared to question 1 of the pilot study, this one had a much stronger impact, since only one example was given and a hypothetical situation was created to put the students in the role of researchers. The idea was to reduce the number of students who had their answers classified as F. In the pilot study, 39% of them had their answers classified as F; this time the percentage was 29.7%. Although this was a small percentage, it was difficult to induce the students to get into the situation proposed by the question.

However, even with the excuse that most of them could not understand the aim of the question it seems there was a strong feeling for answering “yes” (61.5% of the answers). To show this feeling, Table 5.26 was deliberately organised in terms of the distribution of the categories according to the yes/no answer instead of organising it in terms of the categories representing the correct (P), inconclusive (I) and wrong answers (S + E + F + O).

### 5.7.2 Proof in mathematics: question 2

In the second question the students were asked if they could find a counter-example for a theorem; in other words, if they believed that theorems generalise. This question was stated for them as follows:

---

**FIG. 5.5** Question 2: part II

---

2) The Fundamental Theorem of Calculus can be stated in the following way:

“Let  $f$  be a continuous function in the closed interval  $[a, b]$  and  $x$  any number in this interval. If  $F$  is the function defined as  $F(x) = \int_a^x f(t) dt$ , then  $F'(x) = f(x)$ .”

The proof of this can be found in several books on calculus.

Can you give an example of a continuous function in an interval  $[a, b]$  which does not satisfy the Fundamental Theorem?

Yes ☐ No ☐

Justify your answer.

---

### *Grading the question*

The value of this question was the same as question 1: it scored 2. If the students wrote no, affirming that such a function does not exist, they scored 2. If they said they had no knowledge of such a function they scored 1, otherwise they scored 0. The table below displays the frequency of scores by classes.

**TABLE 5.27** Frequency of scores found for question 2 - part II (% in *italics*)

Score	Class						Total
	em	dm	cs	ce	me	ie	
0	19	18	16	15	19	9	96
	<i>86.4</i>	<i>75.0</i>	<i>48.5</i>	<i>60.0</i>	<i>57.6</i>	<i>81.8</i>	<i>64.9</i>
1	0	2	4	5	6	0	17
		<i>8.3</i>	<i>12.1</i>	<i>20.0</i>	<i>18.2</i>		<i>11.5</i>
2	3	4	13	5	8	2	35
	<i>13.6</i>	<i>16.7</i>	<i>39.4</i>	<i>20.0</i>	<i>24.0</i>	<i>18.2</i>	<i>23.6</i>

Table 5.27 reveals that, as in question 1, most of the students scored 0. Only 52 students (35.1%) scored 1 or 2 in the question. The computer science class was the one with the best results (51.5% of the students did not score 0). It was also the class with the highest percentage of students who scored 2 (39.4%). The mean of question 2 was 0.59. Although it was better than the mean of question 1 (0.28), it is still low, under half of the value of the question.

### *Bases for the categories*

The categories set up for this question were about the same as in the pilot study, the only difference being that two pilot categories were grouped together: the one in which students showed they knew that proofs generalise and the one in which students tried to give a proof for the FTC. In the second case it was interpreted that the students also knew that proofs generalise. A description of each of the categories and examples is given below.

#### **- Category G (Proofs are General)**

Category G contains either justifications which made it clear that there are no exceptions to theorems or justifications in which the students said that if the function was continuous on  $[a,b]$ , the FTC would be true. Some students only said that the FTC was true for every function. Although this is a mistake, this kind of answer was still regarded

as G, since, despite lack of care in establishing the hypothesis, the student was convinced that there was no counter-example. Some examples of answers categorised as G are:

“If the theorem is proved, then, unless there is some flaw in the proof, it is impossible to find a counter-example.”

“There is no function which does not satisfy the FTC.”

- Category CE (Counter- examples exist)

CE contains the answers of students who clearly believe that there is a function which does not satisfy the FTC and gave an example of one which they thought could be a counter-example. Some examples taken from the students' questionnaires are:

“ $f(x)=|x|$ ”

“ $y = \ln |x|$ ”

“ $\int_0^{x^2} 2t dt$ ”

“ $f(t) = t^2, F(x) = \int_1^4 t^2 dt = 56/3$  and  $F'(x) = 0$ ”

The commonest counter-example given was a modular function. An analysis of the counter-examples significantly reveals such difficulties in understanding as: (1) what the role of the variables in the function  $F(x)$  is; (2) what the hypothesis of the theorem is; and (3) what a continuous function is.

- Category I (Inconclusive: no knowledge of any examples)

Category I contains justifications of students who seemed to believe there is no such function, but felt shaky about affirming it, as well as others who thought that they did not know enough to provide an example. Their justifications are inconclusive, as we can see by the examples below:

“I do not know of the existence of such an example.”;

“I do not remember any function which does not satisfy the FTC.”;

“I don't have enough knowledge for this.”

- Category O (Others)

Category O contains all sort of answers which do not fit into any of the categories listed above. Some examples are:

“This is the right theorem.”

“I did not study the FTC much in theory; I was only worried about doing the practice. I mean, to do exercises.”

“A continuous function is differentiable at all points.”

The categories are summarised as follows:

G: Proofs are general

I: Inconclusive: no knowledge of any examples of such a function

CE: Counter- examples for the FTC exist

O: Others

X: Not answered

### Discussion of the results

The table below displays the results for question 2.

**TABLE 5.28** Distribution of categories of responses to question 2 - part II (% in italics)

no / yes	Category	Class						Total (n=148)
		em (n=22)	dm (n=24)	cs (n=33)	ce (n=25)	me (n=33)	ie (n=11)	
no	G	3 <i>13.6</i>	4 <i>16.7</i>	15 <i>45.5</i>	5 <i>20.0</i>	8 <i>24.2</i>	2 <i>18.2</i>	37 <i>25.0</i>
	I	–	2 <i>8.3</i>	4 <i>12.1</i>	3 <i>12.0</i>	7 <i>21.2</i>	–	16 <i>10.8</i>
	O	1 <i>4.6</i>	3 <i>12.5</i>	2 <i>6.1</i>	2 <i>8.0</i>	3 <i>9.1</i>	–	11 <i>7.4</i>
	X	15 <i>68.2</i>	5 <i>20.8</i>	3 <i>9.1</i>	8 <i>32.0</i>	8 <i>24.2</i>	1 <i>9.1</i>	40 <i>27.0</i>
Total		19 <i>86.4</i>	14 <i>58.3</i>	24 <i>72.7</i>	18 <i>72.0</i>	26 <i>78.8</i>	3 <i>27.3</i>	104 <i>70.3</i>
yes	CE	3 <i>13.6</i>	9 <i>37.5</i>	9 <i>27.3</i>	7 <i>28.0</i>	4 <i>12.1</i>	6 <i>54.5</i>	38 <i>25.7</i>
	O	–	–	–	–	–	1 <i>9.1</i>	1 <i>0.7</i>
Total		3 <i>13.6</i>	9 <i>37.5</i>	9 <i>27.3</i>	7 <i>28.0</i>	4 <i>12.1</i>	7 <i>63.6</i>	39 <i>26.4</i>
no answer	I	–	–	–	–	–	1 <i>9.1</i>	1 <i>0.7</i>
	X	–	1 <i>4.2</i>	–	–	3 <i>9.1</i>	–	4 <i>2.7</i>
Total		–	1 <i>4.2</i>	–	–	3 <i>9.1</i>	1 <i>9.1</i>	5 <i>3.4</i>

Table 5.28 reveals that the percentage of correct answers (in terms of yes/no) to this question is much higher than for question 1. In fact, 104 students (70.3%) answered



“no”. Of these, 37 students (25.0% of the results overall) gave the correct reason. However a significant number of students (38, i.e. 25.7%) thought that counter-examples to the FTC could be given.

Considering the classes separately, it is clear why the computer science class got the best score. It was the one in which the highest percentage (45.5%) of the students gave “no” as their answer and stated the right reason (category G). On the other hand, the industrial engineering class was the one with the highest percentage of students who had their answers classified as CE, 25.7%. The mathematics classes did not stand out, although the evening mathematics class was the one with the highest percentage of students who answered no (86.4%), most did not justify why (68.2%).

Of the five students who, as was seen in question 1, clearly misunderstood the statement of the theorem, one did not answer question 2. The student who in question 1 affirmed that  $F' = f$  and not  $F' = f''$ , answered “yes” and gave an example with the aim of demonstrating that  $F' = f$ . Her justification was classified as O. Another student answered “yes” and gave a counter- example with several mistakes:

$$\begin{array}{l} "f(x)=2 \Rightarrow \int f(t)dt = x^2 \\ \downarrow \\ f'(x) = 0 \Rightarrow f'(x) \neq f''(x)" \end{array}$$

His answer was classified as CE since he believed the misread theorem and even so gave a counter- example to it. The other two students answered “no” and reaffirmed that  $f'' = F'$ . The initial hypothesis of their argument was that  $f' = F$ . Their answers were classified as G since they believed the misread theorem and also believed that it was not possible to give a counter-example to it.

Table 5.28 also shows the case of one student who gave no answer and yet had his justification classified as I. The reason for this is that in spite of not marking yes or no he wrote:

“I could give such an example, however, I am not able at the moment to formulate what you want.”

He probably believed that there was a counter-example but since he did not state one his justification was not classified as CE, but as I, since it is inconclusive.

From the analysis of question 2 it is probably too strong to affirm that the students believed that “proof is simply evidence”. One factor that has to be taken into account is that they were not used to this kind of question and some of them might have thought that a positive answer was associated with it. It certainly can be said that they are not used to thinking about the process of creation in mathematics.

## 5.8 Summary

Some main issues were raised from examining the results of the questionnaire and comparison across the categories. These issues will be discussed below.

### **The images of a derivative and of an integral**

Questions 1b and 4 show that most of the students did not use the image of an integral as area (category  $R_g$  in question 1b and category A in question 4a) or of the derivative as the slope of the tangent line (category D in question 4a). When they saw an integral, the first image that came to their minds was the elementary primitive of the function and all the formulas of anti-differentiation. This tendency to evaluate the integral is confirmed by the results of categories  $R_p$  in question 2a, P in question 4a and  $W_p$  in question 5. In question 5, even in a situation where it was not possible to find the elementary primitive they still tried to evaluate it.

In question 1b and question 4a some of the answers indicate that some of the students used a rule which works in the case of polynomial functions (the degree of the function increases by 1 after integration) and applied this rule whenever they were integrating a

function, e.g. answers such as  $\frac{|x|^2}{2}$  and  $F(x) = \left[ \frac{f(t)^2}{2} \right]_{-1}^x$  indicate that. This rule

remained in their minds as a general image of integration

### **The definition of a concept**

Question 3 was the one which most closely provoked students to define a concept and use its definition, in this case the concept of continuity. Results showed that it was in connection with the functions  $f_2$  and  $f_4$  (in this case not considering as a mistake the fact of treating the limit at infinity as an existing limit) that the students most used the

definition correctly to justify the correct answer. The fact that the definition was most used for these two functions suggests that when the function is discontinuous (non-examples), the teachers stress the use of the definition to say that one of the conditions is not true, while with the continuous functions they tend not to be very precise. This suggests that students use the definition when they are taught to do so and the teachers stress its use.

Although in question 3 the stress was on definition of continuity, other definitions came up as the result of students mixing up concepts: in this case with the concepts of a function and of a derivative of a function. The mixing up with the derivative of a function was the more frequent and deserves to be treated in a separate topic below. Mixing up arguments related to the concept of a function suggests that verifying that  $\forall x \exists y \in \mathbb{R}$  is so strong in students' minds that some of them think it is sufficient to check and see if the function is continuous or not. In fact, some of them did not even understand the meaning of this sentence, since if they had used this argument consistently, they would have classified all the functions as continuous. In stating a definition, mathematical language itself constitutes an obstacle.

### **The statement of the FTC**

In question 6, students were asked to state the FTC and 40.5 % of students gave no answer. This indicates that although some of these students used the theorem, they could not express in words what it establishes (or perhaps did not know precisely what the FTC is). Also significant is the percentage of students who gave answers unrelated to the FTC (23.7%) and mixed up several concepts and theorems by taking fragmented parts of them. There is clearly some confusion over the difference between the definition of a concept and the statement of a theorem. Some students took a formal definition, e.g. the definition of the defined integral as a Riemann sum and wrote it as if it were the FTC.

### **The relation between continuity and differentiability**

Question 3 revealed that some students mixed up the concept of a continuous function with the concept of a differentiable function. None of the functions was differentiable at all the points of the domain, so they could not use the relation between these two concepts to distinguish the continuous functions. This confusion in a question containing

graphs suggests a disconnection between the understanding of what a definition is and how it can be interpreted graphically.

### **The use of the FTC**

Questions 2a, 4 and 5 show that not even half the students could apply the FTC. The best result was obtained in question 4b, where 40.5% applied it correctly. In a situation where they could choose to find the elementary primitive, as in question 2a, few applied it (27.7%). In a situation where they had no other choice but to apply the FTC, most still tried to solve the integral, while few applied it (30.4%). In question 4a, in which, among other strategies they could have used the FTC directly, by considering  $f(x)$  as the derivative of  $F(x)$ , only 5.4% used it.

### **The role of variables in the FTC**

Questions 2a, 4 and 5 show that some students do not recognise the role of  $x$  and  $t$  in the function  $\int_a^x f(t)dt$ . Category  $W_v$  in questions 2 and 5 provides evidence for this, as well as the fact of some students having written  $F'(x) = f(t)$  in question 4b. Furthermore, they would also have answered question 4a more quickly if they had understood the role of  $x$  in the function  $F(x)$  and replaced some strategic points in  $F(x)$ . This can be interpreted as a more general problem in connection with the concept of a function and in this case with understanding the meaning of  $F(x)$  as a function.

### **Views of the role of proving**

Question 1 of part II of the questionnaire shows that few students, only 5.4%, were used to thinking of proving as a fundamental step in the mathematics. Although there was a low percentage of students who stated that they believed one example was enough to formulate a theorem (8.1%), there was a high percentage of students who answered “yes” to this question (61.5%). On the other hand, in the second question, 25% of the students revealed they knew that proofs are general and 70.3 % of the students said they could not give a counter- example. However, there is still a significant percentage of students who did not believe in the generality of the theorem and thought they were giving a counter-example to it (25.7%).

Regarding the second part of the questionnaire, it has to be taken into account that students are not used to being asked questions concerning the role of proving in mathematics and they could have misinterpreted their aims. However, if they had been firm enough in this role, they would probably have answered the questions correctly.

### **Different patterns of responses**

The categories described in the questionnaire were devised for each of the questions, enabling the different kind of responses found throughout the questionnaire to be characterised. The results of the second part are discussed under the previous heading: views of the role of proving. In order to examine part I as a whole, the questions will be here divided into two groups, one consisting of questions 1, 2, 4 and 5, and the other of questions 3 and 6.

The results of questions 1, 2, 4 and 5 should help to evaluate the students in terms of integrating, applying the FTC and interpreting graphically the meaning of these topics. The responses can be roughly divided into two patterns: one of procedural arguments, and, in contrast, the other of graphical arguments (q. 1b and q. 4a) and arguments using direct approaches (q. 2a and q. 5). The procedural answers were represented in categories  $R_a$  in question 1b,  $R_p$  in question 2,  $P$  in question 4 and  $W_p$  in question 5. The non-procedural approaches in categories  $R_g$  in question 1b,  $R_d$  in question 2,  $D$  and  $A$  in question 4a and  $F$  in question 4b. The results showed that procedural arguments were predominant for groups 2 and 3 in questions 1, 2 and 5, and for all the groups in question 4. Two points should be observed. First that, in relation to question 4, if only the justifications of the students who answered the correct choice (d) had alone been analysed, non-procedural arguments would also have been predominant for group 1. The second point was that, in relation to question 2a, applying the FTC was considered non-procedural because, solving integrals rather than finding the derivative of the integral function is the usual kind of exercise they are asked to do in their textbooks.

Questions 3 and 6 should help to characterise the students in terms of how they deal with the theoretical aspects of the course: definitions and theorems. An overview of the categories enables two patterns of answers to be seen. The first pattern emerged from the students who were precise and paid attention to the theoretical aspects, justifying

their choices by using the definition in question 3 (category  $D_c$ ) and correctly stating the theorem in question 6 (there was no category in this case, as none of them stated it correctly). The second pattern emerged from the students who probably based their answers on the images they had associated with the concept or theorem. These images were either graphical, or contained only partial aspects of the definition or theorem, or reflected confusion with arguments related to other concepts or theorems. This pattern is represented by categories  $D_p$ , M, G and O in question 3 and by categories A, F and B in the case of question 6. It has to be observed that the answers classified as  $D_l$  in question 3 should be an example of the second pattern; however, considering that in many books the treatment of limits leads to the answers represented in this category, they could be classified in the first pattern, and it is this second option that will be adopted in this study.

Regarding question 3, the results show that the second pattern was predominant in all cases. As regards question 6 the second pattern was also predominant and the result would not be affected even if the students who at least stated that  $F'(x) = f(x)$  were included in the first pattern.

These patterns emerged from the results of the questions and they were analysed by questions. It would be interesting to use them to examine each student performance in the questionnaire. This point will be addressed in the next chapter, while examining a sub-sample of the students.

## **CHAPTER 6**

# **THE INTERVIEWS BASED ON RESPONSES TO THE QUESTIONNAIRE**

### **6.1 Introduction**

The interviews followed administration of the questionnaire. The main point in doing them was to find out more about the answers given. During the interviews the students were asked questions designed to make it clear why they had given those particular answers; some were designed on the basis of each questionnaire, while others were the same for all students, with the aim of finding out how the students interpreted mathematical concepts and ideas and how they studied mathematics.

The interviews were conducted after a first look at the questionnaire. They were held in December 1995.

### **6.2 The Sample**

The sample to be interviewed was formed of students who were selected from those who had answered the questionnaire. Twenty six students were interviewed in all and seventeen of those had their interviews analysed. The criteria for selecting these students will be described in section 6.2.1. To provide data and background for the interviews, students' scores in the questionnaire and summaries of their answers will be given in section 6.2.2.

#### **6.2.1 Procedures**

The students to be interviewed were selected from among those whose answers represented the various possible strategies (various categories) for solving the questions and also from among those whose answers were unconventional.

It was planned to interview two or three students from each of the classes, so the

number of interviewees should have been from 12 to 18 students. However, since it is well known that some always fail to come, a larger number were contacted and, at the end, 26 came in all.

The fact that the interview was not the only instrument used in the research, that it had not previously been planned to interview this number of students (26), and there were time constraints as well, led to the decision to select a sub-sample of 17 interviewees for analysis, on criteria described as follows. Two of the students concerned were from the industrial engineering class (Estela and Amanda); two from the mechanical engineering class (Gérson and Geisa); two from the civil engineering class (Beatriz and Ricardo); four from the daytime mathematics class (Alice, Elisa, Fernando and Roberto); three from the evening mathematics class (Cristina, Ivo and Raul); and four from the computer sciences class (Fabiana, Heitor, Hugo and Marcelo).

Various criteria were used for selecting these 17 from among the 26. One was that there should be a maximum of four students from one class. If there were more than four, the interviews which produced most new points for analysis were selected. There also had to be about the same number of males and females. Yet another point concerned only the daytime mathematics class: here, only students who wrote one of the options, teacher training or BSc. courses, were selected. Finally, two other students, Gérson and Alice, were chosen because their teachers recommended them as very good and interested students.

### **6.2.2 Performance on the questionnaire**

The seventeen students whose interviews were analysed obtained the following scores in the questionnaire:



---

**TABLE 6.1** Scores obtained in the questionnaire by the students interviewed

---

<i>Student</i>	<i>Course</i>	<i>Group</i>	<i>Score</i>	
			<i>Part I</i>	<i>Part II</i>
Estela	ie	1	7.8	0
Amanda	ie	1	4.6	0
Gérson	me	1	9.5	0
Geisa	me	1	4.8	0
Alice	dm	2	5.0	0
Elisa	dm	2	3.0	0
Fernando	dm	2	8.3	4.0
Roberto	dm	2	6.0	0
Fabiana	cs	2	6.2	2.0
Heitor	cs	2	2.0	4.0
Hugo	cs	2	5.3	2.0
Marcelo	cs	2	5.4	0
Beatriz	ce	2	3.5	0
Ricardo	ce	2	7.5	0
Cristina	em	3	4.9	0
Ivo	em	3	6.1	1.0
Raul	em	3	1.0	1.0

---

As can be seen, the scores are very different and perhaps represent different competencies as well, although the various strategies of analysis (questionnaire and interviews) were deliberately designed so as not to judge their performance only on the basis of one single strategy.

Each interviewed student's questionnaire will be described below. The general results, the principal features and also strange answers will be mentioned. Some of these answers will be interpreted while describing them, on the basis of the questionnaire alone. At the end of each description a brief summary will be given.

a) Estela

Estela scored 7.8 in the first part of the questionnaire. She gave the right answers to questions 1, 2, 4b and 5. She also recognised, in question 3, which functions were continuous or not. However, she did not use the definition of continuity. She offered a graphical justification, using the argument of having or not having to take the pencil off the paper.

Questions 1b and 2a she solved by using the longest way. In question 1b she solved the

integral of the modular function analytically and in question 2a she first solved the integral (by using the substitution formula) in order to write  $g$  as an elementary function and then she found the derivative of  $g$ .

Even if she did not apply the FTC in question 2a, she did so in questions 4b and 5. In question 4b she applied it correctly, but in question 5 she mixed up the variables, writing that  $h'(x) = e^{-t^2}$ . However, she did not seem to pay attention to the fact that the variables were different and continued to solve the question by replacing  $x$  and  $t$  with  $a$  ( $P = (a,b)$ ), obtaining at the end the correct answer:  $P = (0,5)$ .

She solved question 4a by considering  $F$  as the primitive of  $f$ . However, she treated the integrals as indefinite integrals, finding:

$$\int 2x = x^2 + C$$

$$\int 2 = 2x$$

$$\int -x + 4 \, dx = \frac{-x^2}{4} + 4x$$

At the end her choice was graph (c).

She did not know how to state the FTC, writing in question 6:

“Every differentiable function is continuous”

In Part II she scored 0. She answered “no” in question 1, justifying it by writing:

“Because  $f$  has to be differentiable on  $[a,b]$  and it is not true that every continuous function is differentiable.”

She gave a counter-example in question 2:  $f(x) = |x|$ .

The second part and question 6 indicate that she probably thought that the theorem which states that if  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ , should be added in some way to the FTC. The counter-example she gave is the one which teachers and textbooks use to show that the reverse of this theorem is not true.

### *Summary*

Estela's questionnaire shows that she has mastered the technical procedures to solve

problems in calculus, since she was able to solve most of the questions in the first part of the questionnaire correctly. However, she was confused about theoretical aspects: the role of variables in  $F(x)$ , interpreting  $F(x)$  as a definite integral and recognising what the FTC states were points probably obscure to her. Since she thought there was a mistake in the statement of the FTC given in the second part ( $f$  should be differentiable in the hypothesis of the theorem), it was not possible to judge how she regarded the process of proving in mathematics.

#### b) Amanda

Amanda scored 4.6 in the first part of the questionnaire. She did not understand the meaning of a definite integral, since she added  $C$  to the results she found in question 1:

10 in (1a) and 0 in (1b). In question 1b she knew how to evaluate  $\int_{-1}^1 |x| dx$ , but at the end she wrote that  $\left. \frac{-x^2}{2} \right|_{-1}^0 + \left. \frac{x^2}{2} \right|_0^1 = -\frac{1}{2} - 0 + 0 + \frac{1}{2} = 0 + C$ . Perhaps, what she imagined

the final result should be, namely 0, had some influence on her making this mistake.

She was able to state correctly whether the functions in question 3 were continuous or not, but she got mixed up with continuity and differentiability, justifying all her choices by analysing the graphs to discover whether the functions were differentiable or not in all the points of the domain.

She did not know how to state the FTC (question 6) nor how to apply it in questions 2a, 4b and 5. In question 2a she could not even finish solving the integral. In question 4b she used the result of question 4a, graph d, and analysed it in order to obtain graph (f). In question 5 she solved the integral (wrongly, of course), obtaining at the end  $h'(x) = e^{-t^2}$ .

The only question in which she applied the FTC was question 4a, but she justified her choice in very general terms. She evaluated the slope of the tangent line to the graph (d) only in the interval  $[1,2]$  and wrote that (d) corresponded to the function whose derivative showed a graph equal to the graph  $f(x)$  given.

It is curious that she could not use the FTC in an apparently simpler question (4b). The

fact that she was able to apply the FTC in question 4a was not followed up in her attempts to solve any of the other questions.

In the second part of the questionnaire her score was 0. She was among those who misread the statement of the FTC, interpreting  $f(x)$  at the end of the statement as the second derivative of  $f$ . At least she knew what the FTC is about, since she answered in the first question “no” because “the derivative of the integral is the function itself and not the second derivative of it”.

### *Summary*

Amanda was unable to solve correctly the simpler questions of the first part of the questionnaire. In addition, her knowledge of the FTC was very weak. The fact of being able to use it only in question 4a in some way is inconsistent with the whole questionnaire. The definitions of definite integral, continuity and differentiability were unclear for her. Since she misread the statement of the FTC and did not accept the truth of her misreading, it is impossible to say anything about part II related to the aims of this part.

### c) Gérson

Gérson's score in the first part of the questionnaire was 9.5, the highest in the whole sample. Gérson was considered to be brilliant: he had studied in one of the best schools in Rio de Janeiro and obtained the highest grade, 10, in all the examinations he did in calculus I.

The half point he lost was in question 6. He stated the FTC in the following way:

$$\text{“If } F(x) = \int_a^x f(x)dx \text{ so } F'(t) = f(t).”$$

There is no problem in writing  $F'(t) = f(t)$ , but there is in  $F(x) = \int_a^x f(x)dx$ . He also did not state that  $f$  has to be continuous.

His questionnaire indicated that not only did he know how to solve the questions but he also knew how to solve them by the quickest method: evaluating the integral of the modular function graphically and applying the FTC to questions 2a and 4b.

He used correctly the definition of continuity, seeing if  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a)$  in all the points of the domain. However, although for the function  $f_4$  he stated that  $\lim_{x \rightarrow 0^+} f_4(x)$  did not exist, he added that “it is different from the other limit of the function” (referring to  $\lim_{x \rightarrow 0^-} f_4(x)$ ). The idea that is necessary to find out if  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$  is probably so strongly rooted in students’ minds that, even if they know these limits do not exist, they feel they have to check and compare them.

In question 4 he got mixed up with the notation, sometimes using  $f$  instead of  $F$ , but in the whole context this fact seemed to be only a slip. He solved the question by thinking of  $f$  as the derivative of  $F$  and considering that  $F(-1) = 0$ .

His score for the second part of the questionnaire, 0, did not correspond to his performance in the first part. He doubted the statement of the FTC, writing:

“The derivative is defined in an open interval”.

Perhaps he thought that the questions in part II contained a deliberate mistake, which he was supposed to discover.

### *Summary*

Gérson’s result in the first part of the questionnaire is consistent with what his teacher had said about him: a brilliant student. Although interpreting his questionnaire might have been affected by this knowledge, he proved to be able to use the definition of continuity and apply the FTC in different situations. However, he may have had problems in understanding the role of variables in the function  $F(x)$  and in knowing how to use the hypothesis of continuity in the FTC, as question 6 demonstrates. Nothing can be said in relation to his understanding of proving since he thought there was a mistake in the statement of the FTC: he thought  $f$  should be defined in an open interval.

### d) Geisa

Geisa scored 4.8 in the first part of the questionnaire. She answered questions 1a and 2 correctly and made the correct choices in question 3. She did not give any answer at all to questions 1b and 4.

She probably got mixed up with the definitions of function and continuity. She claimed that  $f_1$  and  $f_3$  were continuous because all the values in the domain were associated with an image. However, when justifying why  $f_2$  and  $f_4$  were not continuous, she no longer used this argument, writing:

“It is not continuous, but is removable” (for  $f_2$ )

“The image of 0 is not finite” (for  $f_4$ )

Examining the second sentence, it is possible to realise that she did not incorporate  $f_4(0) = 0$  in the definition of  $f_4$ .

She could not state the FTC correctly, writing that:  $\int_a^b f(x)dx = F(b) - F(a)$ . She also did not apply it in any of the questions. She is one of the students who solved the integral in  $h(x)$  in question 5, finding  $h(x) = 5 + (-2 t e^{-t^2})$ .

Although she wrote above  $F(x)$  “graph of the area” in question 4a, she did not use the association between integral and area in any question, not even in this one.

In part II her score was 0. She distrusted the statement of the FTC given. She wrote that the result should be:  $F'(x) = f''(x)$  in question 1. In question 2 she tried to give a counter example, but in terms of  $F$ ,  $F(x) = |x|$ . Perhaps she thought:  $F$  is continuous but not differentiable.

### *Summary*

Geisa's questionnaire indicates that she remembered some pieces of what she had learned in calculus, but she was not able to connect them. She could not sustain an argument, moving from one to the other, as she showed by using different definitions of continuity and giving different statements for the FTC (parts 1 and 2). Furthermore, she knew nothing about the FTC and part II confirmed that. She distrusted the statement of the FTC given: she thought it should state that  $F'(x) = f''(x)$ , so nothing can be said in relation to how she regarded proving.

e) Alice

Although Alice was recommended as one of the best students in her class, she only scored 5.0 in the first part of the questionnaire. She came from a university outside Rio de Janeiro (UNICAMP), and in order to be transferred to UFRJ, she had to do calculus I. She was the only interviewee student from the daytime mathematics class whose first option in the entrance examination for UFRJ was teacher training and she wanted to follow that.

She correctly answered questions 1a, 2, 3 and 4b, but did not write anything in the last two questions. In question 3 she used the definition of continuity to classify the functions. Like Amanda, she did not use the FTC in questions 2a and 4b, but she did use it in question 4a. However, unlike Amanda, she gave a clear justification, explaining how she used the fact that  $f$  was the derivative of  $F$ , analysing the intervals where  $f$  was negative or positive in order to see where  $F$  would increase or decrease. She chose graph (c), but she was not sure about it and she wrote “OK” indicating (c) or (d).

She scored 0 in part II. She did not give any answer at all to question 2 and misread the statement of the FTC in question 1. However she answered “yes” and explained why  $F'(x) = f''(x)$ :

“ $F(x)$  is a function whose derivative is  $f(t)$ ; that means  $f(t)$  is the derivative of a function, so the derivative of  $F(x)$  gives me the derivative of  $f(t)$ , which is already a derivative, and therefore the second derivative.”

She argued in circles and got mixed up in creating the second derivative in this context.

*Summary*

Alice's responses to the questionnaire are contradictory and her performance did not correspond to the expectations expressed by her teacher. Questions 3 suggests a correct use of definition and question 4a suggests a good understanding and visualisation of the FTC. The other questions do not sustain this impression. Although she applied the FTC in question 4a, she misread it in the second part of the questionnaire and believed what she misread, which suggests that she was not really sure about what the FTC states. Nothing can be said in relation to how she regards proving in mathematics, she not only did she not understand the aim of question 1, which was the only one of the two

questions she answered, but also she misread the statement of the FTC.

#### f) Elisa

Elisa received a low mark in the first part of the questionnaire: 3.0. She wrote BSc. in the questionnaire, but she said that her first option in the entrance examination to UFRJ was computer science and she would like to be transferred to actuarial studies.

The only question she answered quite correctly was question 1a. She correctly classified  $f_2, f_3$  and  $f_4$  in question 3 and marked the right choice in question 4a.

She gave all sorts of justifications in question 3, mixing up arguments, using them in an inconsistent way, but none of them were correct. She gave a graphical argument for classifying  $f_1$  as a discontinuous function: a corner in the graph. She did not use this same argument for  $f_3$ , which she classified as continuous because the right- and left-hand limits at  $x = 0$  were equal. In  $f_2$ , classified as discontinuous, besides using arguments related to the definition of a function saying that there was a point where the function was not defined, she also said the right-hand limit at  $x = 1$  was 2, while the left-hand limit at this point was  $x$  and therefore they were different. In  $f_4$  she made a wrong and strange evaluation of the limits ( $\lim_{x \rightarrow 0^+} f_4(x) = 0$ ,  $\lim_{x \rightarrow 0^-} f_4(x) = \infty$ ). She showed difficulty in understanding the concepts of limit and function besides continuity itself.

She did not state the FTC in question 6 and did not use it in any of the questions. In question 2 she wrote that  $g'(x) = 2 - t \Big|_2^x = 2 - x$ . She marked (c) as the right graph for question 4b but did not justify her choice. In question 4a she justified the right choice by stating:  $F(x) = f(t) \Big|_{-1}^x = f(x) - 1$  and in question 5 she found  $h'(x) = 0 + \int_0^x e^{-t^2} dt$ .

In part II Elisa scored 0. She did not understand the aim of question 1, answering “yes” because of the FTC. In question 2 she also answered “yes”, arguing that if  $x \notin [a, b]$ , then  $f(x)$  would not satisfy the FTC. She denied the truth of the theorem by not considering one of the hypotheses.

#### *Summary*

Elisa’s questionnaire shows that she could not cope with even the simplest procedures in



calculus, such as finding the derivative of  $g(x)$  by evaluating the integral first (question 2a). She also moved from one argument to another with no consistency, as she demonstrated in relation to her understanding of continuity. Her knowledge of the FTC was poor. In relation to her understanding of proving in mathematics, what can be said is that she did not understand the logical process in a proof, particularly the role of the hypotheses.

#### g) Fernando

Of all the mathematics students Fernando was the one who got the highest score: 8.3 in the first part. His first option was computer science, but since he did not pass he decided to follow a BSc. course and after finishing it, he would then to specialise in computer science.

In the first part of his questionnaire, questions 1, 2, 3 and 4 are correct. In question 5 he showed that he knew how to apply the FTC by answering  $h'(x) = e^{-x^2}$ . However, he did not continue the question. He did not give any answer at all to question 6 (state the FTC).

Except for question 4b, he did not use the quickest approach to solve any of the other questions. In question 4a, for example, he justified the right choice by finding the analytical formula for  $F(x)$  ( he made a small mistake in arithmetic when evaluating the constants).

He made correct use of the definition of a function in question 3; his only slip was to say that  $f_4$  was not continuous because the value of the function at 0 was different from the value of the limit of the function when  $x$  goes to 0.

Fernando understood the aims of the questions in part II and scored 4.0. He answered “no” in both questions, considering that it was necessary to prove a theorem and it was impossible to find counter examples unless the proof is wrong.

#### *Summary*

Apart from some gaps, Fernando did very well in the whole of the questionnaire, parts I and II. He knew the definition of continuity and also knew how to apply the FTC,

although he could not state it. However, he did not apply it to the different situations the questionnaire suggested, sometimes choosing a longer way. His view of proving was perfect.

#### h) Roberto

Roberto's score in the first part of the questionnaire was 6.0. He, like Fernando, would like to have studied computer science, but as he did not pass he said that he wanted to study for a BSc. and later do a specialist course in computer science.

He answered 1, 2 and 4b correctly and made the correct choices in question 3. The fact which stands out in his questionnaire is that he clearly did not understand the role of the variables in the function  $F(x)$  and  $h(x)$  in questions 4 and 5 respectively. Question 2 he solved correctly, but by solving the integral first, so this question cannot be used to examine his problem with variables.

He did not make any choice in question 4a, but he wrote in the space reserved for justifications: " $f(t)$ ?", making it clear that if he knew how to find  $f$  (he probably wanted an analytic formula for  $f$ ), he would use the fact that  $f$  is the derivative of  $F$ . In question 4b he made the right choice, justifying it by stating that  $F'(x) = f(t)$ . In question 5 he wrote  $h'(x) = e^{-x^2}$  but did not continue.

Judging from his answer to question 6, he did not even regard the FTC as being " $F'(x) = f(t)$ "; for him the FTC was the association between area and integral:

"If  $f(x)$  is a continuous function in  $(a,b)$ , it follows that  $\int_a^b f(x)dx$  corresponds to the sum of the areas, between  $a$  and  $b$  of  $f(x)$ "

In question 3 he got mixed up with the definition of function and continuity, checking whether for every  $x$  there was  $y$ , such that  $f(x) = y$ . However, he was not consistent in regard to this criterion. He recognised that it was true for all the functions in question 3, but still marked  $f_2$  and  $f_4$  as discontinuous functions. He probably made his choices without using it.

Roberto scored 0 in the second part. He answered "yes" in both questions. In the second one he gave a counter - example. He probably misread the statement of the FTC and

gave what he thought was a proof, but at the end he obtained  $f''(F(x)) = f'(x)$  and not  $F'(x) = f''(x)$  (as the other students who misread the theorem did). He wrote:

$$\text{"If } f(x) = \int_a^x f(t)dt, f'(F(x)) = f(x) \Rightarrow f''(F(x)) = f'(x). \text{ QED"}$$

This proof in itself is a very rich source of misunderstandings, but the main point is that he used  $f$  and  $F$  in a confused way.

### *Summary*

Roberto's first part of the questionnaire is quite transparent in relation to the main misconception he held: the role of variables in the FTC. He repeated the same mistakes in connection with this problem in different parts of the questionnaire. He also got mixed up with the definitions of function and continuity, as question 3 reveals. In relation to part II, Roberto misread the FTC and thought that the misreading was the truth. He probably understood that theorems need to be proved, since he offered a proof, albeit a false one, for the theorem he had misread in question 1 (part II). On the other hand, he believed that there are counter-examples for theorems.

#### *i) Fabiana*

Fabiana scored 6.2 in the first part of the questionnaire. She answered questions 2 and 4b correctly, made the right choices in question 3 (without justifying them) and found the derivative of  $h(x)$  in question 5 (but did not finish the question).

In question 1 she added  $C$  to the answers she found (10 and 0). She applied the FTC in questions 4b and 5. In question 4a, she marked (b), considering  $F$  as the primitive of  $f$ , but her justification was very vague, only in terms of the degree of the polynomial function in each interval. So, for example, in the interval  $1 < x < 2$ , she argued that as the degree of a constant function is 0,  $F(x)$  should be a straight line. She probably had in mind the image of the derivative of a constant function being the constant itself.

The statement of the FTC she wrote in question 6 had to do with the association with the definition of the definite integral as a Riemann sum.

In part II she scored 2.0. She did not understand the aim of question 1 but she correctly answered and justified question 2. She answered "yes" to question 1, and tried to explain

the FTC in her own words:

“ $F(x)$  is the primitive of  $f(t)$ , so the derivative of  $F(x)$ ; in other words,  $F'(x)$  is the  $f(t)$  itself. As the integral is defined between  $x$  and  $0$ ,  $f(t)$  is  $f(x)$ .”

It is interesting to note how she interpreted  $f(t)$  and  $f(x)$ . She probably thought of  $f(t)$  as a function in which  $t$  varies over the domain (the whole graph) and  $f(x)$  as the value of  $f$  at a specific point and concluded  $F'(x) = f(t)$ .

### *Summary*

Fabiana's first part of the questionnaire could be classified on the whole as very vague. She did not trouble to go into details in her justifications. This particularly affected question 3; nothing can be said about how she defined a continuous function, since she did not give any justification at all. She applied the FTC in the questions in which its application was most evident: 4b and 5. She had problems with variables, as she demonstrated in part II, and perhaps thought that in the last line of the proof of the FTC  $f(t)$  is replaced with  $f(x)$ . The only thing that can be said about her conception of proof is that she knew that proofs generalise.

### j) Heitor

Heitor's score in the first part of the questionnaire was very low: 2.0. He got 1 point in question 1a and 1 in question 3 for marking  $f_2$  and  $f_4$  as discontinuous functions. All the other questions were wrong.

Examining his questionnaire, it is not difficult to discover the obstacles he came across in trying to solve the questions. In question 1b he considered  $|x^2|$  to be the primitive of  $|x|$ , but that was a difficult question for most of the students. In questions 2, 4, 5 and 6 he made it clear that he regarded the FTC as being:  $F(x) = f(x)$ . As a result of applying this misconception, he obtained  $g'(x) = 2$  in questions 2a and 2b; graphs (f) and (e) for questions 4a and 4b, respectively; wrote  $h(x) = 5 + e^{-x^2}$  in the first line of his solution to question 5; and stated the theorem as quoted below:

$$\text{“} \int_a^x f(t) dt, a < x = f(x) \text{”}$$

In question 3, he wrote arguments related to differentiability in connection with the

functions  $f_1$  and  $f_3$  and concluded they were discontinuous functions. He made the correct use of the definition for justifying his choices for functions  $f_2$  and  $f_4$ .

He understood the aims of the questions in part II, and scored 4.0. He marked “no” in both, answering that it was necessary to prove the theorem and that there were no exceptions to theorems.

### *Summary*

Heitor’s answers in the questionnaire were consistent and indicated that he basically holds two misconceptions: thinking that the FTC states that  $F(x) = f(x)$  and mixing up continuity with differentiability. He correctly applied the wrong result to the questions. In part II he made it clear that he has a perfect conception of what a proof means.

### k) Hugo

Hugo scored 5.3 in the first part of the questionnaire. He correctly answered questions 1a, 2 (by finding the primitive), made the right choices for  $f_2$  and  $f_4$  in question 3 and also correctly marked the graphs in question 4.

He is one of the students who knows that  $f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$  and used this information.

but wrongly. He found two solutions instead of one: 0 for  $\int_{-1}^1 -x dx$  and 0 for  $\int_{-1}^1 x dx$ .

He marked all the functions in question 3 as discontinuous for the same reason: he thought that the right- hand limit was different from the left- hand limit at  $x = 0$  in  $f_1, f_3$  and  $f_4$ ; and  $x = 2$  in  $f_2$  (in this case he probably got confused and used 2 instead of 1).

In question 6 he stated the FTC exactly as it was stated in the second part of the questionnaire. He probably copied it from a classmate who had already solved part II. He justified the answers to question 4 by writing “by the FTC”. Of course, knowing the statement correctly could have given him a hint on how to answer question 4b, but it was not clear how he used this information in question 4a. However, he could not apply the FTC in question 5. He solved the integral and found that:  $\int_0^x e^{-t^2} dt = -\frac{1}{2}e^{-x^2} - \frac{1}{2}$ .

His score in part II was 2. He correctly answered question 2, but not question 1. He

decided to “prove” the result given in the example, as follows:

$$\text{“Ex: } F(x) = \int_0^x 2t dt \rightarrow t^2 \Big|_0^x \rightarrow \text{substituting } x \text{ for } t,$$

$$x^2 - 0 = x^2 \rightarrow \left. \begin{array}{l} F(x) = x^2 \\ F(x) = 2x \end{array} \right\} \text{ then, } F'(x) = f(x).”$$

He used the corollary of the FTC to prove the FTC was valid in this particular example.

### *Summary*

Hugo’s first part of the questionnaire is distinguished by the fact of his knowing the relevant information and yet not being able to use it properly. He probably copied the statement of the FTC from one of his classmates but could not apply it in the different situations given in part I. He also knew how to define the function  $|x|$ , but he could not solve question 1b correctly. In relation to part II, Hugo understood that proofs generalise (question 2), but in trying to give a proof (question 1), he used a circular argument.

### 1) Marcelo

Marcelo scored 5.4 in the first part of the questionnaire. He was the only student in the group of 17 students interviewed who was not from the daytime mathematics class and was doing calculus I for the second time. The reason why he was chosen is because he represented a group of students from his class who were repeating calculus I and had solved question 1b graphically. However, this was the only question he solved using a graphical approach. He did not use the association between area and integral in question 4a. He marked graph (c) in 4a, thinking of  $F$  as the primitive for  $f$ .

He correctly answered questions 1, 2, 4b and part of question 5. He knew how to apply the FTC, although he did not use it in question 2, but he did so correctly in questions 4b and 5. However, he gave no answer to question 6.

He got mixed up with continuity and differentiability in question 3. He answered that  $f_1$  and  $f_3$  were non-differentiable at 0, and therefore discontinuous. However, he classified  $f_2$  and  $f_4$  as continuous, saying they were differentiable at all points. So he did not classify

any of the functions correctly.

His score in the second part was 0. He misread the statement of the theorem and was one of the students who agreed with the misread statement, answering “yes” and confirming that  $F'(x) = f''(x)$ . In the second question he answered “no” and tried to prove that  $F'(x) = f''(x)$ . His answer follows:

$$“F(x) = \int_a^b f(x)dx$$

$$f'(x) = F(x)$$

So, the function being continuous,

$$f''(x) = F'(x)”$$

In his justification, instead of writing  $F'(x) = f(x)$ , he changed it to  $f'(x) = F(x)$  (second line).

### *Summary*

Marcelo demonstrated by his questionnaire that, in spite of being able to apply the FTC, he did so automatically without thinking of what he was doing, since he found a proof for the statement which he had misread in part II. He also mixed up the relation between differentiability and continuity, and neither of these concepts was clear to him. Nothing can be said about whether he believed that examples are enough to generalise, but he probably knew there were no counter-examples for a theorem.

### m) Beatriz

Beatriz scored 3.5 in the first part of the questionnaire. This was the first time she had done the course, but she failed and was going to repeat it. She was the only student in this situation in the group of 17 interviewed, since all the others passed in the course just finishing. The reason why she was chosen was that she raised in the interview many interesting points about the course itself which help in analysing the results of the questionnaire.

Beatriz correctly answered questions 1a, 2 and 4b. The only question in which she used the FTC was 4b. In question 4a she marked b, without giving a proper justification. In

question 5 she solved the integral, finding  $h(x) = 5 + e^{-x^2} - 1$ .

The statement she gave of the FTC was very confused:

“Every anti-differentiable function is continuous in a closed interval  $[a,b]$ , differentiable in an open interval  $(a,b)$  and its integral varies from  $a$  to  $a$  (for example  $\int_a^a A(t)dt = 0$ ) will be null.”

All the choices Beatriz made in question 3 were wrong. She justified all of them by checking to see if the functions were differentiable or not (she considered that  $f_1$  and  $f_3$  were not differentiable but that  $f_2$  and  $f_4$  were).

She scored 0 in the second part and both answers deserve to be quoted. She answered “no” in the first question because:

“Probably because even before we learn the subjects, the theorems are thrown at us in such a way as to press us to memorise them and not to reflect. The time is very short and there is no time to make analyses of this type.”

And she answered “yes” in the second question because:

“If I integrate a function as  $F(x) = \int_x^x f(t)dt$ , the area (which is what determines the integral) will be null and not a function of  $x$ .”

In the first she criticised the structure of the course itself, without realising that she did not even need to know the theorem in order to answer the question. In the second she found a counter-example. It is interesting that she thought that since the final result is  $f(x)$ , the variable  $x$  had to appear and so  $F(x) = 0$  would not be a function of  $x$ .

### *Summary*

Beatriz's questionnaire shows that what she remembered at the end of the course were pieces of definitions and statements of theorems which she related in a confused and completely wrong way. She was only able to use the FTC in question 4b. She mixed up continuity and differentiability and did not even have an idea of what a function being continuous or differentiable means in graphical terms. Also, as suggested by part II, she did not have a good grasp of variables in a function. The answers to part II suggest that she did not understand the role of proving in mathematics.



n) Ricardo

Ricardo scored 7.5 in the questionnaire. He correctly answered questions 1, 2b, 4b, 5. He also made the right choices in question 3 and was one of the two students who mentioned both: that  $f$  has to be continuous and  $F'(x) = f(x)$  in question 6.

The thing that stood out in his questionnaire was that the FTC was regarded as a rule; and he made this clear by writing in question 2a:

“By the first fundamental rule of Calculus,  $g'(x) = (2x-1) dx$ ”

Also in question 4b:

“By the first fundamental rule of Calculus  $F'(x) = f(x) dx$  (or he might have written  $F'(x) = F(x) dx$ )

$$F'(x) = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} ,,$$

He knew how to apply the FTC correctly; the problem is that he memorised it, adding the  $dx$  at the end. Another problem is that he was not worried about writing  $F$  or  $f$  in such a way as to distinguish them clearly. This point was illustrated not only in question 4b, but also when he stated the FTC (question 6):

“Let  $f$  be a continuous function in the interval  $[a,b]$ ,  $\forall x \in [a,b]$ , and let  $F$  be the function  $F(x) = \int_a^x f(t)dt$ , then we have  $F'(x)$  will be equal to  $F'(x) = f(x) dx$  (or  $f'(x) = F(x) dx$ ).”

In brackets is the version of what he might have written at the end.

He chose (a) in question 4a, but did so by using an unusual argument. Based on the relation that if  $f$  is continuous then  $f$  is integrable, he concluded that: if  $f$  is discontinuous then  $f$  is not integrable and looked for discontinuous graphs. He eliminated graph (e), but then he stopped arguing because he realised that all the other were graphs of continuous functions.

He was also unconventional when justifying his answers to question 3 (for  $f_2$  and  $f_4$ ). He made the right choices for question 3. For the functions  $f_1$  and  $f_3$  he wrote that they were continuous all over the domain, but for  $f_2$  and  $f_4$  he wrote that they would only be

continuous if the domain was  $\mathbb{R} - \{1\}$  and  $\mathbb{R} - \{0\}$  respectively. He was absolutely right, considering the definition of continuity in which only the points in the domain should be analysed. However, these are not justifications at all.

Ricardo's score for the second part of the questionnaire was 0. He answered "yes" in both questions. In the first one he copied the example given in the statement to show that  $F'(x) = f(x)$  by doing the following: first he wrote  $F(x)$  as a polynomial function and then he found the derivative of it, obtaining  $F'(x) = 2x$ . In the second question he gave an example of a graph of a function  $f(x)$  which was not differentiable, indicating that  $f'(x)$  did not exist at a certain point.

### *Summary*

Ricardo's questionnaire is unconventional. He used unconventional justifications in questions 3 and 4a. In question 3 he was right; he worked out what the domain should be in order for  $f_2$  and  $f_4$  to be continuous. His answers point to a clear conception of continuity (although they were not justifications). In (4a), he failed by using an argument not logically correct (from  $f$  is continuous  $\Rightarrow f$  is integrable, he inferred that  $f$  is not continuous  $\Rightarrow f$  is not integrable). He regarded the FTC as a rule, which he memorised, adding the  $dx$  at the end. As for the second part of his questionnaire, he believed that it is sufficient to use examples in order to state a theorem and that there are counter-examples to theorems.

#### o) Cristina

Cristina's score for the first part of the questionnaire was 4.9. She did not give any answer at all to questions 1b, 5 and 6. She solved question 2 by writing the primitive as a polynomial function and the only question in which she (perhaps) applied the FTC was question 4. In question 4a she chose (b), explaining that:

"The derivative of a constant is a constant and the derivative of a parabola is a straight line."

Probably she considered  $f$  as being the derivative of  $F$ . Judging from her choice (graph (b)) and the justification above, she had the idea that the derivative of a constant is the constant itself.

In question 4b she chose graph (f) because:

“The derivative of the derivative is the initial function.”

This changing a word is very interesting in itself. Was her putting it this way a slip or had she wrongly memorised the sentence “the derivative of the integral is the function itself”, which students use to memorise the FTC?

Cristina was able to recognise which functions in question 3 were continuous and which were not. The argument she used was a visual one. She looked to see whether the graphs were all in one piece (curve) or not.

Cristina scored 0 in Part II. She answered “yes” in the first question because of the FTC and “no” in the second one without justifying it.

### *Summary*

Cristina was confused about what the FTC is. She memorised it using the wrong words. She reasoned visually in two questions (3 and 4a), but in her case this probably indicates lack of the proper tools (knowledge of definitions and theorems), rather than understanding of the visual aspects linked to the questions. Nothing can be said about her in relation to the aims of part II.

### p) Ivo

In the first part of the questionnaire Ivo scored 6.1, which was one of the highest scores in his class. Only two other students got a better mark than he did, the highest score being 6.4. A glance at his questionnaire reveals at once that in some parts he was very careful in the more theoretical aspects while in others he made mistakes which show he did not understand some fundamental points in calculus.

He made correct use of the definition of continuity (question 3); he wrote in question 2a “by the fundamental theorem of Calculus” and he included in the statement of the theorem that  $f$  had to be continuous and  $F'(x) = f(x)$ . On the other hand, in the same question 2a, for example, his final answer was  $g'(x) = 2t-1$  and in question 5 he replaced  $t$  with  $a$  in  $h(a) = 5 + \int_a^x e^{-t^2} dt$ .

He did not use a visual approach for any of the questions in which he could have done so. He gave the wrong answer to question 4a: (b). He justified it by finding the analytic formula of each of the three lines in the graph of  $f(x)$  and trying to find the elementary primitive. However, he did not know how to express  $F(x)$  as a function; he wrote, for example that  $F(x) = \int_1^2 2dx = 2x|_1^2$  in the interval  $[1,2]$ .

He marked (f) as the correct answer to question 4b. His approach to this question was very unusual. He used the analytic formulas of each of the lines he had found in the previous item, but did not reason from the graph; instead he “applied” the FTC to each of the integrals below, not realising that these integrals represent numbers:

$$\text{“ Because } F(x) = \begin{cases} \int_{-1}^1 xdx \Rightarrow F'(x) = x, x \in [-1,1] \\ \int_1^2 2dx \Rightarrow F'(x) = 2 \Rightarrow y = 2 \\ \int_2^5 -x + 4 \Rightarrow F'(x) = -x + 4 \end{cases} \text{”}$$

The statement for the FTC he gave was:

“Let  $f(x)$  be a continuous function in the interval  $[a,b]$  and  $g(x)$  be a primitive of  $f(x)$ , then we define  $F(x) = \int_a^b f(x)dx = g(b) - g(a)$  and also  $F'(x) = f(x) = g'(x)$ .”

He mixed up the statement of the FTC and of the corollary of the FTC. The main point is that he did not understand how  $F(x)$  behaves as a function. The other point is that he used  $g'(x) = f(x)$  as the hypothesis and thesis of the theorem.

Ivo's score in part II was 1. He answered “no” in question 1. Although he recognised that the result was not necessarily valid for every function, he did not explain why he recognised that. Regarding his justification it is not possible to say whether he thought that because it was necessary to test for more examples or because it was necessary to prove.

He gave a counter-example in question 2. However, it was very difficult to understand his handwriting in this question. It was possible to see that his counter-example was a function in the form  $F(x) = \int_a^x f(t)dt$  and the final result was  $F'(x) = f(t)$  and also that he said something about “family of primitive functions”.

### *Summary*

Ivo's questionnaire is contradictory. He probably regarded the theoretical aspects of calculus as important: he nearly stated the corollary of the FTC, and he made correct use of the definition of continuity. However, he came up against an obstacle which appeared in several parts of the questionnaire: his failure to understand  $F(x)$  as a function, with all the consequences that implies, like the role of variables in  $F(x)$ . In relation to the second part, he believed that it was possible to give a counter-example but did not make it clear why he thought it is not possible to generalise from one example only.

#### q) Raul

Raul's score for the first part of the questionnaire, 1.0, was together with that of four other students, the lowest in the whole sample (148 students). The point he got was in question 3, marking correctly that  $f_1$  and  $f_2$  were continuous and discontinuous functions respectively. However, his justifications were vague, being expressed only in terms of whether the function followed (or not) a certain pattern or rule.

In the other questions Raul made elementary mistakes. Even when evaluating the integral in question 1a:  $\int_0^2 (3x^2 + 1)dx$ , he replaced  $x$  with the upper and lower limits of the integral, obtaining  $(3 \cdot 2^2 + 1) - (1)$ .

He knew absolutely nothing about the FTC or how to apply it. When he tried to state it he wrote:

“In evaluating an integral, the derivative of a function  $f$  has to be the integral which one has to find. So, by differentiating a function  $f$  being equal to  $g$ , the integral of  $f'$  has to be equal to  $g$ .”

Raul scored 1 in part II. He answered in the first part:

“Because the 1st integral was defined from 0 to 2, and the second integral from 0 to  $x$ , it is not possible to conclude anything”

By “1st” he meant the first integral:  $\int_0^x t dt$  and by “second” he meant  $\int_a^x f(t) dt$ . He probably regarded the second as one more example.

He answered “no” in the second question, and it is difficult to understand his answer. He

said that it was necessary to give the derivative of  $f(x)$  because of the definition of an integral, which apparently makes no sense.

### *Summary*

Raul's questionnaire reveals that he did not know even how to apply the simplest formulas of anti-differentiation to solve integrals. He had no idea of what a definite integral is and even less of what the FTC is. Perhaps he had some image of the concept of continuity, since he marked two of the functions correctly, giving a vague graphical explanation. As for the second part, the only thing that can be said is that perhaps he regarded the general representation of  $F(x) = \int_a^x f(t)dt$  in the statement of the theorem as just one example.

### Overall comparison based on patterns of responses

In order to give an overview of the students, comparing their performances in the questionnaire by means other than the score, an attempt was made to characterise their responses across the categories and questions using the patterns of responses described in Chapter 5, section 5.8, this time by students. Three sets of questions were analysed for each of the students: 1b and 4a; 2, 4b and 5; and 3 and 6. In each of these sets the following questions were examined: (i) would a student who used a procedural argument in one question use procedural arguments in all the other questions?, (ii) would a student who used a non-procedural argument in one question use a non-procedural argument in all the other questions as well?, (iii) would a student who used the definition in question 3 state the theorem in question 6 correctly?

Examining the students who solved question 1b correctly, it was found that 2 students used procedural arguments in questions 1b and 4a: Ivo and Fernando. One, Gérson, used graphical arguments in both, while the other, Márcio, solved 1b graphically and 4a in a procedural way.

Closer attention was paid to the way the students solved questions 2a and 5. It was found that 10 out of the 17 students solved question 2a by finding the elementary primitive first and 1 solved it directly. Of these 10, only two, Fernando and Fabiana, were able to apply the FTC in question 5. They both solved question 4b also by applying

the FTC. This meant that, although they solved question 2a by a procedural mechanism, they had the tools to solve it directly. Only one student in the sample, Gérson, solved all these three questions by applying the FTC.

Examining questions 3 and 6 showed that only two students, Gérson and Ivo, used the definition in question 3 correctly and also that, although they did not state the FTC correctly they had the idea of what the FTC is about. Two more students used the definition correctly in question 3, Fernando and Alice. However, Fernando did not answer question 6 and Alice wrote something that had nothing to do with the theorem.

The results showed that Gérson, who was the student who got the highest score, was the only student who used non-procedural arguments in all the questions concerned. Fernando, who got the second highest score, used procedural arguments in the same questions (except for question 5, which he solved correctly). However, although Fernando seemed to have a procedural way of reasoning, examination suggests that he would be able to solve at least question 2a directly. Gérson and Ivo were the ones who showed in questions 3 and 6 that they paid attention to the theoretical aspects. It should be remembered here that Gérson and Fernando were the ones with the best scores in the whole questionnaire while Ivo got the best score of his class.

The results of all the students who have not been mentioned here are very mixed, some of them with answers left in blank, others classified as O (others), others as only W (wrong) and others using various kinds of strategies in the questions. However, one general conclusion can be stated: the predominant kind of strategy is procedural and of argumentation, non-theoretical.

The results of part II have not been examined here, since they were very bad and it was difficult to draw any conclusion from them based only on the questionnaire. Only Heitor and Fernando solved the two questions correctly, suggesting that they knew the role of proving in mathematics, while 11 students solved none of them correctly.

### **6.3 Interview schedule**

The interviews were organised in the same way as in the pilot study (described in section 4.3.2): some questions differed for each student according to their particular answers,

while some were common, each of the interviews lasting between 30 minutes and 1 hour. There were some slight differences in relation to the pilot interviews, some questions being included, which will be described below.

In connection with question 3, the students were asked to define what a continuous function is. If they gave a definition contradictory to that given in the questionnaire, they were shown one example (chosen from the four functions in the questionnaire) and were asked to explain how both would work for that example. The example chosen was always one in which the two of them did not work well together.

At the end of the part of the interview concerned with part I of the questionnaire the students were asked to compare the integrals  $\int_2^4 (2x-1)dx$  and  $\int_2^x (2t-1)dt$  and say what each of them represented. This was added in order to see if they realised that the first resulted in a number, while the second was a function.

These two questions concerned points which the pilot study showed as potential obstacles for the students: continuity of a function and the function  $F(x)$ .

#### 6.4 Methodology of analysis

The interviews were tape-recorded. Full transcriptions were made of the nine students who were chosen to work with the computer later. These transcriptions were used to devise a framework for analysis and examine interesting points that the interviews raised. The way found was to draw up 15 tables. Nine of them related to the questions in the questionnaire: 1b, 2a, 3, 4a 4b, 5 and 6 from part I; and 1 and 2 from Part II. The other six were entitled: First image of a derivative and of an integral, Convincing,  $F$  and  $F'$  (in connection with question 4), Comparison between the integrals  $\int_2^4 (2x-1)dx$  and  $\int_2^x (2t-1)dt$ , Study habits, and FTC.



**FIG. 6.1** Frame of one of the tables used to analyse students' interviews.

Question 1b

[illegible]

The table in Fig. 6.1 helps to describe the general frame of the tables. In all of them the first two columns identify the student and the last column is reserved for interesting quotations about the question concerned. In the tables on the questions in the questionnaire there was a column with the categories described in Chapter 5 representing how each student's answer was classified, and another column showing whether the student changed the solution in the interview. The other columns of the tables (if there are any) are on points related to the theme of the table. In the example given, if the students had not given a graphical solution to question 1b, they were asked if they knew the graphical meaning of  $\int_{-1}^1 |x| dx$ . If they said no, they were asked to sketch the graph of the function  $f(x) = |x|$  from  $x = -1$  to  $x = 1$  and to relate that graph to the integral in the question. This made it possible to find out if they connected the area under the graph with the integral in that situation.

The interviews of each of the nine students who had their interviews fully transcribed were read carefully and the points relating to each of the tables were marked in the proper spaces. Following that, the interviews of the remaining eight students were listened to carefully and with pauses, in order to complete the tables.

The tables were first looked at one by one to obtain an overall view of the group of interviewees in relation to each table. After that they were carefully examined for each student, this time comparing them with the student's questionnaire and seeing what the

interview had added in relation to it.

The headings chosen for the summary of the questionnaire (section 5.8) served as a guide for analysing each student's interview. Apart from these headings two new points of interest appeared in the interviews. One was about students' appreciation of proof. In the interviews they were asked questions directly related to this, e.g. interpreting the proof of the FTC. The other was about their study habits. They were also asked questions concerning this point, e.g. if they had studied the proofs and if they had used lecture notes or textbooks to study. The results of this analysis follow in the description of each student's interview.

## **6.5 Results**

Each student's interview will be examined in turn. The headings from the summary of the questionnaire (except for the heading "different patterns of responses") will be used in the description of each interview. These headings are: images of a derivative and of an integral; the definition of a concept; the statement of the FTC; the relation between continuity and differentiability; the role of variables in the FTC; the use of the FTC; views of the role of proving. However, only new points, interesting or unexpected facts, and explanations of obscure items in the questionnaire will be mentioned. The intention is to look for what can be added to the analysis of the questionnaire. Thus not all these headings will be mentioned for each student. In some of the interviews, two headings will be grouped together, when there is very close interaction between the themes in those interviews. Besides these headings, new ones, introducing themes only explored in the interviews, will be added in each description. They are: an appreciation of proof and study habits. No mention will be made in any of the interviews about the questions as to whether the students remembered what the mean value theorem for integrals was or whether they had thought about the graphical meaning of the FTC, since no student replied in the affirmative.

### 6.5.1 Estela

#### Images of a derivative and of an integral

From the questionnaire, it is clear that Estela did not use the association of an integral with area and a derivative with the tangent line, but in the interview she stated these associations, although she was still unable to use them.

#### The definition of a concept

In the questionnaire Estela used the argument of the function being continuous in the case of sketching its graph without taking the pencil off the paper, while in the interview she gave a definition of “ $x$  having an associated image”. However, she spoke as if there were exceptions to definitions; for example she added that it could happen that a function was not continuous even if every point of the function had an associated image. At the end, she said she did not know how to give a definition of a continuous function.

#### The statement of the FTC

Estela did not state the FTC in the questionnaire and during the interview it was clear that this was because she did not know which theorem was the fundamental one, as she herself explained. Her statement, “Every differentiable function is continuous”, for example, could be the fundamental theorem.

#### The relation between continuity and differentiability

Estela revealed that she was confused about continuity and differentiability in question 1 of part II of the questionnaire. She had said it was necessary for  $f$  to be differentiable and not continuous, and in the interview she did not change her mind. The reason became clear only in the interview. She did not return to what she had written in question 6 (“Every differentiable function is continuous”), but she got mixed up with the theorem: if  $f$  is continuous on  $[a,b]$ , then  $f$  is integrable on  $[a,b]$ , a theorem given when teaching definite integrals, usually two or three classes before the teaching of the FTC. She probably memorised that the relation was: if  $f$  is differentiable on  $[a,b]$  then  $f$

is integrable on  $[a,b]$  ( that is also true but a weaker result) and thought that it was true to say that if  $f$  is not differentiable on  $[a,b]$  then  $f$  is not integrable on  $[a,b]$  . From the questionnaire it was interpreted that Estela was confused about the concepts of continuity and differentiation, but the interview showed more confusion, not only about these two concepts but about integration as well.

### The role of variables in the FTC

Estela was asked about variables when applying the FTC, using the answer she had given ( $h'(x) = e^{-t^2}$ ) in question 5 and comparing it with the result of question 2. The following dialogue indicates that she thought it did not matter which variable to use:

- Researcher: Why did you answer  $g'(x) = 2x-1$  and here  $h'(x) = e^{-t^2}$  ?
- Estela: Are you asking me why I didn't write the  $x$  here?
- Researcher: Yes
- Estela: I could have written  $x$ .
- Researcher: Do you think that it doesn't matter which you write?
- Estela: I don't think it does.

She had also said previously in the interview, that she would not be able to solve these questions, if the  $x$  was in the lower limit instead of being in the upper limit of the integral.

### The use of the FTC

In order to apply the FTC, Estela created a rule which helped her to memorise it and put out of her mind any theoretical aspect linked to the FTC. After examining question 2 in the questionnaire, which she had solved finding the elementary primitive first, she immediately realised she could have solved it directly. She explained why by using the following algorithmic procedure that worked for her: cross out the integral sign and at the end replace  $t$  by  $x$  (she explained she could cross it out because differentiation and integration are inverse processes, and one would cancel out the other one).

### Views of the role of proving

From the questionnaire it was not clear how Estela regarded proving in mathematics, since she thought there was a mistake in the statement ( $f$  should be differentiable). In the interview she was asked the question again. She was questioned on how she would

answer question 1 if the statement included all the conditions she believed should be included. She answered “yes”, without explaining much. She also used an experimental argument when asked how she would convince someone else that the FTC was true: she would integrate a function and so find the derivative of it.

### An appreciation of proof

Estela contradicted herself at two different moments of the interview when asked if she had ever read the proof of the FTC. First, when asked if the teacher had proved the FTC in class she said she was not sure, but later on when examining the proof in her class notes she said that she had read it. Probably she had just read it but not tried to make any sense of it. For example, she did not know where the condition of continuity was used, and made it very clear that she did not care about it because she never needed to use this information:

Estela: Normally this thing of the function being continuous, being differentiable, we will never need to use it. They give us the exercises to solve. We do not see if the function is continuous or not, we see nothing at all. I only pay attention to what I have to solve.

It seems that the obstacle to her appreciating the proof was the fact that she did not care about proofs.

### Study habits

Estela used the lecture notes to study and the book was only used to do some exercises. She said she had read the proofs.

### *Summary*

The summary of Estela’s questionnaire shows that she could solve most of the questions of Part I despite being confused about some theoretical aspects, namely: variables in  $F(x)$ , understanding of  $F(x)$  as a definite integral, statement of the FTC and distinction between continuity and differentiability.

The interview showed that she did not understand how the function  $F(x)$  works. The fact of saying that it did not matter how to write the variables and that she did not know what would happen if  $x$  was in the lower limit of the integral, confirms this point. She also made it clear that she had no idea which result corresponds to the statement of the FTC.

She got mixed up with some other theorems of calculus. She was confused about the theorems which establish the relations between continuity, differentiability and integrability. She was not sure which relations could be established between these concepts or how to create logically correct equivalent sentences in the form “if ... then” to establish these relations. She was also not sure how to define a continuous function.

In the questionnaire, it was not clear how she regarded the role of proving and the interview showed that she probably thought of proving as an experimental process, although it may also be the case that she did not care about formalism, since she did not need it to succeed in the course. She said she had read the proofs in the class notes, but the interview suggested that if she had read she had not tried to understand them, at least in the case of the proof of the FTC.

The interview with Estela suggested that she tried to be a practical person in relation to how she coped with calculus. In order to avoid the more theoretical aspects, she attempted to make some sense of them by creating algorithmic procedures to memorise the results she needed to solve the questions of calculus.

### 6.5.2 Amanda

#### Images of a derivative and of an integral

Amanda's first image of a derivative was  $\frac{dy}{dx}$  and of an integral was the symbol  $\int$ .

Nevertheless, when she explained in the interview how she had reasoned to solve question 4a in the questionnaire, she made it clear that she had reasoned applying the image of the tangent line to all possible graphs of  $F$ .

Amanda's answer to question 1b of the questionnaire was  $0 + C$ . In the interview she obtained it correctly by evaluating the area under the graph of  $f(x) = |x|$ . However, she did not know how to use this image to explain why the definite integral should not have the constant added. She just said it should be added but she did not remember why.

Another fact that demonstrated her not understanding the meaning of the definite integral was the answer she gave to  $F(0)$ ,  $F$  being defined as in question 4. She said it would be

equal to:  $\int_{-1}^0 f(t)dt = f(0) - f(-1)$ . Summed up, these answers suggest that her images of a derivative and of an integral were very poor.

### The definition of a concept

Like Estela, Amanda held different definitions of the concept of continuity of a function. These definitions did not always agree with each other when she used them to classify a function as continuous or not. She presented three definitions (or images, perhaps) in the interview: (1) a function is continuous if it is differentiable; (2) a function is continuous if it is possible to sketch its graph without taking the pencil off the paper and (3) a confused mixture of two ideas, that to every  $x$  corresponds a  $y = f(x)$  and that there should be no “empty spaces”. In her words :

Amanda: It is a function which is defined in an interval and it does not have... If it is a straight line, there will be no empty space, all the points will have an associated image. This point here could have a small hole in it and the function be defined here. It would not be continuous because .... Mathematically it is more difficult.

Definition (1) she used in the questionnaire and in the interview and (2) and (3) only in the interviews. When she stated in (3) that “all the points have an associated image” she could see by examining the graph of  $f_2$  that it was not the same as being continuous because the function could have a “hole” in it and still be defined at all the points. In the process of trying to give a definition, she had to include all the possible cases and images, which made her return to the definition given and try to incorporate these cases.

She also said that if she used the definition (1) she would have to classify  $f_1$  and  $f_3$  as discontinuous as well, but she still thought they were continuous using argument (2).

### The statement of the FTC

Amanda did not state the FTC either in the questionnaire or in the interview. She only knew that she was applying the result: “the derivative of the integral is the function itself”.

### The relation between continuity and differentiability

Amanda mentioned the relation between continuity and differentiability in two parts of the interview: when defining what a continuous function is and when trying to find where



it mattered that  $f$  had to be continuous in the proof of the FTC. In the first part she said a function is continuous if it is differentiable and in the second part she said that in order for  $f$  to be differentiable, it had to be continuous. She did not really know what the relation between continuity and differentiability was.

#### The role of variables in the FTC

Amanda used  $t$  instead of  $x$  in  $h'(x)$  (question 5) and changed it only when she was confronted with the use of these two variables.

#### The use of the FTC

From the questionnaire we know that she only applied the FTC to question 4a, without explaining clear reasons. Reviewing questions 2a, 4b and 5 in the interview, it turned out that she could have solved them by applying the FTC.

#### Views of the role of proving

Amanda's performance in part II changed substantially between the questionnaire and the interview. After acknowledging that she had misread the theorem, she answered "no" to both questions, for the correct reasons. She affirmed she was convinced of the theorem because of its proof and she would use the proof to convince someone else.

#### An appreciation of proof

Amanda said that she thought proofs are important to understand theorems. However, when asked to explain where the condition of continuity was used, she said that in order for  $f$  to be differentiable,  $f$  had to be continuous. Thus she considered the proofs important, but probably did not try to reconstruct them.

#### Study habits

Amanda said she used her lecture notes more than the book. She said that she usually read the proofs, but added "not on the days before the test", as she knew she did not need them.



## *Summary*

Amanda's performance in the questionnaire was poor. However, in the interview she changed the answers she would give to some of the questions. In part I she was able to correctly apply the FTC to questions 2a, 4b and 5 although she had not known how to apply it in the questionnaire. In part II she answered the questions correctly, after the exact wording of the statement of the FTC had been carefully read to her (she had misread it in the questionnaire), showing that she knew what the role of proofs is in mathematics.

Despite this improvement, Amanda still held fundamental misconceptions. She presented three definitions (some could be named images) for continuous functions without noticing they were inconsistent. Moreover, even though she knew the role of proofs and attributed importance to it, she did not seem to know how to reconstruct the proofs; she could not, for example, say correctly where the fact that  $f$  has to be continuous was used in the proof of the theorem.

### **6.5.3 Gérson**

#### Images of a derivative and of an integral

Gérson's first image of a derivative was the tangent line at a point and of an integral was anti-differentiation and sum. In question 1b, his solution had been classified as correct graphically ( $R_g$ ), since he had drawn the graph. However, he said he had not in fact thought in terms of area under the graph, but had used the fact that the function was even. This was why he wrote at the side of the graph:  $\int_0^1 x dx + \int_0^1 x dx$ . He used the image of area in two other points of the interview: when evaluating  $F(0)$  and when interpreting what  $\int_2^x (2t - 1) dt$  meant.

#### The statement of the FTC and the role of variables in the FTC

In Gérson's interview, the way he stated the FTC was linked to the role of variables in  $F(x)$  and this is why these two issues are being tackled together.

In question 6 of the questionnaire, Gérson showed that he was confused about variables. In the interview, he rewrote the FTC in the following way:

$$\text{“ If } F(x) = \int_a^x f(t)dt \text{ so } F'(b) = f(b) \text{”}$$

The reason why he used  $x$  and  $b$  in this confused way was because he thought of  $b$  as a point, although it could have been any point; while  $x$  was a variable which gives the general form of the function. He explained why he used  $b$ , in the following way:

Gérson: Why wouldn't it be  $x$ ? I don't know, because here you would be substituting the  $b$  for the  $x$ . For example,  $x^2$ ,  $f(x) = x^2$ , so  $f(b)$ ....  $f(3) = 3^2$ . So the  $b$  would be equal to 3.

Researcher: Do you regard  $b$  as a point?

Gérson: Yes.

He did not add that  $f$  had to be continuous either in the questionnaire or in the interview.

Surprisingly, near the end of the interview, Gérson interpreted the statement of the FTC as the definition of an integral itself:

Gérson: I think that the definition of an integral is this (*he means the FTC*). So, you say you define like that, integration is anti-differentiation. This is going to be the definition, the beginning of everything. Even what I had told you about area has nothing to do with it.

Researcher: Do you think that integral begins with the theorem?

Gérson: Yes, I do.

Probably at this part of the interview, as integration and the FTC were being emphasised, he thought that the FTC was the definition of an integral itself.

### Views of the role of proving

In the questionnaire Gérson doubted the statement of the FTC, writing that the interval where  $f$  was defined should be open and not closed. After it had been explained how to consider a derivative at the end points of an interval, Gérson answered “no” to both questions for the correct reasons.

### An appreciation of proof

Gérson said he had not studied the proof of the FTC. He could not say where the condition of  $f$  being continuous was used and he declared he was not convinced of the result of the FTC.

### Study habits

Gérson said that he had used both: books and lecture notes to study. However, he made it clear that he had used the book to do exercises more than to study theoretical matters such as definitions and theorems. He also stated he had studied some proofs.

### *Summary*

Gérson is a student who is certainly concerned about the theoretical aspects of calculus. His questionnaire had already indicated this and the interview confirmed it. He is a little confused about variables in functions, although in some ways he was very clear about separating the meanings of variables. However, there is apparently no particular problem associated with  $F(x)$ . Perhaps, because during the interviews great stress was laid on the FTC and on integration, he became confused at some point of the interview and said that he regarded the FTC as the definition of an integral itself.

Although he was a good student, he did not seem to try to reconstruct the proofs and he could not say where the condition of  $f$  being continuous was used. In relation to the role of proving, after the explanation about the restriction he had placed on the statement of the FTC, he gave the right answers to the questions in the second part.

#### **6.5.4 Geisa**

### Images of a derivative and of an integral

Geisa's first image of a derivative was the slope of the tangent line and of an integral the inverse of the derivative. In the questionnaire she had written "graph of the area" above  $F(x)$  in question 4a, but in no part of the interview did she demonstrate recognition of this association: she did not use any graphical image to solve any of the questions either in the questionnaire or in the interview.

### The definition of a concept

In the questionnaire Geisa showed she was confused about the definitions of function and of continuity. The interview showed that she not only mixed up these two concepts but also confused continuity with differentiability.

Another point concerned her understanding of mathematical language and dealing with the logic behind it. In the questionnaire, she had answered that  $f_1$  and  $f_3$  were continuous because for every point in the domain there was an associated image. In the interview, the following passage shows that she did not know how to negate a sentence which would symbolically be represented as  $\forall x \exists y / f(x) = y$ .

Researcher: If I told you to sketch the graph of a function in which this was not true, I mean, it was not true that every point of the domain had an associated image, would you be able to? In which some of the points did not have an associated image?

Geisa: Some of the points or every point?

### Relations between continuity and differentiability

Geisa mixed up continuity with differentiability. She thought of a continuous function as a differentiable function.

### The statement and use of the FTC

Geisa did not state the FTC correctly either in the questionnaire or in the interview. In both situations she recognised that there was a relation between integration and differentiation. She distrusted the statement of the FTC in part II of the questionnaire (writing that it should state that  $F'(x) = f''(x)$ ) and in the interview she still distrusted the statement, but for another reason, saying that the final result should be:  $F'(x) = f(x) - f(a)$ .

However, she did not use this result when she applied the FTC in question 2a. In the questionnaire, she found the correct answer to this question by first solving the integral and in the interview by correctly applying the FTC. Although unable to solve question 4b in the questionnaire, she did so in the interview by applying the FTC, although without saying that she was using the FTC.

### Views of the role of proving

Even though Geisa distrusted the statement of the FTC as given, it was possible to check her views of proving. She did not see any problems in generalising based on only one example. She also thought that it was possible to give a counter-example to the FTC: she said that as well as the modular function she had given in the questionnaire, she could have written an exponential function. She was not convinced of the truth of the FTC because she believed there were counter-examples to it.

### An appreciation of proof

Geisa thought that she understood the proof of the FTC after examining it in her textbook, but she could not even say correctly where the condition that  $f$  had to be continuous had been used. She said that if the function was not continuous “the infinitesimal sum would be very complex”. She was certainly confused about concepts.

### Study habits

Geisa’s study habits were not clear from her interview.

### *Summary*

The interview with Geisa gave even more support to the supposition that she remembered pieces of calculus but was not able to connect or use them in a consistent way. This is supported by the fact that in spite of not believing the statement of the FTC given in part II, she used this same statement in questions 2a and 4b, although it was not clear if she knew that she was using the FTC. Moreover, she gave another definition of continuity of a function, this time mixing up continuity with differentiability. The other interesting point that the interview revealed was that she had problems in logic and how it is expressed in mathematical language: she did not know how to negate a sentence of the kind  $\forall x \exists y / f(x) = y$ . The interview also suggested that Geisa held an experimental view of the process of proving in mathematics.

### 6.5.5 Alice

#### Images of a derivative and of an integral

Alice's first image of a derivative was the slope of the tangent line and she had no image associated with an integral. This inability to associate area and integral was confirmed when, in spite of being able to sketch the graph of  $f(x) = |x|$  in the interview, she did not know how to use it to solve question 1b.

#### The statement and the use of the FTC

Alice's view of the FTC was confused. In the questionnaire she did not answer question 6, and in the interview she showed that she knew that there was a kind of connection between differentiation and integration, but was not sure what it was. As she said:

Alice: I think there is... wait a minute....the derivative of an integral... I think there is but I do not remember if it is the integral of a derivative or the derivative of an integral.

However, even though she got confused in the interview, she did not change the answer to question 2a, which she had found by applying the FTC correctly.

#### Views of the role of proving

Alice had misunderstood the statement of the FTC in the questionnaire, but in the interview she made it clear that she knew that examples do not generalise and it is necessary to prove. However, she was not so convinced about the fact there would be no counter-examples.

#### An appreciation of proof

Alice had not studied the proof of the FTC. She did not know where the condition that  $f$  has to be continuous was used when seeing the proof in her textbook.

#### Study habits

Alice said that she had studied the theory using her lecture notes and that she had only used the book to solve the exercises. She said that at the end of the calculus course she had not studied it very much, since she had to devote her time more to another subject: linear algebra.

## *Summary*

The analysis of Alice's questionnaire had suggested that she was not sure about what the FTC is, and the interview confirmed this impression, since she could not even establish the correct association between differentiation and integration. The questionnaire had also suggested that she believed that examples are enough to state a theorem but the interview did not sustain this impression and she expressed the necessity of proving.

### **6.5.6 Elisa**

#### Images of a derivative and of an integral

In the questionnaire Elisa did not use the association of integral and derivative with the images of area and tangent line, and in the interview she made it clear that these images were not in her mind when asked what her first image was of a derivative and an integral. However, although she had given a wrong solution to question 1b in the questionnaire, she found the correct solution graphically in the interview after being asked to sketch the graph of  $f(x) = |x|$  and interpret graphically the meaning of  $\int_{-1}^1 |x| dx$ . Nevertheless, she did not know how to explain why the analytic solution she had given in the questionnaire was wrong. She could not associate the graphical and analytical solution, in fact she was unable to give a correct analytic solution. Moreover, she did not use the image of area associated with integration in any other part of the questionnaire. The fact of being able to solve only one question graphically in the context of the interview indicates that Elisa had no graphical images associated with the derivative and the integral of a function.

#### The definition of a concept

In the questionnaire Elisa used a graphical argument and partial aspects of the definition of continuity to classify the functions, none of them correct. In the interview, besides the arguments used in the questionnaire, she added one more: she said it was a function which had no "open interval, no point of discontinuity". However, when she was pressed further to give a formal definition, she expressed it taking the three-condition definition into account.

Researcher: Can you give me a more formal definition?

Elisa: There is that that thing about the limits.

Researcher: What thing about the limits?

Elisa: The right-hand limit being equal to the left-hand limit. The  $f(x)$  being equal to the limit of  $f(x)$  and this  $x$  has to be in the domain of the function.

Although she had some knowledge of the definition she did not apply it in the questionnaire. This suggests that she knew only roughly how to say it without understanding it.

### The use of the FTC and the role of variables in the FTC

In Elisa's questionnaire the matter of the variables in the FTC did not come up, but it appeared strongly in the interview. She had answered questions 2a, 4a and 5 wrongly in the questionnaire and said she had not known how to correct them because she had not understood how the variables  $t$  and  $x$  worked in the FTC. In relation to questions 2a and

4a she said she had answered them as follows:  $g'(x) = 2 - t \Big|_2^x = 2 - x$  and

$F(x) = f(t) \Big|_{-1}^x = f(x) - 1$ , because she only knew she had to transform the variable  $t$  into

$x$ . The FTC was for her like a transformation rule.

### Views of the role of proving

In the questionnaire Elisa answered "yes" to both questions, arguing in the first that it was due to the FTC and in the second that if  $x$  was not in the interval  $[a,b]$  the FTC would not be satisfied. The second answer showed her failure to understand what a theorem states and the logical process in a proof, and this was confirmed in the interview when, after being re-asked the questions with some more details, she said in connection with question 1 that she could not generalise on the basis of a single example, she had to "prove the contrary". When asked to explain what she meant by that, she showed that she did not understand what a proof by contradiction was. She thought she had to negate  $F(x)$ , and what she meant by that was to consider another  $F(x)$ . In relation to question 2, she changed her mind in the interview, saying there could not be any examples of such a function, but she said that without much conviction.



### An appreciation of proof

Elisa did not understand the proof of the FTC when it was shown to her.

### Study habits

Elisa said she had only used her lecture notes and done the lists of exercises given by the teacher. She had not used the textbook. She admitted she did have the habit of reading the proofs but only the final results, which in her words were the “the formulas”.

### *Summary*

Elisa’s questionnaire had already revealed a weak performance that did not change much in the interview. Although she was able to give correct solutions for some of the questions she had got wrongly, she did not do so with much conviction. There were, in fact, three cases to cite. In question 1b she gave the correct graphical solution in the interview, but she was not able to say why the analytic solution given in the questionnaire was wrong. In relation to defining the concept of continuity, connected with question 3, she first stated a graphical argument and later moved to the three-condition definition, but she had not applied it in the questionnaire and did not return to it to see how it would work; she said it mechanically. In question 2, part 2, she changed her mind and said there would be no counter-examples, but she did not seem sure about this.

The interview confirmed the fact she had difficulty in understanding the logical process in mathematics. She probably thought of a proof as a proof by contradiction, but not understand this either. A new fact that the interview brought up was her saying that did not understand the role of the variables in the function  $F(x)$ , which was an obstacle to her applying the FTC.

### **6.5.7 Fernando**

#### Images of a derivative and of an integral

Fernando had not used any graphical image connected with differentiation and integration to solve any of the questions in the questionnaire. However, in the interview

he showed that he knew these images. Not only did he say that his first image of a derivative was a tangent line and his first image of an integral was area, but he also used the image of area to solve questions 1b and 4a, which he had solved analytically in the questionnaire.

### The statement of the FTC

In the questionnaire Fernando had not given any answer to question 6, but in the interview he said that  $F'(x) = f(x)$ . However, when asked if the function could be any function, he said it could. He did not recognise that  $f$  had to be continuous.

### The role of variables in the FTC

In the questionnaire, the fact of Fernando not being able to complete question 5 (he only found  $h'(x)$ ), suggested that perhaps he did not know how the variables worked in the function  $F(x)$ . In the interview this impression was shown to be wrong. He solved question 5 without any interference.

### The use of the FTC

In the questionnaire Fernando applied the FTC in questions 4 and 5, but solved question 2a by evaluating the integral first. However, in the interview he solved question 2a by applying the FTC.

### An appreciation of proof

Fernando said that he had not studied the proof before, but had just followed it in class. When he was asked in the interview where the fact that  $f$  had to be continuous was used, he said that if the function was discontinuous, it would have a vertical asymptote and it would be impossible to define the area under it. This is true in the case of functions like  $f_4$  in question 3 of the questionnaire (the limit does not exist at a point), but not true in the case of functions in which the right- and left-hand limits are different from the value of the function at a certain point (like  $f_2$  in the same question), or functions in which the right-hand limit is different from the left-hand limit at a certain point. Although Fernando was not correct in saying that it was true for any function, at least he realised why  $f$  had to be continuous in one of the cases.

### Study habits

Fernando said he had used both: the textbook and his lecture notes to study. He said he finds it important to read the proofs, but convinced himself of the truth of the FTC by examples.

### *Summary*

In the questionnaire Fernando had already had a very good performance, and this improved in the interview. Some of the questions he had solved correctly in the questionnaire by using an analytical approach, he was able to solve in the interview by reasoning graphically: 1b and 4a. He finished question 5, which he had just answered partially. He said that  $F'(x) = f(x)$  in question 6, which he had not answered in the questionnaire. There is nothing to add in relation to part II, which he had already answered correctly in the questionnaire. He was certainly a good student and recognised the value of proving in mathematics.

### **6.5.8 Roberto**

#### Images of a derivative and of an integral

Roberto's first image of a derivative was a tangent line and his first image of an integral was area. He apparently did not make any direct use of these associations in the questionnaire, but he did in question 1b in the interview when evaluating the area under the graph of  $f(x) = |x|$ . However, he answered question 4a using a graphical reasoning which apparently had nothing to do with area or tangent line. In the questionnaire he had not answered this question and in the interview he chose (b) because the graph was, as he said, "broken". Apparently he compared the patterns of  $f(x)$  and  $F(x)$ , and (b) was the graph which had corners in  $x = 1$  and  $x = 2$  (although this was the case of graph (f) as well).

#### The definition of a concept

In the questionnaire Roberto got mixed up with the definitions of a function and of a continuous function, writing the argument  $\forall x \exists y / f(x) = y$ . In the interview he said he

would give another sort of argument more related to examining if the function followed a certain pattern, as can be seen from his own words:

Roberto: The function would be continuous if you did not find any point outside this function. when you applied the function the point did not come out where you expected it to.

The expression “outside this function” revealed a view of a function as a single formula. He also said, confirming this impression, that in  $f_2$ ,  $y$  was different from  $f(x)$ , referring to  $x=1$ . Probably for him,  $f_2(1) = 1$  and  $y = 2$  at  $x=1$ .

### The statement of the FTC

In the questionnaire, instead of stating the FTC, Roberto mentioned the association between area and integral, adding that  $f$  had to be continuous,  $f$  being the function to be integrated. In the interview he changed his answer to:

$$“\int_a^b f(x) = g(b) - g(a) \quad f(x) = g'(x)”$$

Among other mistakes the followings deserve to be mentioned: he missed the condition that  $f$  had to be continuous and he did not write  $g'(x) = f(x)$  as the hypothesis of the corollary of the FTC. In fact, he wrote pieces of the theorem, but did not connect them in a formal way.

### The role of variables in the FTC

From the questionnaire it is known that the main misconception Roberto held was in relation to the use of the variables  $t$  and  $x$  in the FTC. The interview did not help to clear up this point much, he did not explain why he wrote  $F'(x) = f(t)$  in question 4b and  $h'(x) = e^{-t^2}$  in question 5. When he was asked to compare  $\int_2^4 (2x-1)dx$  and  $\int_2^x (2t-1)dt$  he explained the graphical meaning from the first integral but not from the second one. Furthermore, he did not know how to complete question 5; he did not know how to use the fact that  $h'(a) = 1$ . All this suggest that his confusion with variables is possible related to his not understanding how  $F(x)$  works as a function.

### The use of the FTC

From the questionnaire it seemed that Roberto had applied the FTC in questions 4b and 5, although he was confused about variables. In the interview he showed he knew how to solve question 2a by applying the FTC. However, the interview suggested that he perhaps had not applied the FTC to solve question 5. He said that question 5 was wrong because after solving the integral, he should have realised that the limits of the integral were 0 and  $x$  and should have applied them in the answer. Interpreting his words, it seemed that he “solved” the integral. It has to be considered in this interpretation that he might have been affected by the interview process and forgotten how he had reasoned in the questionnaire.

### Views of the role of proving

Roberto misread the statement of the theorem in the questionnaire and believed the misreading was correct. However, since he offered a “proof” for the theorem as misread, besides giving a counter-example for it, it was interpreted that he probably believed theorems needed to be proved and that there are counter-examples for theorems. The interview supported the second impression but not the first. After being set right about the correct statement, he still thought that there might be a counter-example for the FTC. However, when he tried to explain what had to be done before the result found could be published, he said the theorem should be submitted to experiments. He also said that to try to convince somebody else of the truth of a theorem he would ask for a counter-example. If no counter-example was found, the theorem was true. He probably had a view of mathematics as an experimental science. In his own words:

Roberto:        To publish a theorem? You have to be able to prove the theorem irrespective of examples, to submit the theorem obviously to experiments, to counter-examples, to see if it holds firm.

### An appreciation of proof

Roberto had not studied the proof of the FTC. He said he had only followed it in class. He did not know to explain correctly where the condition that  $f$  had to be continuous was used. He thought that  $f$  had to be continuous in the FTC because it had to be differentiable.

### Study habits

Roberto said he had used the textbook to do exercises, and studied the theory from his lecture notes.

### *Summary*

Roberto's performance in the questionnaire did not change much in the interview. His only improvement was in question 2a, which he was able to solve directly. One of the important points his interview indicated was how he interpreted a function. He showed he had a view of a function as being defined by a single formula. In relation to part II, it was interpreted from his questionnaire that he probably believed it was necessary to prove the theorem. However the interview suggested he was not sure about how happens the process of generalising in mathematics and probably held an empirical view of it.

### **6.5.9 Fabiana**

#### Images of a derivative and of an integral

From the questionnaire it is clear that Fabiana did not use any graphical images associated with differentiation and integration. to solve any of the questions. In the interview she said her first image of a derivative was the symbol  $\frac{dy}{dx}$  and of an integral the symbol  $\int$ . When she was asked to sketch the graph of  $f(x) = |x|$  and give a graphical interpretation for it, she said "area", but still did not know how to use this association to solve the question. Probably she just said "area" mechanically, without understanding fully what this meant. This impression was confirmed by the fact that when she was asked why she added  $C$  to her answers to question 1, she said it was because the  $C$  indicated an approximate area, as the following passage from her interview shows:

Fabiana: This +C to tell you the truth I never really understood what it meant. But it is some approximate thing, I know...The book always adopt it when we evaluate... this +C. Perhaps, as I understood, the area is some approximate thing....

### The definition of a concept

In the questionnaire Fabiana classified the functions correctly, but did not give any justification for her choices. In the interview she first justified them by using the argument that  $\forall x \exists y / f(x) = y$ . However, when she was asked to test this argument for  $f_2$  and  $f_4$ , she got mixed up, realising it could not be applied consistently for all the functions. She moved to another argument: whether it was possible or not to take the pencil off the paper. When she was asked to check and compare these two arguments, she said she was no longer able even to classify the functions, so confused had she become.

### The statement of the FTC

In the questionnaire the statement Fabiana had given for the FTC had nothing to do with it. In the interview she said she did not remember what the FTC was.

### The use of the FTC

In the questionnaire Fabiana had applied the FTC in questions 4b and 5, but not in question 2a which she had solved by evaluating the elementary primitive first. She said in the interview she did not know any other way of solving this question. She was asked to compare questions 2a and 5 but she did not connect them, she did not know if in question 2 the result was the function inside the integral only by chance. It was strange that she should have applied the FTC in questions 4b and 5 and not related them to question 2a.

### Views of the role of proving

In the questionnaire Fabiana showed that she did not understand the aim of the first question in part II, but she correctly answered question 2. In the interview she said she would have to test with other functions in order to publish the theorem and later she said she would need to do “something more general”. However, in trying to explain what she understood by this expression, she said she meant by “general” that no value should be attributed to  $x$ . She probably said this mechanically without realising that no value was being attributed to  $x$  in the example given.

### An appreciation of proof

Fabiana did not know how to explain any part of the proof when she saw it. She said she had accepted the result of the FTC and as she could solve the questions without needing to understand the proof, she did not attach any importance to it.

### Study habits

Fabiana had used the textbook only to do exercises, the theory she had read in her lecture notes.

### *Summary*

Fabiana's questionnaire did not offer much evidence of how she reasoned, particularly in question 3, in which she classified the functions without justifying her choices. In the interview, when she was asked to give a justification, she gave two different and non-equivalent arguments. This caused her to become confused and say she could not even classify the functions any more. She was also confronted with a different situation: comparing questions 2a and 5. Surprisingly, she did not make any connection between the results of these two questions.

In relation to part II she was not sure about how the process of generalising in mathematics occurred. Although she said she needed to use a general example, her first reaction in the interview was to say she needed to test with more examples, an experimental view.

## **6.5.10 Heitor**

### Images of a derivative and of an integral

From the questionnaire it is clear that Heitor did not use the images of a tangent line and area in any of the questions. In the interview he referred to these images when he was asked what his first image of a derivative and of an integral were. Despite mentioning these images, he was not able to use them correctly in question 4a. He had chosen (f) as correct choice in the questionnaire and in the interview he chose (b), arguing that it was



the only graph in which there was a parallel line to the  $x$ -axis from  $x = 1$  to  $x = 2$ , as the following passage demonstrates:

Heitor:        Because if I look here (*he was examining the graph of  $f(x)$* ), at the interval from 1 to 2 the tangent line is constant, isn't it? So, from 1 to 2 it is parallel to the  $x$ -axis and the only one which corresponds to this idea is (b).

He was able to make correct use of the association between area and integral only in question 1b, which he had solved wrongly in the questionnaire.

Heitor recognised the graphical associations with the concepts of a derivative and an integral, but he did not know how to make full use of these associations.

#### The definition of a concept and the relation between continuity and differentiability

From the questionnaire it is clear that Heitor got mixed up with arguments about the definitions of continuity of a function and derivative of a function when he classified  $f_1$  and  $f_3$  as discontinuous. In the interview he corrected these answers, justifying them by using the definition. He did not make any more mention of his confusion. The functions  $f_2$  and  $f_4$  he had already classified and given the correct justification in the questionnaire.

#### The statement of the FTC

In the statement given in the questionnaire Heitor wrote that  $F(x) = f(x)$ . In the interview he realised his mistake and changed it to  $F'(x) = f(x)$ . However, he did not add that  $f$  had to be continuous.

#### An appreciation of proof

Although Heitor recognised the role of proving in mathematics, as his questionnaire showed, he admitted he had not read the proof but only memorised the result. When he saw the proof, he could not explain correctly where the fact that  $f$  had to be continuous was used. He thought it had to be continuous in the FTC because it was differentiable.

#### Study habits

Heitor said he had mainly used the textbook to study. He also said that before becoming an undergraduate student, he had been studying the proofs and trying to understand

them. However the rate of work at the university did not allow him to go deeper into the subject.

### *Summary*

Heitor questionnaire had indicated two misconceptions that he held: thinking of the FTC as  $F(x) = f(x)$  and mixing up continuity with differentiability. Both misconceptions he cleared up in the interview. He was certainly a good student but did not know how to make much use of graphical reasoning, as his solution to question 4a showed.

### **6.5.11 Hugo**

#### Images of a derivative and of an integral

In the questionnaire Hugo did not use the images of tangent line and area. He said in the interview that his first image of a derivative was the tangent line but he had no graphical image associated with integration; he connected it with anti-differentiation. He did not use graphical reasoning in any part of the interview and did not even associate area with the graph of the function  $f(x) = |x|$ .

#### The definition of a concept

From the questionnaire it was clear that Hugo had used only partial aspects of the definition, evaluating the right- and left-hand limits at one particular point to see if they were equal. He also demonstrated that he did not know how to evaluate these limits, since he had written that they were different in the case of  $f_1$ ,  $f_2$  and  $f_3$  ( $x$  equal to 0, 2 and 0 respectively). In the interview he only recognised that these limits were equal in the case of  $f_1$  and of  $f_3$ , but did not do so immediately. In the case of  $f_4$  he made it clear that he knew that the right- and left-hand limits at 0 did not exist. Hugo was not able to use the definition correctly either in the questionnaire or in the interview.

#### The statement and the use of the FTC

The questionnaire gave the impression that Hugo had copied the FTC from a classmate. In the interview he confirmed this impression, recognising he had cheated. But even with

the correctly written statement of the FTC in front of him, he was unable to use it in any of the questions, either in the questionnaire or in interview.

#### Views of the role of proving

From the questionnaire, it is known that Hugo answered question 2 in the second part correctly, but not question 1. In the interview he changed the solution to question 1, arguing that the answer was “no” because the theorem needed to be proved. He was thus able to answer correctly both of the questions in part II.

#### An appreciation of proof

In Hugo’s interview the proof of the FTC was not shown to him. He said he had not read it and if he tried to convince someone else it was true he would use examples. In no part of the interview did he show awareness of the importance of the theoretical aspects of mathematics.

#### Study habits

Hugo said he had used both: textbook and lecture notes to study. However, the interview left the impression that he had not used the textbook much.

#### *Summary*

The questionnaire had led to the suspicion that Hugo had copied the FTC from a classmate and the interview confirmed this - neither in the interview nor in the questionnaire did he know how to use the theorem. He also did not know on either occasion how to give a correct definition of continuity; he used only partial aspects of it. The only part Hugo did well in was part II. In the questionnaire he had answered question 2 correctly but not question 1, in the interview he answered both correctly.

### **6.5.12 Marcelo**

#### Images of a derivative and of an integral

In the questionnaire Marcelo had used the association between area and integral to solve question 1b and this was the reason why he was chosen for the interview. He said he had

solved the question in that way because the previous teacher (he was doing calculus I for the second time) had emphasised the use of graphs during the course. His first image of a derivative was a tangent line and of an integral was area. However he did not use these images in question 4a of the questionnaire, which he answered incorrectly, marking (c), by thinking of  $F$  as the primitive of  $f$ . He did not change this answer in the interview. Although Marcelo said there had been an emphasis on graphical reasoning in the previous course, he only applied it in one specific situation: question 1b.

#### The definition of a concept and the relations between continuity and differentiability

In the questionnaire Marcelo had shown himself confused about the definitions of a continuous function and a differentiable function. In the interview this confusion persisted. He said he did not know how to give any definition for a continuous function. He also added that if  $f$  is continuous, then  $f$  is differentiable.

#### The statement of the FTC

Marcelo had not answered question 6. In the interview he wrote the following on a separate sheet:

“Since  $f(x)$  is a continuous function in an interval,  $F(x)$  is named the antiderivative of  $f(x)$ , such that  $\int f(x) = F(x)$ ,  $F'(x) = f(x)$ .”

Although he did not state the FTC correctly, at least he wrote that  $f$  had to be continuous and that  $F'(x) = f(x)$ .

#### The role of variables in the FTC

In two parts of the interview Marcelo demonstrated that he did not understand how  $F(x)$  behaves as a function: when he stated the theorem and in question 5, he was not able to use the fact that  $h(0)=1$ . He said he did not understand the role of the variable  $t$  in  $F(x)$  and this was for him an obstacle to completing question 5.

### The use of the FTC

From the questionnaire it was clear that Marcelo knew how to apply the FTC, although he did not apply it in question 2a. In the interview he demonstrated that he knew how to solve this question directly. He said in the interview that when he had finished solving it in the questionnaire by evaluating the elementary primitive first, he realised he could have solved it directly. His first reaction when seeing an integral was to evaluate it; if he was not able to do so (question 5), he used the FTC.

### Views of the role of proving

Marcelo had misread the statement of the FTC in the questionnaire. When the problem with the quotation marks was explained to him, he answered “no” to both questions. The reason he gave for answering “no” in the first question was that he thought it was necessary to try to negate the theorem and to prove its negated form. If it was not possible to give a proof in this way, the theorem itself was correct. The following passage illustrates his argument:

Marcelo:       Take the theorem in its more general form and try to prove the contrary of it; if you manage to prove the contrary it means that it is wrong.

He did not believe that there was a counter-example for the FTC just because so far nobody had ‘proved the contrary of the FTC’. In spite of answering “no”, he is very confused about what proving is.

### An appreciation of proof

Marcelo had no understanding of the proof of the FTC. He did not know how to explain where the fact that  $f$  had to be continuous was used. He also showed that he did not understand the meanings of the limits which appeared in the proof.

### Study habits

Marcelo said he had used the book only to do exercises; the theory he had read in his lecture notes. He also did not read all the proofs; for example, he had never read the proof of the FTC before the interview.

## *Summary*

The questionnaire had already indicated that Marcelo was confused about the relationship between continuity and differentiability and the interview only confirmed this. Although he had solved question 1b graphically in the questionnaire, this was an isolated case; he was not able to solve any other question graphically, either in the questionnaire or in the interview. In relation to part II he had misread the statement of the FTC. After this point had been cleared up in the interview, the answers he gave suggested he thought the theorem was valid since it was not possible to “prove the contrary” of it. His words suggested he was referring to a proof by contradiction, although he showed he had not even understood what it meant.

### **6.5.13 Beatriz**

#### Images of a derivative and of an integral

Beatriz said she associated a derivative with slope and an integral with the opposite of a derivative. She had not used any graphical reasoning in the questionnaire and in the interview she used the association between area and integral to solve question 1b only after being asked to sketch the graph of  $f(x) = |x|$ . However, in another part of the interview, she showed that she did not understand the meaning of this association, when she answered that  $F(0) = 0$ ,  $F$  being defined as in question 4a.

#### The definition of a concept

Beatriz's questionnaire revealed that she mixed up continuity with differentiability of a function. She classified all the functions incorrectly, analysing them to discover whether they had a derivative at the points in the domain. In the interview she mixed up the definitions of a function and of a continuous function. She stated that a continuous function was one in which every point in the domain had an associated image. However, when asked to apply the different arguments (from the interview and from the questionnaire), she became confused, and was not sure of any of the arguments.

### The statement and the use of the FTC

The statement Beatriz wrote in the questionnaire had nothing to do with the FTC and in the interview she did not change it. She neither knew how to state it nor how to use it. The only question in which Beatriz had used the FTC in the questionnaire was 4b. She had solved question 2a by evaluating the elementary primitive first, and in the interview she said she did not know any other way of solving it. She had solved question 5 in the questionnaire wrongly and her first reaction when she saw this question in the interview was not to change it. Later, in the interview she said the result should be  $h'(x) = e^{-t^2}$ , which showed she had problems with the use of variables.

### The role of variables in the FTC

In the questionnaire the problem with variables appeared in the “counter-example” Beatriz gave in question 2, part 2 ( $F(x) = \int_x^x f(t)dt$ ). In the interview this problem came up when Beatriz was asked to solve question 5. After answering  $h'(x) = e^{-t^2}$ , she was questioned about the use of  $x$  and  $t$  and answered that she did not understand how to use these variables. This problem is certainly linked to the fact of not understanding  $F(x)$  as a function, as she revealed in the second part of the questionnaire and in question 4a of the interview, when she said, in a rather mechanical way, that  $F(0)$  should be 0.

### Views of the role of proving

In the questionnaire Beatriz showed that she had not understood the point of the first question and offered a counter-example in the second question. In the interview she changed both answers. However, she still did get question 1 wrong, answering “yes” because the derivative of an integral should be the function itself. In the second question she answered “no”, and her justification seemed to imply that she thought the theorem should always work.

Although it could be interpreted from question 1 of the questionnaire that she did not understand its aim, when this question was re-asked in the interview, she still got it wrong, which makes it seem more likely she did not know how the process of generalising in mathematics occurs.

### An appreciation of proof

Beatriz was quite unable to explain the proof of the FTC. She said  $f$  had to be continuous in the FTC in order to be differentiable.

### Study habits

Beatriz said she had used her lecture notes to study the theory and the textbook only to do exercises. In more than one part of the interview she repeated that the rate of work at the university was very difficult for her and she did not have time to assimilate anything.

### *Summary*

The interview with Beatriz confirmed the impression left by the questionnaire: she remembered only pieces of the course and did not know how to relate them. Her performance in the questionnaire did not change much in the interview. She obviously did not understand what the FTC is about and how  $F(x)$  works as a function. The only question she had solved wrongly in the questionnaire but was able to solve correctly in the interview was question 2 of the second part: she changed about the possibility of offering a counter-example for the FTC. She felt that the new pace of the university work was an obstacle to her development in the course; several times she repeated it was too much for her.

#### **6.5.14 Ricardo**

### Images of a derivative and of an integral

Ricardo had not used the images of tangent line and area in the questionnaire and in the interview he said that his first image of a derivative and of an integral were the symbols:

$\frac{dy}{dx}$  and  $\int$ , respectively. He had correctly solved question 1b in the questionnaire

analytically and in the interview he showed he knew how to solve it graphically. He tried to use the idea of slope of the tangent line to solve question 4a in the interview, but he got mixed up; he said he only knew how to find the graph of  $F(x)$  if he was given the analytic form of  $f(x)$ .



### The definition of a concept

In the questionnaire Ricardo had marked all the functions correctly in question 3. He justified his choices by writing that  $f_1$  and  $f_3$  were continuous all over the domain and  $f_2$  continuous in  $\mathbb{R} - \{1\}$ , and  $f_4$  continuous at  $\mathbb{R} - \{0\}$ , which were not justifications, although they gave the impression that he knew the definition of continuity in which only points in the domain should be analysed.

This impression was not confirmed in the interview. He made it clear that he had given these justifications because he thought that a continuous function was one with no restrictions and, in the domain he specified  $f_2$  and  $f_4$  would have no restrictions. It would be interesting to know what he would have answered if he had been given the function  $f(x) = x$  if  $x \neq 1$  and  $f(1) = 1$ . Ricardo admitted that he did not care about definitions; he said he did not study them much.

### The statement of the FTC

Ricardo was one of the students who wrote in the questionnaire both: that  $f$  had to be continuous and that  $F'(x) = f(x)$  (although he did not write in that way, but as  $F'(x) = f(x) dx$ ). From the questionnaire it was not clear if he wrote that  $F'(x) = f(x) dx$  or  $f'(x) = F(x) dx$ . In the interview he made it clear that he was wrong to write  $dx$  and that the correct relation was  $F'(x) = f(x)$ .

### Views of the role of proving

From the second part of the questionnaire it was inferred that Ricardo believes that examples are enough to state a theorem and that there are counter-examples for theorems proved. The interview confirmed this impression. He did not change his answer to question 2, but did change the one to question 1. He said he would answer “no” because it was not enough to test for only one function, but for a “variety of functions”, as he put it.

### An appreciation of proof

Ricardo did not remember anything about the proof when he saw it. He did not know how to explain any part of the proof and he said he had just accepted it.

### Study habits

Ricardo said he had used his lecture notes more than the textbook to study. He admitted he had not studied the proofs and had only verified the truth of the theorems by testing examples.

### *Summary*

The interview with Ricardo confirmed the impression left by his questionnaire: he only memorised the FTC as a rule, although in the questionnaire the rule was  $F'(x) = f(x) \, dx$  and in the interview it changed to  $F'(x) = f(x)$ . He admitted he was not interested in going deeper into theoretical aspects. Indeed, contrary to the impression given by his questionnaire, he thought of a continuous function as a function with no restrictions at all. As the questionnaire had already suggested, Ricardo believed that examples were enough to state a theorem and that there were counter-examples for theorems.

## **6.5.15 Cristina**

### Images of a derivative and of an integral

Cristina's questionnaire suggested that she used the image of a tangent line in question 4a, thinking of  $f$  as the derivative of  $F$ , although holding the misconception of regarding the derivative of a constant as the constant itself.

She said in the interview that she connected derivative with slope of the tangent line and integral with area and volume. However, she did not know how to use the image of area to solve question 1b. In the questionnaire she had not given any solution to this question and in the interview, in spite of being able to sketch the graph of  $f(x) = |x|$  and relate  $\int_{-1}^1 |x| \, dx$  to the word "area", she did not know how to evaluate the area above the graph

of  $f(x)$ . It seems she used “area” as a word without having any mathematical interpretation for it, as she was not able to use this information.

### The definition of a concept

In the questionnaire Cristina had used a visual argument for classifying all functions as continuous or not: she looked to see if all the points of the function belonged to a single curve. The interview suggested that she was probably thinking of  $f_2$  or  $f_4$ , each one as two functions, because when she was asked what a continuous function was, she answered:

Cristina: In the case of a discontinuous function there are going to appear two functions, three... the idea I have is more or less this. There isn't a general thing, do you understand?

However, this idea of more than one function is not linked to the idea of more than one formula to define a function, since she classified  $f_3$  as continuous. It was probably linked to the idea of not being able to sketch the graph without taking the pencil off the paper.

### The statement and the use of the FTC

Neither in the questionnaire nor in the interview did Cristina answer question 6. From the questionnaire it was interpreted that she was confused about the wording of the FTC, memorising it as the “derivative of the derivative is the function itself”, as she said in question 4b to justify her choice, graph (f). The interview indicated that this was not only a slip and in fact these words did not make any sense to her. As in the questionnaire she had referred to differentiation twice, in the interview she spoke about integrating twice.

She had solved question 2a in the questionnaire by solving the integral first, but did not know how to do it directly in the interview. She had not attempted question 5 in the questionnaire and in the interview she showed surprised at having to find the derivative of an integral function, as the following passage of her interview shows:

Cristina: Finding the derivative of the integral?

Researcher: Why are you surprised?

Cristina: I don't know, because I've never found the derivative of an integral. I am speaking seriously.

Probably, in order to find the graph (f) of question 4b, Cristina reasoned correctly, but was unable to express her reasoning in words.

## Views of the role of proving

From Cristina's questionnaire nothing could be said about her view of proving in mathematics. The interview suggested that she did not have a clear view of how to generalise in mathematics. In connection with question 1, she said she could not publish a theorem based on only one example; she would need to follow what she called "the process", though she did not make it clear what she meant by that, as we can see from the following passage of her interview:

Researcher: But if you test for several examples can you publish?

Cristina: No. That's not what I am saying ...If it is true for five you can publish it. But if it is only true for 1, hum... I'll see the whole process, it's this, this and this. The second one, the process is the same... it is something like that. If it is true for all it's because I don't know how to say what I think, it's something like that, I wouldn't publish for one. Not just based in one example, no.

In relation to the second question, she made it clear why she had answered "no" in the questionnaire without justifying it. She said that if the theorem was stated a long time ago, she did not feel confident enough to give an example which would destroy the theorem. Although she had a feeling that there were no counter-examples for theorems, the reason for that was probably because she did not believe that counter-examples could be found after the theorem had already existed for so many years.

## An appreciation of proof

Cristina understood no part of the proof when it was shown to her.

## Study habits

Cristina said she had used more her lecture notes to study than her textbook. She also said she did not like to study proofs in calculus, but she liked to study them in algebra. She probably connected the need to prove more with teaching of algebra than with teaching of calculus.

## *Summary*

The questionnaire had led to the suspicion that Cristina was confused about what the FTC is and the interview confirmed this impression. She had never thought about having to differentiate an integral function and she did not know even how to state correctly in

words the connection between derivative and integral because the words had no meaning for her, they were only a kind of word game. She had problems not only with the concept of continuity, but also with a more basic one, the concept of a function itself. She probably regarded a discontinuous function as more than a function. In relation to the second part, the interview suggested that she did not know how the process of formalising in mathematics occurs. She did not know how to make a generalisation and she probably believed that there were no counter-examples for the FTC because many years had passed since the theorem was proved and probably any possibility of counter-examples had already been exhausted.

#### 6.5.16 Ivo

##### Images of a derivative and an integral

In the questionnaire Ivo had proved not to have used any graphical image to solve the questions in which he could have used one. In the interview he said that in connection with a derivative he had two images: rate of change and secant lines approximating to the tangent line at a point. In connection with an integral he held the image of area.

In the questionnaire Ivo had sketched the graph of the modular function in question 1b, but had solved the question analytically. In the interview he was able to solve it graphically. He made it clear that he had only sketched the graph in the questionnaire to see if the result of the integral would be positive or negative because he had never solved an integral of a modular function before. He also demonstrated in the interview that he knew how to make the connection between the graphical and the analytical solutions.

However, if he had a good graphical reasoning in question 1b, he did not have one in question 4a. He had marked graph (b) in the questionnaire, reasoning by finding the elementary primitive of  $f(x)$ , and he said in the interview that he did not know any other way of solving it. He probably did not know how to connect the definite integral with area when the integral itself was a function. This impression was supported by the fact that when he was asked to compare  $\int_2^4 (2x-1)dx$  and  $\int_2^x (2t-1)dt$  he associated the first one with area but he did not state any association of the second one with anything.

### The definition of a concept

In the questionnaire Ivo had used the definition to classify the functions as continuous or discontinuous in question 3. Only two points needed to be cleared up in relation to this question: (1) he wrote that  $f_4$  was discontinuous because the limit at 0 was different from  $f(0)$  and (2) he justified  $f_1$  and  $f_3$  being continuous because for every  $a$ , the  $\lim_{x \rightarrow a} f(x) = f(a)$  and also because  $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$ . In relation to (1), he said in the interview the limit at 0 did not exist, and in relation to (2) it was not clear to him that he did not need to write the second condition because it was already included in the first one.

### The role of variables in the FTC

Ivo's questionnaire suggested that he did not know how  $F(x)$  behaved as a function: he got mixed up with variables in question 2a, he answered incorrectly questions 4a and 5, and he wrote in the statement of the FTC in question 6 that  $F(x)$  was equal to  $\int_a^b f(x) dx$ . The interview not only confirmed this impression but also gave more data for it. He did not change any of the answers to these questions; he answered incorrectly what would be the values of  $F(-1)$  and  $F(0)$ ,  $F$  defined in question 4; and he did not know what would be the value of  $h(0)$ ,  $h$  defined in question 5.

### Views of the role of proving

From the questionnaire Ivo's view of the role of proving was not clear. He wrote that the result of question 1 was not necessarily valid for every function, but did not explain why, and he gave a counter-example for question 2. In the interview he said that in relation to question 1 he needed to prove, but he said he would have to prove for every  $F(x)$ , which confirms that he was confused about the functions in the FTC. In relation to question 2 he said there would be no counter-example, but still showed he was confused about the function  $F(x)$  when he tried to explain the counter-example he had given in the questionnaire.

### An appreciation of proof

Ivo said he had studied the proof of the FTC before. However, he did not even understand the role of the limits in the proof of the FTC. When he was asked where the condition that  $f$  had to be continuous was used, he answered that  $f$  had to be continuous in order to delimit the area to be evaluated.

### Study habits

Ivo said he had used only his lecture notes to study.

### *Summary*

The interview with Ivo did not add much to the impression conveyed by the questionnaire in relation to part I. Except for question 1b, in which he demonstrated that he knew how to solve it graphically as well, and question 3, in which he said the limit at 0 for  $f_4$  did not exist, he did not change any of the solutions he had given for the other questions. The interview confirmed Ivo's difficulty in understanding the role of the variables in the FTC, which it is probably due to his not understanding the role of the functions  $f$  and  $F$  themselves in the FTC.

Ivo changed the solutions he had given in the second part of the questionnaire. He had answered "no" to the first question although without explaining the need to give a proof. In the interview he did explain it. He had answered "yes" to the first question, and in the interview he said "no". As regards these answers he understood the role of proving in mathematics.

## **6.5.17 Raul**

### Images of a derivative and of an integral

In the questionnaire Raul had not used any graphical images to solve the questions about integration and differentiation. In the interview Raul said he had no image associated with a derivative and he thought of an integral only as the opposite of a derivative. He mentioned the word "area" in the interview in connection with questions 1b and 4a, but he did not know to use this information. Surprisingly, he stated correctly the graphical

meaning of the integrals:  $\int_2^4 (2x - 1)dx$  and  $\int_2^x (2t - 1)dt$ . He also said the result of the first one would be a number, while the result of the second one an expression with  $x$ .

### The definition of a concept

Raul correctly classified the functions  $f_1$  and  $f_2$  in question 3 as continuous or discontinuous, but his justifications were vague; he only examined them to see if they followed a certain pattern or not. In the interview he said it was not clear to him what a continuous function was. He did not want to state any definition.

### The statement of the FTC

Raul showed by his answer to question 6 that he did not know how to state the FTC. Although he mentioned the words derivative and integral, he expressed the relation between them in a confused and wrong way. In the interview he wrote the following in a piece of paper:

$$“\int g(x)dx = f(x), f'(x) = g(x)”$$

Although he only characterised an indefinite integral, at least he did not get mixed up connecting integration and differentiation.

### The use of the FTC

Neither in the questionnaire nor in the interview did Raul use the FTC. He had answered questions 2a and 4b incorrectly and had not attempted question 5. In the interview he only changed the answer to question 2a; he did so correctly, but by solving the integral first. Like Cristina, he expressed great surprise at having to find the derivative of an integral function.

### Views of the role of proving

From Raul's questionnaire it could only be said that he probably considered “ $F(x) = \int_a^x f(t)dt$ , then  $F'(x) = f(x)$ ” in the wording of question 1 as one more example in which the FTC works. The interview showed that this interpretation was probably correct. He



said that in order to state the theorem it was necessary to test for “a group of functions”. Probably he meant testing with examples of several kinds, varying the limits of integration and varying the function inside the integral. Further in connection still with question 1, he spoke about proof, but he did not know what a proof means, since he spoke about proving for several cases. He used “to prove” with the same meaning as “to test”. The following passage of his interview illustrate this interpretation:

- Researcher: What would be necessary to publish it in a book?
- Raul: More functions, a group of functions, many things... It is not only because of a function ... It's a more complicated thing, isn't it?
- Researcher: Do you think if we had tested for more functions would be enough?
- Raul: It would have to prove... depending on a series of factors.
- Researcher: Do you think it would be necessary to prove?
- Raul: Yes, that too.
- Researcher: That too? What more?
- Raul: To prove ... to work with the derivative and the integral.
- Researcher: Would the proof be sufficient ?
- Raul: Sufficient? I don't know if it would be sufficient but ... I would see if the result of this integral here (*in question 1*) , I would see the result of the derivative of this ...
- Researcher: From this particular case?
- Raul: From this and from other ones as well. One cannot write something based on only one case. It is necessary to change the cases, change the area, different areas to be able to evaluate a formula that it is possible for all of them.

He said he did not know how to answer question 2. He had answered “no” in the questionnaire with a confused justification that he was unable to explain in the interview.

### An appreciation of proof

Although Raul said in the interview that he had understood the proof of the FTC in class when the teacher proved it, in the interview he showed he had not understood it at all (the proof was shown to him in his lecture notes).

### Study habits

Raul said he had only used his lecture notes to study.

## *Summary*

The interview only confirmed the impression left by the questionnaire, Raul could not cope with even the simplest concepts of calculus. His performance in the questionnaire had been poor and he did not improve much in the interview. The only part of the interview in which he did well was in interpreting correctly the meaning of the two integrals:  $\int_2^4 (2x - 1)dx$  and  $\int_2^x (2t - 1)dt$ , but this was a very isolated case. In relation to part 2, Raul's interview suggested that he probably regarded proving as the same as testing.

## **6.6 Summary**

The same headings used for the summary of the questionnaire (section 5.8) will be used in the summary of the interviews besides the following: An appreciation of proof and Study habits.

### **Images of a derivative and of an integral**

In the questionnaire most of the students did not use the graphical images associated with a derivative and an integral to solve any of the questions in which they could have used them. In the interview, when asked about their first image of a derivative and of an integral, only 13 of the 17 students interviewed stated the connection between derivative and slope of a tangent line, while only 7 stated the connection between integral and area. However, the fact of being able to state these associations did not mean they knew how to use them. It was difficult to use any of these images, particularly in question 4, where the integral was itself a function.

In the case of three students, an interesting fact appeared in question 4. These students thought graphically of the derivative of a constant as the constant itself. They mixed up the graph of the slope of the tangent line with the tangent line itself.

Some of the students changed the way they had solved question 1b in the questionnaire, using the area under the graph of it in the interview. However, it has to be considered that they were first asked to sketch the graph of  $f(x) = |x|$ , which may have influenced

them to solve this particular question graphically. If by understanding how to use the graphical images is meant to be able to use them in more than one part of the interview and to be able to connect the graphical and analytical solutions, only two students can be considered to have understood how to use them in the interview: Gérson and Fernando.

### **The definition of a concept**

The main point that arose in the interview in relation to how the students define a concept was the fact that they used different and non-equivalent definitions in the questionnaire and in the interview, sometimes giving more than one definition in the interview. They could, in the questionnaire, have used the argument of  $f$  being continuous if it was possible to sketch the graph of it without taking the pencil off the paper, and in the interview the argument of  $f$  being continuous if every point in the domain had an associated image. When asked to compare these two arguments in one particular case:  $f_2$  for example (it is continuous by the second argument but not by the first one), some students either tried to reconstruct the definition given or incorporated particular cases into the definition given but in such a way as if there were exceptions to it. However, at the end of this process, most of the students in this situation said they no longer knew what a continuous function was. They had great difficulty in establishing any link between the different arguments and if a graphical justification was given they did not know how to link its image with any non graphical argument also given.

Another point is concerned with the mixing up of concepts. There was no consistency in this mixing up in the questionnaire and in the interview; there were students who, in the questionnaire mixed up continuity with arguments related to the concept of a function, and in the interview mixed up the relation between continuity and differentiability.

### **The statement of the FTC**

The interview made it clear that the fact of some students not having written anything on question 6 did not mean that some of them did not hold at least the idea of differentiation and integration as inverse processes. From the sample interviewed, 6 students had not written anything on question 6 and of these 4, tried successfully or not, to state a relation between integration and differentiation.

In the questionnaire, two of the 17 students interviewed stated that  $f$  had to be continuous and in the interview, one more stated that. This suggests they did not see why continuity is important in the theorem and where it is used. However, the point which stands out in the analysis of the answers given is that it is very difficult for the students to state the hypothesis and the thesis of a theorem in a correct way. They are not used to doing this.

### **The relation between continuity and differentiability**

The relation: if  $f$  is differentiable then  $f$  is continuous is certainly in students' minds, perhaps not in the correct order or with its purposes not fully understood. The questionnaire had shown that they try to relate differentiability and continuity but in doing so mix them up; the interview showed that this mix-up can involve integration as well. The students had great difficulty in understanding the logic behind the sentences, particularly of the if-then sentences. This in part can explain why they are confused over the theorems: if  $f$  is differentiable on  $[a,b]$  then  $f$  is continuous on  $[a,b]$  and if  $f$  is continuous on  $[a,b]$ , then  $f$  is integrable on  $[a,b]$ . There was certainly a disconnection between the understanding of a definition of a concept and its graphical meaning, as already indicated in the summary of the questionnaire, but there was also a lack in students' minds of crucial graphical examples which would have helped them to find the correct relations between these concepts.

### **The role of variables in the FTC**

The interviews supplied more data for the interpretation based on the analysis of the questionnaire that the students' problem with variables was linked a more general problem with the concept of a function, in this case exemplified by an integral function, which they were not used to seeing. Most of the students who proved to have problems with variables in the questionnaire found it difficult to find the values of  $F(0)$  and  $F(-1)$ ,  $F$  being defined as in question 4 and also to interpret  $\int_2^x (2t - 1)dt$  as a function. Since they did not understand the variables in  $F(x)$ , they did not understand how they worked in the FTC.

### **The use of the FTC**

The questionnaire had shown that few students applied the FTC in the questions concerned with it (2a, 4 and 5). In the interview this situation did not change much. Only in question 2a was there a significant change. Of the 17 students interviewed, only one had applied the FTC correctly in the questionnaire, and in the interview 8 applied it (5 of these had their answers classified as  $R_p$  in the questionnaire). In question 4b, 8 students interviewed had used the FTC in the questionnaire and only 2 more used it in the interview. In question 5, 7 students had used the FTC in the questionnaire (using the variables correctly or not) and three more used it in the interview. There was no change of situation in question 4a.

The results showed that although in question 2a there was a significant change of situation in the number of students who applied the FTC, it was not as great as expected, considering the simplicity of the question. There was an expectation that all of the interviewees who had solved the integral by first finding the primitive would realise they could solve it directly when seeing the correct answer at the end, but this did not happen. The FTC was not strongly rooted in the students' minds. Two obstacles came out in the interview to applying it: not understanding the role of the variables and memorising it as a meaningless rule.

### **Views of the role of proving**

The analysis of the questionnaire showed that the students were not used to thinking about the process of proving in mathematics and the interviews gave more support to this impression. The questions were re-asked and some students changed their answers, but this did not affect this general conclusion.

As regards question 1, in the questionnaire only 2 of the students interviewed stated it was necessary to prove the theorem, while 2 wrote that one example was enough to state the theorem. In the interview 6 more students said it was necessary to prove, but on the other hand 5 more showed that they held an experimental view of mathematics (perhaps considering that not only one example was enough to state the FTC). As regards question 2, 4 students demonstrated that they knew that proofs generalise, while 6

thought they were giving a counter-example for the FTC. In the interview this situation improved: in all, 8 students showed that they knew that theorems generalise while 4 gave a counter-example.

Although question 2 was answered better in the interview, the improvement was not as great as expected, considering that the question was re-asked in such a way to make it clearer. There was great difficulty in understanding what a proof is. In spite of using the verb “to prove”, the meaning of it for some of them was very similar to the verb “to test”. It was not easy to explain how they would prove theorems in mathematics; bits of sentences such as “to every  $x$ ” and “to prove the contrary”, show that they had in mind some aspects of proving, but repeated them, correctly or not, without understanding their context.

Another aspect to be mentioned was that, although some students said that there were no counter-examples for the FTC, they said it without much conviction. It seemed that they thought that since the theorem had been stated a long time ago and nobody had ever found a counter-example, they would not be the ones to say there was a counter-example.

### **An appreciation of proof**

The interviews showed that none of the students understood the proof of the FTC. They did not know how to reconstruct it and none of them explained correctly where the condition that  $f$  had to be continuous was used. Two students at least said it was important in order to be able to define the area under the graph of  $f$ , which would be true in the case of some discontinuous functions. However, there was a tendency to associate continuity with differentiability and not realise that  $f'$  was not even evaluated in the FTC: four students said that  $f$  had to be continuous in order to be differentiable.

The fact that most of the students said it was important to study the proofs (only two explicitly admitted not caring about proofs), did not mean they had really studied them.

## **Study habits**

The students clearly used their lecture notes more than their textbook to study, disregarding the course they were following. Fourteen students admitted that: 11 of them said they used both, most of these using the textbook only to do exercises, while 3 used only the lecture notes. One student did not tell her study habits and 2 others said they used both: lecture notes and textbook equally; however, the interview of one of them gave the impression that he in fact used only his lecture notes.

This result was relevant in the context of the research, since lecture notes are never detailed, the teacher not always concerned with presenting the theory in detail on the blackboard. The fact of not using the textbook to study certainly would adversely affect the understanding of the theoretical aspects of calculus.

## **Different pattern of responses**

The results described in section 6.22 under the heading “Overall comparison based on patterns of responses” were based on the patterns of responses discussed in Chapter 5, section 5.8, but only the analysis of responses to the questionnaire was used. The general finding described in this section was that procedural and non-theoretical arguments were predominant. This situation changed slightly during the interviews in the case of the procedural answers. Six out of the 10 students who solved question 2a correctly by finding the primitive first, did so in the interview by applying the FTC directly. The other question in which the results changed was 1b. Eight of the students who had used procedural reasoning in this question (correctly or not) showed they knew how to solve it graphically. However, as already mentioned under the heading “images of a derivative and of an integral” in this same section, it has to be considered they were asked to sketch the graph of the modular function first. No significant change happened in relation to questions 4a. It is true that sketching the graph of the modular function may have given them a hint to use the association of area. However, comparing the results of questions 1b and 4a, one more factor has to be taken into account; while the students were probably used to being asked to find the area under the graphs of given functions in the course of calculus I, they were not used to solving exercises connecting graphical images with the function  $F(x)$ .



Although there was some change from procedural to non-procedural answers, there was no significant change from non- theoretical to theoretical answers (questions 3 and 6). The results of question 3 deserves to be examined. Only two students from the interview sample had solved question 3 using the definition in the questionnaire correctly and in the interview, when asked what a continuous function was, one other, Heitor, stated the definition correctly and another, Elisa, got close to stating it correctly, but without much conviction. Most of the students replaced the images used in the questionnaire with other images. This result suggests that the reason why they justified their choices by using other sorts of arguments was not because they were not asked to state the definition of continuity, but because in fact they did not know to state it.

The fact that the results changed only in the case of some questions in which procedural strategies were used suggests that, in these sorts of questions, it is possible to realise, after finding the correct answer, that other strategies can be applied, which shows, in some way, flexibility of reasoning. However, that is not what happens in the case of stating definitions and theorems: translating images into formal language is a very difficult task and is even more difficult is if the images are not correct.



## CHAPTER 7

### THE COMPUTER-TASK BASED INTERVIEWS

#### 7.1 Introduction

The last phase of the main study consisted of task-based activities explored with Graphic Calculus (Tall, Blockland and Kok, 1990). They were conducted in January 1997 after the interviews based on the answers given to the questionnaire. Activities were planned aiming at providing data that would complement the analysis already carried out on the basis of the questionnaire and of the subsequent interview. This time, advantage was taken of the graphical potential of the computer to evaluate students' visual understanding of continuity, differentiation, integration and the FTC itself.

The computer was not used in this research with the specific aim of teaching or of evaluating the software. If there was any teaching involved, it was a consequence of the interview itself.

#### 7.2 The Sample

The activities explored with Graphic Calculus were conducted with 9 students selected from the group of 17 students described in chapter 6. The criteria for the choice of these 9 out of the 17 students were availability (January is a vacation month, so they came to the university only to be interviewed), representativity (there was to be one student per class), number of students by sex (about the same number of males and females).

As a precaution, some might fail to come, more than 6 students were summoned for interview and in the end the following students took part in this last phase of the study: Estela, Gérson and Ricardo from each of the engineering classes; Heitor and Hugo from the computer science class; Alice, Elisa and Roberto from the daytime mathematics class and Cristina from the evening mathematics class. The interviews with Hugo and Elisa were not analysed for the purpose of this research. Hugo seemed to be very tired and not interested in any of the activities from the beginning. Elisa was the only student from the

mathematics students interviewed who did not want to follow a BSc. or teacher training course. At the end it was decided that there should be two students from the daytime mathematics class, so the sample finally comprised 3 mathematics students and 4 students from the other areas.

### 7.3 Description of the activities

As explained in Chapter 4, section 4.5.2, three sets of activities were planned. They are described below.

#### Set 1: Students' image of differentiation and integration

Set 1 consisted of 7 activities. The activities from 1 to 5 aimed at familiarising the students with the program “Gradient”, exploring how they associate a derivative with the slope of a tangent line and exploring their image of  $f'(x)$  as a function. In the first two they were asked to work with polynomial functions, the second differing from the first by a constant ( $f(x) = x^2$  and  $f(x) = x^2 + 2$  respectively), in the third with a trigonometric function  $f(x) = \sin x$ , in the fourth with a continuous but non differentiable function, and in the last with a discontinuous function  $f(x) = \operatorname{sgn} x$ . Besides these general aims, activity 4 and 5 aimed at investigating how students interpret visually what happens at the points where the function is not differentiable, and activity 5 what happens at points where the function is discontinuous.

Activities 6 and 7 aimed at familiarising the students with the program “Area”, exploring how they associate an integral with the area above the graph of  $f(x)$  and familiarising them with the building of the graph of  $F(x)$ . In activity 6 the area was to be evaluated using the computer, while in activity 7 students were asked to find the result by using paper and pencil first and the computer later in order to check the result. Activity 6 was taken from Anton (1995, p.304).

The activities from set 1 were the following:

---

**FIG. 7.1** Set 1 of the computer-task based activities

---

In activities 1 to 6 we will work with option 4 of the main menu: Gradient.

1) a) Type function  $f(x) = x^2$  ( $x^2$ ), domain  $D = [-3, 3]$ . Select F1 and then option 1 "drawing chords through a selected point". Give  $x$  the value 1. Press W (Wait) and P (plot). Follow the table which is being created and interpret the values which are being given. You can modify the initial distance from the fixed point ( $x = 1$ ) to the following point, either by touching D and typing the distance you wish or by using  $<$ ,  $>$  or  $-$ . Calculate  $f'(1)$ , trying to obtain an accurate answer.

b) Keep the same function but this time select option 2 to build up the derivative function. Choose two distinct values for  $h$  in order to compare the results obtained. What does  $h$  mean? Evaluate  $f'(1)$ .

2) Repeat the same activities, but this time with the function  $f(x) = x^2 + 2$ . Compare with the results obtained in activity 1.

3)  $f(x) = \sin x$ . Using options 1 and 2, evaluate  $f'(\pi/2)$  and  $f'(0)$ . Use Zoom and interpret the results obtained on the screen.

4)  $f(x) = \text{abs } x$  ( $|x|$ ). Repeat the same activities but this time give  $x$  the values -1.5; 1.5 and 0. Can you predict which graph will appear on the screen as a result of zooming in on each of these cases?

$$5) f(x) = \text{sgn } x = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ 1 & (x > 0) \end{cases}.$$

Repeat the same activities calculating if possible,  $f'(-1)$ ,  $f'(1)$  and  $f'(0)$ . Interpret the results obtained on the screen.

In activity 6 we will work with options 5 and 6 of the menu, respectively: Area (Integration) and Magnify. In activity 7 we will work only with option 5.

6) Find the area of the region in the first quadrant that lies below the given curves and above the  $x$ -axis:

a)  $x + x^2 - x^3$

b)  $3 \sin x - x - 0.5$

7) Find the area of the region that lies below the curve  $f(x) = |x|$  and the  $x$ -axis in the domain  $D = [-1, 1]$ .

---

## Set 2: Students' understanding of $F(x)$

Set 2 was made up of three activities exploring students' understanding of the area function. In activity 1 they were given the analytical formula of six different functions. For each of them they were asked to sketch the graph of  $f(x)$  and of  $F(x)$  in two different settings: paper and pencil first and computer later. The aim of it was to see which approach they would use to sketch the graph of  $F(x)$ : if they would build  $F(x)$  as they had seen in activities 6 and 7 from set 1 (in "Plot Area Function") or if they would use anti-differentiation formulas. After being asked to sketch the graph with paper and pencil they were asked to check the results on the computer, explaining the differences in the results (if there was any). In activity 2 they also had to sketch the graph of  $F(x)$ , but this time the graph for  $f(x)$  was given instead of the analytical formula.

In activity 3 students had to solve an equation in which the function was an integral function. The aim was to explore their understanding of  $F(x)$  as a function and the role of the variables  $x$  and  $t$ . Items (a) and (b) were taken from Anton (1995, p.304).

The activities in set 2 were as follows:

**FIG. 7.2** Set 2 of the computer-task based activities

1) Sketch with pencil and paper the graph of  $f(x)$  and  $F(x)$  for each  $f$  defined above,  $F(x)$  being the function  $F(x) = \int_0^x f(t)dt$  for (b) and  $F(x) = \int_{-1}^x f(t)dt$ , for (a), (c), (d), (e) and (f).

a)  $f(x) = 2$        $D = [-1, 3]$

b)  $f(x) = 2$        $D = [0, 3]$

c)  $f(x) = x^2$        $D = [-1, 3]$

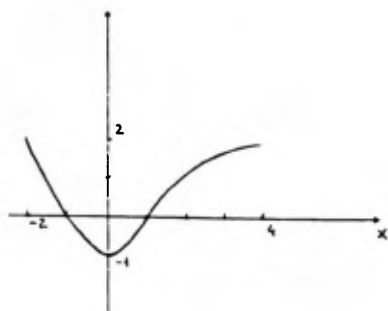
d)  $f(x) = x^2 + 1$        $D = [-1, 3]$

e)  $f(x) = |x|$        $D = [-1, 3]$

f)  $f(x) = \text{sgn}(x)$        $D = [-1, 3]$

Use the computer to draw these same graphs and compare the graphs on the screen with the ones you have drawn.

2) Sketch without using the computer the graph of  $F(x) = \int_{-2}^x f(t)dt$ ,  $f(x)$  being the function the graph of which is drawn above,  $D = [-2, 4]$ .



2) Find all values of  $x$  that satisfy the equations:

a)  $\int_1^x (t^3 - 2t - 1)dt = 0, x > 1.$

b)  $\int_0^x (t^2 + \sin t)dt = 3, x > 0.$

c)  $\int_{-2}^x (|t| - 2)dt = 4, x > -2.$

### Set 3: Students' image of the FTC

Set 3 consisted of eight activities. It was the longest one not only on account of the number of activities but also because of the level of difficulty involved. In activity 1 the students were asked to write  $F(x)$  as an integral function ( $F(x) = \int_a^x f(t)dt$ ). Although this

question was not directly related to the FTC, it was deliberately put in the last set because the idea was to see if, after working with the area function and seen  $F(x)$  written as  $\int_a^x f(t)dt$  in the previous exercises, the students would be able to write it by themselves.

The next activities: 2, 3, 4 and 5 aimed at exploring students' interpretation of the fact that "differentiation and integration are inverse processes" (taking care with the constant), and activity 5 explored the fact that  $f(x) = |x|$  is not differentiable at  $x = 0$ . One therefore has to be careful when considering that differentiation and integration are inverse processes.

Activities 6 to 8 are more directly related to the proof of the theorem. In activity 6 the students were asked to explain how the FTC worked for a constant function. In activity 7, with an appropriate changing of scales, they could see that, the function being continuous, it can be stretched horizontally and so behaves as a constant function in a small interval. Finally, in the last activity students were asked to extend the FTC to a not necessarily constant function. Activity 7 was suggested in the Manual of Graphic Calculus (1990, p.21).

Although the activities more related to the proof of the FTC were 6, 7 and 8, the idea was, depending on how the interview went, to ask questions to see how the students interpreted  $F(x+h) - F(x)$  and  $\frac{F(x+h) - F(x)}{h}$  from exercise 2 onwards. They were to compare their interpretations of  $\frac{F(x+h) - F(x)}{h}$  using "Area Picture" (area above the graph of  $f(x)$ ) and "Plot Area Function" (slope of the function  $F(x)$ ).

---

**FIG. 7.3** Set 3 of the computer-task based activities

---

- 1) Plot area function of  $f(x) = x^2 + \sin x$ ,  $D=[0,3]$ , which we will name  $F(x)$ . How would you symbolically represent  $F(x)$  as an integral?
- 2) Use the program "Area" to draw the graph of  $f(x) = x+2$ ,  $D=[0, 3]$ . Use the Input primitive to verify the analytical expression for the area function obtained. Go to option 4 of the menu, "Gradient", and draw the graph of the area function  $F(x)$  and  $F'(x)$ . Interpret the result.
- 3) Do the same in inverse order: first use "Gradient" to draw  $f'(x)$  and then "Area" to draw the area function of  $f'(x)$ . Compare with the result obtained in activity 3.
- 4) Repeat activities 2 and 3, but this time considering the domain  $[-1,3]$ .
- 5) Repeat activities 2 and 3 for the function  $f(x) = |x|$ ,  $D=[-1,3]$ . Compare with the results obtained.
- 6) The Fundamental Theorem of Calculus relates differentiation and integration, concepts geometrically associated respectively with tangent line and area. Explain how this relation works for a constant function.
- 7) Draw in the computer the graph of  $\sin x$  from  $x = 1$  to  $1.001$  and  $y = -2$  to  $2$ . Discuss the result which appears on the screen. Is the result the same for any function?
- 8) Observe the proof of the Fundamental Theorem of Calculus and try to explain how the relation you have already explored in exercise 5 works for a function which is not necessarily constant.

Does the Fundamental Theorem of Calculus work for any function?

---

The following convention will be adopted when we want to refer to an activity: the pair  $(a, \text{set } b)$  means activity  $a$  from set  $b$ , e.g.  $(3, \text{set } 1)$  means activity 3 from set 1.

## 7.4 Development of the interviews

The interviews lasted from 3 to 4 hours each with students working individually at the computer. They were asked to solve the activities in sets 1, 2 and 3. The role of the researcher in the interviews was to observe them doing the activities, and to intervene when judged necessary, explaining some questions or asking new questions in order to

follow how the students were reasoning. The only question in which no intervention was to be made was in (1, set 2), while the students were drawing the graphs with paper and pencil. The students could do all the activities without stopping or, if they wanted to, could stop to eat or drink something before returning to the interview.

The structure of the interviews changed after the first one had been completed. The reason was that it proved to be very long. Two activities were taken out (3b, set 2) and (4, set 3). Although they were interesting, their aims had already been explored in other activities of the interview. One activity was simplified (2, set 2); the students no longer had to draw the graph of  $F(x)$ , but only describe orally how it would be. This change was decided on because two activities sketching graphs (1 and 2 of set 2) would take a long time and it was more important for this research that the students should complete (1, set 2), from which it would be possible to see which strategy they would use if they had the opportunity of choosing between using formulas or a graphical reasoning. Finally, in set 3 they were asked to represent symbolically  $F(x)$  using the function  $f(x)$  defined in activity 2 and not longer the function  $f(x)$  defined in activity 1, so it was not necessary for them to plot one more graph on the screen.

Although these were task-based interviews, they were not strict. According to how each interview developed, it was possible with some students to go further in the theory while with others it was not possible to do all the exercises in set 3. It should be said that only Heitor and Roberto did activity 7.

### **7.5 Methodology of analysis**

The interviews were tape-recorded and video-taped. They were analysed taking into account the fact that the activities around the computer were designed to complement and make clear some points related to the previous analysis. It was decided to fully transcribe only one interview and, on the basis of this one, design a framework for listening to the others.

The interview chosen to be transcribed was the one with Gérson, since he was a good student as his questionnaire and previous interview showed and also because he expressed himself very clearly. On the basis of Gérson's interview, for each activity or



group of activities, the key questions were established which were to be considered and answered when listening to the other interviews. Obviously, this process did not eliminate the transcription of particular points in each interview that were considered to be interesting to analyse. It should be noted that to establish these questions the headings used in the summaries of chapters 5 and 6 were kept in mind (sections 5.8 and 6.6).

In activities 1 to 5 from set 1 the key questions involved the students' images of a derivative. They were:

- . How did the students interpret the images of a derivative at a point and of a derivative function?

- . Did they use these images to answer the questions posed or did they use differentiation formulas? For example, in activity 2, the aim was to find if they interpreted the fact that the derivative function was the same as in question 1 because the derivative of a constant was 0 or because the curve was translated, and so the slope of the tangent line at each point did not change.

Specifically to activities 3 to 5 and related to their images of a differentiable function, the question was:

- . How did the students react when they saw the result of zooming in at the points at which the function was differentiable or non-differentiable? Did they use the word "differentiable" in explaining what they had seen ?

And related only to activities 4 and 5:

- . Did the students know how the program worked and how it should work for sketching the graph of the derivative function? In this case the program did not show that  $f'(0)$  did not exist when building up the graph of the derivative functions  $f(x) = \text{abs } x$  in activity 4 and  $f(x) = \text{sgn } x$  in activity 5. However, the students could see that the values of the slopes of the tangent lines, when getting close to 0, differed according to whether the approach was from the left or the right. They were 1 and -1 for  $f(x) = \text{abs } x$ , and successively greater positive or negative numbers for  $f(x) = \text{sgn } x$ .

In activities 6 and 7 the question was concerned with students' image of integration. It was:

. How did the students interpret the image of the area under the graph of  $f(x)$  and the building up of the graph of the function  $F(x)$ ?

The activities in set 2 were linked to the students' images of integration and to their understanding of  $F(x)$  as a function (in particular the role of the variables in the function  $F(x)$ ). The key question connected with activities 1 and 2 was:

. What approach was used to build up the graph of  $F(x)$ ? Did the students benefit from the previous activities done with the computer or did they use only anti-differentiation formulas?

And in relation to activity 3:

. Was there any difficulty connected with the use of the variables  $x$  and  $t$ ?

In set 3, except for question 1, the other activities were related to the use of the FTC, the students' understanding of differentiation and integration as inverse processes, and their understanding of the proof. The key questions for activities 2 to 5 were:

. Did the students know to apply the FTC to predict the correct results?

. Did they understand the aims of these questions (namely that differentiation and integration have to be carefully seen as inverse process)?

And for activities 6 to 8:

. Did the students understand why  $f$  has to be continuous in the FTC?

In addition, as observed in section 7.4 in connection with question 2, the students were asked to interpret the meaning of  $F(x+h) - F(x)$  and  $\frac{F(x+h) - F(x)}{h}$ . The key questions for this interpretation were:

. Were the students able to visualise that when  $h \rightarrow 0$ ,  $\frac{F(x+h) - F(x)}{h} \rightarrow f(x)$ ?

. Were the students able to see the connection between “area” and “slope” (they could see the image of area with the graph of the area under the graph of  $f(x)$  using the “Area Picture” and think of the slope of  $F(x)$  using the “Plot Area Function”)?

The aim of asking these questions was not that they should all be answered in each interview and the results of these answers described, but that they should be used to see if there was any result that could help to explain and add some new point to the analysis already described in chapter 6 for each of the students (section 6.5).

## 7.6 Results

The description of each interview will follow the same structure as in chapter 6, section 6.5. Some of the headings in the summary of the interview will be used in these descriptions, as follows: images of a derivative and of an integral; the relation between continuity and differentiability; the use of the FTC; the role of variables in the FTC; an appreciation of proof. The other headings were not mentioned because the subjects they dealt with were not directly explored in this last phase of the study. It should be mentioned that under the heading the use of the FTC, not only the use of the FTC will be treated, but also, in some way, the use of its corollary, since will be described how students understood the idea that one has to be careful when declaring that differentiation and integration are inverse processes.

Just as in chapter 6, not all these headings will be mentioned for each student; a heading will not be mentioned if there was nothing new to be added to the previous analysis.

### 7.6.1 Estela

#### Images of a derivative and of an integral

The questionnaire and the interviews showed that even if Estela knew that an integral and a derivative could be associated with the images of area and tangent respectively, she did not use this information. However, the extent of this association in her mind still needed to be investigated, and the activities with Graphic Calculus helped to clarify this point.

It was discovered that she was used to two images: the secant lines being built up and getting close to the tangent line and the pictures of the rectangles under the graph of  $f(x)$ . However, she tended not to use these images in favour of a more algorithmic approach.

In the case of the graphs of  $f'(x)$  and  $F(x)$ , she was not even familiar with their images at all. She conveyed this very well when she commented on the graph of  $f'(x)$ :

Estela: I had never thought about these dots. We all think like this: either the function at a point or the derivative of the function right away.

By dots she meant each point of the graph of  $f'(x)$ . What she was saying was that she was not used to being asked about  $f'(x)$  as a function of  $x$ , in which  $x$  is not something general but represents each point in the domain. The word “each” is very important in this context. About the graph of  $F(x)$  she said:

Estela: I never realised that there could be the graph of an integral; the teacher had never mentioned that.

In the activities with paper and pencil (1, set 2), she showed clearly her difficulty in visualising  $F(x)$  when, in order to sketch the graph of  $F(x)$  in items (c) and (d) she chose a number of points, and evaluated the area under the graph up to these points but did not analyse the behaviour of the graph of  $F(x)$  in the interval between them. In the other items she used anti-differentiation formulas (without adding any constant to the elementary primitive). She also revealed this same sort of problems in (3, set 3) when she predicted that the result would be  $f(x) = x + 2$  and not  $f(x) = x$ . When interpreting the result on the screen, she justified it because of the “so-called constant”, without thinking that  $F(0) = 0$ .

If it was difficult for Estela to interpret  $F(x)$  graphically, it was even more difficult for her to interpret what  $\frac{F(x+h) - F(x)}{h}$  meant when she was asked about that in set 3. In fact, she could not give any interpretation of it at all.

### The use of the FTC

Estela regarded the FTC as a rule, which she memorised by using an algorithmic procedure which worked for her, as her interview demonstrated. The interview around the computer did not change this situation at all. In (2, set 3), she predicted correctly

what the final result would be, but did not know how to explain graphically why it worked.

Although she predicted correctly the result of (2, set 3) , she was unable to do so with (3, set 3). She said the final result was  $f(x) = x+2$  and did not know how to interpret the correct result on the screen, as already mentioned above in “The images of a derivative and of an integral”. In (5, set 3) she was so confused that she got mixed up trying to establish a relationship between the area function and the derivative of it and in the end was unable to give a correct answer.

Estela’s use of the FTC was limited to memorising, without understanding, that the derivative of the integral is the function itself . When she was asked to interpret this memorised rule and think how it would work in the inverse order, she got completely confused.

#### The role of variables in the FTC

In the questionnaire Estela had shown she was confused about the variables when she answered that  $h'(x) = e^{-t^2}$  in question 5 and when in the interview she said it did not matter whether she was using  $x$  or  $t$ . In the interview around the computer, this confusion remained, when she was asked to represent symbolically  $F(x)$  as an integral in set 3. She wrote that  $F(x) = \int_0^3 x + 2dx$ . She was asked to explain how the function represented in that way would work, and then she realised that she was wrong, as the following passage from her interview shows:

Researcher: Is this the area function?

Estela: No, this will result in a number. I have to write something that will result in an expression. Do I really have to use the integral? If I put that  $t$  thing...

She changed her answer from  $F(x) = \int_0^3 x + 2dx$  to  $F(x) = \int_0^x (t + 2)dt$ . Even if she was not sure about the notation, at least she realised that her first representation would result in a number.

### An appreciation of proof

The previous interview had shown that Estela did not care about proofs and she demonstrated that she did not understand the proof of the FTC. Besides her not caring about proofs in general, there were clearly some obstacles preventing her from understanding the proof of the FTC. Her difficulty in understanding  $F(x)$  and in interpreting the definition of the derivative has been mentioned above. There was also the way she operated with limits as she demonstrated in (8, set 3), when she examined the proof in a textbook and analysed the following limit:  $\lim_{\Delta x \rightarrow 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}$ . She said that when  $\Delta x \rightarrow 0$ ,  $A(\Delta x) \rightarrow 0$  or for a constant (in this case  $A(x)$  is the area function). Probably she worked out the answer as  $A(x + \Delta x) - A(x) = A(\Delta x)$ .

### *Summary*

The questionnaire and the previous interview suggested that Estela knew how to apply the procedures but that it was not easy for her to interpret them. One of the points which was an obstacle for her and became clear in this last phase of the study was her difficulty in interpreting the definition of the derivative of a function  $(f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h})$ . Another point already realised in the earlier phases of the study and now duly confirmed was that she did not understand  $F(x)$  as a function, with an associated graph. Even when working with the computer, she still insisted on using anti-differentiation formulas. The way she operated with limits and with functions was also an obstacle for her, as was realised when she was asked to examine the proof of the FTC. In Estela's case the task-based activities made her aware that she had so many conceptual problems in her mind that at the end she could no longer do anything correctly.

### **7.6.2 Gérson**

#### Images of a derivative and of an integral

In the questionnaire Gérson had used the association between area and integral to solve question 4a. In the interview he mentioned the image of the slope of a tangent line when

he was asked about what his first image of a derivative was. However, it was not possible to say if he knew how to use it. The interview around the computer showed that both graphical associations were, in fact, very clear in his mind and that he systematically made use of them.

He predicted correctly the results which would appear on the screen for almost all the activities. As for the ones he predicted wrongly, he could at least interpret the results on the screen correctly. For instance, in the case of  $f'(0)$  with  $f(x) = \text{sgn } x$ , he said the result would be 0 on the computer, although he knew that  $f'(0)$  did not exist. He was correct in saying the derivative did not exist but was not correct in saying that the computer would give 0 as the solution. However, he correctly interpreted the result which appeared, thinking in graphical terms:

Gérson: I had never thought about that. It (the program) takes the point 0 and one point here (he showed one point on the curve). Each time it (the point which approaches 0) comes here (shows 0), the more it (the tangent line) goes to  $90^\circ$ , this angle here; and the tangent of  $90^\circ$  goes to infinity.

Although he did not use formulas in performing the activities in set 1, surprisingly he did (1, set 2) with paper and pencil using only anti-differentiation formulas. He knew how to do the activities graphically, as he showed when he was asked to examine the results on the screen.

Gérson had a clear view of the definition of the derivative. When he was asked to predict the result of  $f'(0)$  with  $f(x) = \sin x$ , he remembered that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and interpreted this limit graphically, associating it with  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ . He also correctly interpreted graphically that  $\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$  would give  $f(x)$  (by examining the graph of  $f(x)$ ) and  $F'(x)$  (by examining the graph of  $F(x)$ ), although he was given some hints to help him make these interpretations.

The only graphical image he found it difficult to understand at first was that of a “locally straight” function (image of a differentiable function). He predicted wrongly the result of zooming in at the graph of  $f(x) = \text{abs } x$  at  $x = 0$ , saying that the result would be a straight line. However, he later understood the reason why he was wrong.



### The use of the FTC

Gérson used the FTC correctly in the questionnaire. In the interview around the computer he showed he knew how to use it in the activities of set 3. The inverse process, “the derivative of the integral” however, he did not understand very well. He predicted that the final result of (3, set 3) would be  $f(x) = x + 2$  and he did not know how to interpret geometrically this process of finding the derivative of the integral.

### The role of variables in the FTC

In the questionnaire Gérson wrote that “If  $F(x) = \int_a^x f(x)dx$  then  $F'(t) = f(t)$ .”, and in the interview that “If  $F(x) = \int_a^x f(t)dt$  then  $F'(b) = f(b)$ ”, showing that he was confused about variables. In set 3, when he was asked to represent  $F(x)$  as an integral function with  $f(x) = x + 2$ , he first wrote that  $F(x) = \int_0^x (x + 2)dx$  and later, after being asked to explain his answer, that  $F(x) = \int_0^x (t + 2)dt$ . However, his problem was only in writing  $F(x)$  correctly; he did not have any difficulty during the interview in using  $F(x)$  as a function.

### An appreciation of proof

The previous interview had shown that Gérson did not realise that  $f$  had to be continuous and could not understand why when he saw the proof. While solving set 3, he first thought the FTC was valid for any function. However, remembering the functions he had worked with in set 1, particularly  $f(x) = \text{abs } x$  and  $f(x) = \text{sgn } x$ , he said it was not valid for the second one because if  $f$  was not continuous it would not be possible to find  $F'(x)$ . He got to the right point and said:

Gérson: Oh ... so if the function is continuous the area won't suddenly ... give a jump like this. If the area does give a jump, the integral will have a corner like this.

In spite of not being able to express his thought correctly, the image of the graph of  $F(x)$  with a corner was correct.



He seemed to have understood the proof of the FTC and expressed it with great conviction saying:

Gérson: I had never thought about that. I did not know why the integral returns. Now I understand, I really understand.

### *Summary*

The questionnaire and the interviews revealed that Gérson was a good student, concerned with the study of theoretical aspects of calculus. The interview with the computer confirmed this impression. It also showed that he connected derivative and integral with the images of a tangent line and area, and, more than that, he made use of these connections. The only part he did not make use of these images was when using paper and pencil in (1, set 2); perhaps he trusted more in algorithmic approaches than in graphical ones. Although he was able to follow and make correct connections which allowed him to understand the proof of the FTC, this was only possible by his being given some hints; he had never thought about many aspects of it before.

### **7.6.3 Alice**

#### Images of a derivative and of an integral

The questionnaire and the previous interview had shown that Alice did not make use of the association between area and integral. As for a derivative, she knew about its association with the slope of the tangent line. However, it was not possible to say if she knew how to use it. The interview around the computer showed that Alice had very weak graphical reasoning in general. The results described below confirm this point.

In (1, set 1) she thought the computer was evaluating the derivative using anti-differentiation formulas, even when she saw the table with the values of  $\Delta x$  and the corresponding values of the slope in front of her. She was also very surprised when she saw the graph of  $f'(x)$  being built up on the screen, although in this case she seemed to have understood how it was done. In (2, set 1) she used the fact that the derivative of a constant is 0 to explain why the result of  $f'(x)$  ( $f(x) = x^2 + 2$ ) was the same as that of  $f(x) = x^2$  (1, set 1). She could not explain graphically how the software acted in the case of  $f'(0)$  with  $f(x) = \operatorname{sgn} x$  (5, set 1).

If she found it difficult to think of slope associated with a derivative, she found it even more difficult to understand and use the association of an integral with area and how this association worked. She took some time to understand that the convention was that a minus sign was put in front of the real area when it was evaluated in the intervals where the graph of  $f(x)$  is below the  $x$ -axis (the step being a positive number). In the activities with paper and pencil (1, set 2) she used mixed strategies: anti-differentiation formulas and the image of area. In using area, she was in doubt whether she should accumulate areas, as she recognised the computer did, or evaluate each area at each interval (area of each trapezium) and mark a corresponding point for each of the areas. She expressed this by saying:

Alice: This was my doubt: when we used the computer, the computer added the areas, right? I was in doubt if I should add the areas or if I should calculate the little areas and should mark each little area. I was trying to do that, but in one of the exercises I saw that this does not work.

The difficulty in dealing with the graphical meanings of a derivative and an integral increased when she had to mix them and interpret  $\frac{F(x+h) - F(x)}{h}$  when  $h \rightarrow 0$ . She did not interpret it when the graph of  $f(x)$  was drawn on the screen and the area under it was marked. However, when she was asked to interpret this same expression with the graph of  $F(x)$  in front of her she immediately associated it with “slope” (as she said). In the first case an attempt was made to help her to first think of the area below the graph of  $f(x)$ . However, this supposed a good understanding of  $F(x)$  as a function, which she did not have. In the second case, even without fully understanding the association of a derivative with the slope of a tangent line at a point, she could have remembered the definition of a derivative and just said “slope”.

#### The relation between continuity and differentiability

The previous phases of the study did not indicate that Alice mixed up continuity with differentiability. However, in this last phase she showed that she was confused about the definitions of both concepts. When in (4, set 1) she had to justify why  $f'(0)$  did not exist, with  $f(x) = |x|$ , the following dialogue took place:

Alice: The derivative exists when the right -hand and the left-hand limits.... Don't they have to be the same?

Researcher: Which limits?

Alice: Limits of the function, of the absolute value function.

She exhibited the same confusion over the limits in both definitions as Estela did.

### The use of the FTC

Alice's use of the FTC in the questionnaire was limited to question 4a. In the previous interview she had shown herself confused about the connection between a derivative and an integral: whether it was the derivative of the integral or the integral of the derivative. In the interview around the computer she recognised she was applying the FTC and predicted the results correctly in the case of applying it (the derivative of the integral). However, this did not happen with the inverse order (the integral of the derivative): in (3, set 3) she predicted that the final result would be  $f(x) = x + 2$ .

### The role of variables in the FTC

Alice's questionnaire and previous interview did not suggest that she had any difficulties changing variables. However, in the interview around the computer, her first representation of  $F(x)$  in set 3 was  $\int_0^3 x + 2dx$ . She was not asked about that and she did not change this answer until the end of the interview. The other part where she got mixed up with variables was in (3, set 2), when she was asked to solve  $\int_1^x (t^3 - 2t - 1)dt = 0, x > 1$ . She first thought she had to find the value of  $t$  which would satisfy the equation. Her problems with variables in  $F(x)$  seem a consequence of her failure to understand  $F(x)$ , already mentioned in "The images of a derivative and of an integral".

### An appreciation of proof

Alice's knew nothing of the proof of the FTC, as the previous interview had demonstrated. The interview with the computer confirmed this. One of the obstacles has already been mentioned: she did not understand  $F(x)$  as a function. Moreover, when she was asked why  $f$  had to be continuous in the proof of the FTC, she also revealed that she did not understand the concept of limit: she said  $f$  had to be continuous so that it would be defined both in  $x$  and in  $x+h$ .

### *Summary*

Alice had been recommended as one of the best students in her class. However, the three phases of the study did not support this impression. The first two phases showed she had problems even in establishing in simple words the association between differentiation and integration. In the interview around the computer it was clear she had some basic difficulties, as for example: mixing up the limits in the definitions of continuity and differentiability, and not using the graphical images associated with a derivative and an integral.

Building up the graphs of the derivative and integral functions was new to Alice. However, the second one presented an extra difficulty: it had to be interpreted as an accumulative process. For these reasons she could not finish the activities of set 3 satisfactorily; in fact, it was impossible for her to go through the proof of the FTC in one single interview.

#### **7.6.4 Roberto**

### Images of a derivative and of an integral

The previous phases of this study had shown that although Roberto had said that his first image of a derivative was a tangent line and of an integral was area, he did not know how to make full use of these associations. The interview around the computer confirmed this result; every activity and interpretation seemed to be new to him. One example was when he was asked to predict the graph of the derivative function of  $f(x) = \text{abs } x$ . He said it would be  $f'(x) = 1$  and seemed never to have thought that if the

derivative of a function is negative for all points in an interval, then the function decreases on that interval. However, in other parts of the interview he seemed to learn from the activities. He was, for example, able to explain correctly why the result of (2, set 1) would be the same as that of (1, set 1), stating that the parabola was translated but that this did not alter the evaluation of the slope of the tangent line at each point on the curve.

Roberto was confused about how to evaluate the integral function. He was asked to predict what the result of (7, set 1) would be with the graph already drawn in the computer in front of him. He said it would be 0, because  $x$  was negative in one interval, and positive in another one and the “area” in one interval would cancel the “area” in the other. In his own words:

Roberto: If I evaluate the area for a negative  $x$ , it is going to give a negative number, and adding it to the other one, the result will be 0.

Roberto solved the activities with paper and pencil using the graphical image of area. However, he did not use this image correctly; he evaluated the area at certain points (some correctly and others not) and joined the points without reasoning how the function would behave in the interval between these points. It was doubtless difficult for him to see how  $F(x)$  behaved as a function. Only in set 3 he seem to have suddenly woken up and realised how it was.

However, even if in set 3 he began to make connections, it was not possible for him to go further and link slope with area. While doing the activities in set 3, he stated that  $\frac{F(x+h) - F(x)}{h}$  was the formula which led to the derivative of  $f$ . It was only with some help that he could see that it would led to  $f(x)$  (when  $h \rightarrow 0$ ).

Roberto’s graphical reasoning was very poor, but it seems that it was because he was not used to working thinking in graphical terms.

### The use of the FTC

The only question in the questionnaire in which Roberto used the FTC for sure was question 4b. In the previous interview he had shown that he knew how to apply it as well in question 2a. In the interview with the computer he correctly predicted what the result

of the activities 2 and 5 (first part of this one) would be. However he did not predict correctly which function would appear on the screen with the inverse process, “the integral of the derivative”. Both for  $f(x) = x + 2$  and for  $f(x) = |x|$  he said the result would be the  $f(x)$  itself.

### The role of variables in the FTC

The previous interview had suggested that Roberto mixed up the variables  $x$  and  $t$  in the questionnaire (he wrote  $F'(x) = f(t)$  in question 4b and  $h'(x) = e^{-t^2}$  in question 5), because he had failed to understand  $F(x)$  as a function. The interview with the computer supported this conclusion, as already described above in “The Images of a derivative and of an integral”. He once more showed in this last phase of the study his confusion over variables: he was one of the students who represented symbolically  $F(x)$  in set 3 as  $\int_0^3 x + 2 dx$ .

### An appreciation of proof

It is clear from the previous interview that Roberto did not understand the proof of the FTC and could not explain correctly where the condition that  $f$  had to be continuous was used. He said that  $f$  had to be continuous in the FTC because it had to be differentiable.

The interviews around the computer showed that among other obstacles to his understanding of the proof of the FTC there was a weak visualisation of a derivative and an integral and difficulty with the concept of a function. He did activities 6 and 7 of set 3, but not 8. In activity 6 he chose to work with paper and pencil, and sketched in the same axis the graphs of  $F(x)$ ,  $f(x)$  and  $f'(x)$ ,  $f(x)$  being a constant function. He wrote below the graphs:

$$\text{“} \frac{F(x+h) - F(x)}{h} = \operatorname{tg} \theta = F'(x) = f(x) \text{”}$$

$$A(x, x+h) = h. F(x) = \text{”}$$

As can be seen from the last line, he mixed up  $f(x)$  with  $F(x)$  when evaluating the area and did not finish his reasoning. In (7, set 3) he did not understand the change of scales. He thought it was the same as when using the zoom. When he was asked if the result (a straight line parallel to the  $x$ -axis) would appear for every function, he decided to experiment for  $f(x) = \text{sgn } x$  in the neighbourhood of 0 and he concluded it would not work for discontinuous functions. However, when he was asked if the FTC was valid for every function he said it was. Making connections in order to acquire a full understanding of the role of the hypothesis in the theorem is, in fact, a difficult process.

### *Summary*

Roberto's previous interview had revealed his difficulty with the concept of a function, which he had viewed as being defined by a single formula. The most important point that his interview with the computer brought up concerned this same concept. It was difficult for him to realise there was a cumulative process in building up  $F(x)$  and also to see that in order to construct the graph of a function it was not sufficient to find its value at certain points and link them. It was also difficult for him to interpret what the meaning of  $\frac{F(x+h) - F(x)}{h}$  was, since he needed to have very clear in his mind what the graphs of  $f(x)$  and  $F(x)$  were. At the end of the interview he began to make connections. He was interested in the more theoretical aspects of calculus, but was not used to working with this sort of activities.

### **7.6.5 Heitor**

#### Images of a derivative and of an integral

The questionnaire and the previous interview had shown that Heitor did not make use of the graphical images linked to differentiation and integration. However, in the interviews around the computer this changed completely. He and Gérson were the only students who consistently used geometrical reasoning first in all the activities concerning differentiation in list 1. Gérson did so from the beginning and Heitor after activity 2.

He was the only student who correctly sketched all the graphs of (1, set 2) using paper and pencil. He was also the only one who used the same approach as the computer



program to sketch the graph of  $F(x)$  for each of the functions  $f(x)$  given. He had a good view of rate of change and proportionality and these images helped him to make the graphs. The two quotations which follow illustrate how he made use of them; both refer to (1, set 2), the first specifically to item (a):

Heitor: When  $x = 1$ , of course the area will be 0. However, as the function is constant, the area will be a rectangle. Thinking like this, when  $x$  varies, increases, this area will increase uniformly. Thus, the graph of this area will be a linear function; it will be a straight line.

And the following, referring to item (c):

Heitor: I realised the following: from -1 to 0 the area, it is like if ... the rate of increase ... is negative, so it will increase with less rapidly.

The fact that he was given a hint before beginning set 2 might have influenced him to think about how the computer worked. He was told:

Researcher: You are going to sketch the graph of the area function. Think about the work we have done up to now using the computer.

Although this hint should not have been given, perhaps it did not influence him much, since he was very clear in explaining how he had reasoned to make the graphs.

The association of rate of change and proportionality with a derivative helped him to interpret graphically the meaning of  $\frac{F(x+h) - F(x)}{h}$  in a constant function in (6, set 3).

He used this association to see that the graph of  $F(x)$  was linear. In his own words:

Heitor: As  $x$  grows the rectangle will grow uniformly, I mean it varies linearly... it is proportional, this is the word I wanted to find, the value of the area is proportional to the function.

Heitor's interview with the computer completely changed the impression that the questionnaire and the previous interview had left in relation to his use of graphical images. He was certainly not used to them, but from the moment he was shown them on the computer he was the student who made most use of graphical reasoning.

### The relation between continuity and differentiability

Heitor got mixed up with arguments related to the definitions of continuity of a function and of the derivative of a function in the questionnaire (for functions  $f_1$  and  $f_3$ ) and in the interview he had corrected his answers, causing the impression that this



misunderstanding had been cleared up. However, in the interviews around the computer his confusion appeared again in a different situation. In (4, set 1), when he was asked to predict the result of zooming in at  $x = 0$ ,  $f(x) = |x|$ , the explanation he gave showed he was confused over the definitions of continuity and differentiability of a function. The following dialogue shows this point:

Researcher: How do you classify this function?

Heitor: It is not continuous... no, it is continuous. But it does not have a limit ... no, it has a limit at this point 0.

Researcher: What does that mean, doesn't have a limit?

Heitor: Because the right-hand derivative is different from the left-hand derivative at this point 0. The left-hand limit is different from the right-hand limit. (...) The right-hand derivative would be 1 and the left-hand derivative would be -1.

It was not clear from his statement which limit he was referring to; at first he seemed to be referring to the limit in the definition of a continuous function, but the second time to the limit in the definition of a derivative of a function.

### The use of the FTC

Heitor's questionnaire had shown that he thought that the FTC stated that  $F(x) = f(x)$ , but in both interviews he used the correct relation:  $F'(x) = f(x)$ . In connection with the use of the FTC in the last interview, the only activities he could not follow were those of set 3, which aimed at showing that one has to be careful when saying that differentiation and integration are inverse processes. In all of them he predicted the final result would be  $f(x)$ .

### The role of variables in the FTC

At two points in the interview around the computer Heitor showed himself to be confused about variables. Neither in the questionnaire nor in the previous interview had he revealed any difficulties with that. The first point came in (3, set 2), when he was asked to solve the equation:  $\int_{-2}^x (|t|-2)dt = 4$ ,  $x > -2$ . He wrote that  $x-2 = 4$  and so  $x = 6$ . On being questioned about this solution, he did not know to explain it, and said he was not able to solve the question any more. It was strange that he was able to do the items (a) and (b) using the computer without difficulty. However, this activity cannot be

isolated from the whole context of the interview and perhaps he was just tired and let his old intuitions come back to his mind to solve the question.

The other point was when he was asked to represent symbolically  $F(x)$  in set 3. He first wrote  $\int_0^x (t^2 + \sin t) \Delta x$ , but then he wrote  $\int_0^x (t^2 + \sin t) \Delta t$  and much later in the interview corrected this to  $\int_0^x (t^2 + \sin t) dt$ . In fact, more than problems in understanding the variables in  $F(x)$ , this revealed problems in understanding the differential element in an integral.

### An appreciation of proof

Heitor did not understand the proof of the FTC, as the previous interview had shown and he, like Roberto said that  $f$  had to be continuous in the FTC in order to be differentiable. In the interview around the computer, he was able to solve (6, set 3) well, using proportionality as already mentioned in “The images of a derivative and of an integral”. He also concluded that in (7, set 3), the graph would be a horizontal line only for continuous functions (he compared it with  $f(x) = \operatorname{sgn} x$ ). However, he still could not explain why  $f$  had to be continuous. He said that the theorem was not valid for discontinuous functions because it was impossible to find the derivative of  $F$  without the result being a continuous function. This is, of course, not necessarily true. In his own words:

Heitor: Can the derivative of a continuous function not be continuous? For example, if you have a primitive function and you find the derivative of it, this derivative has to be continuous, doesn't it?

Although Heitor was able to make several correct connections in the interview, the full understanding of the proof was still very difficult for him, because, among other reasons, it requires a good comprehension of concepts such as differentiability and continuity.

### *Summary*

The interview with the computer showed that Heitor still mixed up continuity with differentiability, a misconception which his questionnaire had demonstrated and that he had apparently cleared up in the previous interview. His confusion was probably due to his not understanding the limits in the definitions of these two concepts. It also suggested

that he had not used visual reasoning in the previous phases of the study because he was not used to it, but when he did use it, he was the student who most used graphics to solve the questions. He had a strong image of rate of change linked to differentiation, which proved to be helpful in analysing some of the aspects involved in the FTC, such as understanding the meaning of  $\frac{F(x+h) - F(x)}{h}$ .

### 7.6.6 Ricardo

#### Images of a derivative and of an integral

The previous phases of this study have shown that Ricardo had poor graphical reasoning. The interview with the computer confirmed that. All the time he used differentiation and anti-differentiation formulas. He also held some misconceptions connected with his graphical images. First he said that the “slope” ( he used this word in a general sense) is positive for  $x$  positive and negative for  $x$  negative. This same sort of reasoning he employed later when interpreting integration linked to the evaluation of the area under the graph of a function: he said the area took a negative sign in the case of  $x$  being negative. Using this reasoning, like Roberto he predicted that the result of evaluating the area under the graph of  $f(x) = |x|$  (7, set 1) would be 0.

He did (1, set 2) using only anti-differentiation formulas and did not do (2, set 2) correctly. The cumulative process in the building of the area function was not understood by him. In some parts of the interview he thought that, in order to evaluate the value of  $F(x)$  for some  $x$ , he had to evaluate the area under the graph of  $F(x)$  in an interval containing  $x$ , and the value of that area would be the value of  $F(x)$ .

If it was difficult for him to think of the graphical images of a derivative and an integral separately, it was impossible for him to link them. At the end of the interview he made it very clear that he was not used to thinking geometrically and thought it made the subject more difficult. In his own words:

Ricardo: Well, geometrically, it complicates everything. As I said, I only memorise formulas. I don't know calculus, I only know how to memorise formulas, memorising formulas it works.

### The relation between continuity and differentiability

The questionnaire and the previous interview did not indicate that Ricardo was mixing up continuity and differentiability. However, the interview around the computer showed that perhaps he shared the confusion already mentioned in other interviews about the use of the limits in both definitions. However, as with Heitor, it was not clear if in fact he got confused over the limits or if he had in mind the correct limit in the definition of a derivative of a function when he said:

Ricardo: Oh ... yes. The left -hand limit is going to be -1, so there is no limit at this point 0, there is no derivative at this point 0.

### The use of the FTC

The questionnaire showed that Ricardo knew how to apply the FTC correctly, regarding it as a rule. In the interview he once again used this word while doing (3, set 2), in which he tried to use the FTC ( which was a possible approach). However he was not sure if by differentiating  $F(x)$  he would find  $F'(x)$  or  $F(x)$ . He seemed to know how to apply the FTC correctly but he did not know how to describe precisely what he was applying. In the questionnaire it has already been mentioned that he used  $f$  and  $F$  in a confused way. Probably this was due not to his handwriting but to the fact that he really did not know the correct use of them.

All the activities of set 3 using the FTC he did correctly, but he did not understand the idea of using the inverse process to show that differentiation and integration have to be seen with care as inverse processes.

### The role of variables in the FTC

Ricardo was the only student who wrote the integral function in set 3 correctly at the first attempt. He wrote:  $\int_0^x (t + 2) dt$ ,  $D = [0,3]$ . However, in his case this did not mean he had acquired a complete understanding of how  $F(x)$  worked as a function. Examination of his interview shows that after set 2 he seemed to be able to interpret correctly how  $F(x)$  behaved graphically, but still held some misconceptions, as for example thinking that  $F(x)$  was built up locally and not accumulatively (a problem already mentioned).

### An appreciation of proof

In the previous interview Ricardo said he had merely accepted the proof of the FTC without taking the trouble to understand it. From the interview around the computer nothing more can be added, except that while trying to do (6, set 3) he took a paper and pencil and “proved” that  $F'(x) = F(x)$ . He used  $f$  and  $F$  for that in a confused way.

### *Summary*

The impression caused in the previous phases of the study was confirmed in this last interview: Ricardo had memorised the FTC as a rule which he knew how to apply. This interview also showed that although he knew how to apply it, he did not know to describe it; in trying to describe it he mixed up  $f$ ,  $F$  and  $F'$ .

Ricardo also interpreted  $F(x)$  wrongly, as the other interviewees did, thinking of  $F(x)$  being built up locally. This appeared very clearly when at one moment he thought it would be impossible to evaluate the area at a single point and, later on, that the area at a point should be 0 (although it was not the area at a point that was being evaluated but the area function up to this point). Ricardo was a student clearly clung to formulas and used them without trying to interpret them.

### **7.6.7 Cristina**

#### Images of a derivative and of an integral

Cristina’s interview suggested that, in spite of using the words “slope” and “area” in connection with a derivative and an integral, she probably just said them but did not know how to use these images in the process of differentiating and integrating. The interview with the computer confirmed this, and, in fact, showed that she remembered pieces of the course, but as they had no meaning for her, did not know how to use them. In activities 1 to 5 in set 1, she referred constantly to the formula of the slope of the

straight line :  $\frac{y - y_a}{x - x_a}$ , where  $(x_a, y_a)$  is a fixed point, and tried to use it in activity 4 to

find  $f'(1.5)$ , by writing  $\frac{y - 1.5}{x - (-1.5)}$ . It did not help her.

She found it very strange that there could be a function  $F(x)$  and she was the only student that did not even try to do the activities with paper and pencil in (1, set 2) and was unable to do any of the exercises in (3, set 2). The interview with the computer confirmed the impression left by the previous phases of the study, namely, that Cristina's graphical reasoning was very poor or even absent.

### The use of the FTC

The questionnaire and the interview showed that Cristina was confused about what the FTC was. In the questionnaire she said that the "derivative of the derivative is the initial function" and in the first interview she spoke about twice integrating. In the interview with the computer she repeated again the same expression, "the derivative of the derivative" to explain what the activities of set 3 were about. Obviously, Cristina did not understand what the FTC was about and even less how to use it.

### The role of variables in the FTC

Neither in the questionnaire nor in the first interview did this question of variables appear. In the final interview Cristina did not know how to represent  $F(x)$  in set 3. More than any problems related to the role of variables, this only confirmed that she was even unable to make any attempt to represent  $F(x)$ , which to her was a completely unknown function.

### An appreciation of proof

Cristina understood nothing about the proof in either of the interviews. At the end of the interview around the computer she even said that she did not need to know the proofs to succeed in the course.

## *Summary*

The interview with the computer did not add any new information to what was already known from the previous phases of the study. Cristina was confused not only about what the FTC was, but about all the concepts in calculus. She merely repeated certain words without understanding how to use them and what they meant.

### **7.7 Summary**

Looking back at the computer-task based interviews, it is important to consider that they were all gone through in one day, so at the end the interviewees were visibly tired. Even taking this limitation into account, there were certainly some new points to be added to the summary of the interviews described in chapter 6 (section 6.6). They are mentioned now under the following headings: the images of a derivative and of an integral, the relation between continuity and differentiability, the role of variables in the FTC, the use of the FTC, an appreciation of proof.

#### **Images of a derivative and of an integral**

In the summary of chapter 6, mention was made of the fact that even if the students were able to state the associations between the slope of the tangent line and derivative, and area and integral, this did not mean that they knew how to use them. The interview around the computer showed that even if they were familiar with the images of the secant lines going to the tangent line and with the image of the area under the graph of a function, this still did not guarantee that they would use them. There was a strong tendency to use differentiation and anti-differentiation formulas. The images were not new, but the use of them was new. To most of the students the images seemed to perform the function of just illustrating the concepts.

The graphs of the derivative function ( $f'(x)$ ) and of the integral function ( $F(x)$ ) were not familiar to the students. That had already been shown by the results of question 4a in connection with the integral function. It should be observed that it was more difficult for the students to understand how the integral function was built up than the derivative function. While to evaluate the value of  $f'(x)$  the students only had to analyse the slope of



the tangent line at the point  $x$ , to evaluate the value of  $F(x) = \int_a^x f(t)dt$ , they had to calculate the value of the area from  $a$  to the point  $x$ . The cumulative process in  $F(x)$  was certainly an obstacle to their understanding how  $F(x)$  works.

### **The relation between continuity and differentiability**

The questionnaire had revealed that some students mixed up the concepts of continuity and differentiability (question 3) and the first interview that they were confused about the theorem: if  $f$  is differentiable on  $[a,b]$ , then  $f$  is continuous on  $[a,b]$ . The final interview showed that one of the points which probably contributes to this confusion is mixing up the limits in the definitions of these two concepts. A function  $f(x)$  is said to be continuous at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$  and a function  $g(x)$  is said to be differentiable at  $a$  if

$\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$  exists. Each of these limits exists if the corresponding right-hand and left-hand limits exist. Although the limits in these definitions have different shapes and meanings, they were mixed up in some students' minds. This was probably the case with three students, Alice, Heitor and Ricardo. However, one factor which interfered in this analysis was the lack of precision in the use of language; at some points in the interview these students used the word "limit" without making clear which limit they actually meant.

### **The role of variables in the FTC**

The results of the questionnaire and the interviews suggested that the difficulties the students had with variables in  $F(x)$  was linked to a general problem with the concept of a function. They were not used to thinking of different kinds of function, in particular of the integral function. In the interview with the computer the difficulty with variables appeared again when they were asked to represent  $F(x)$  as an integral function. Their initial answers to this question were very significant.

Three of them (Estela, Alice and Roberto) answered  $\int_0^3 x + 2dx$ , which is, to some extent, not surprising, since the image which students commonly have of an integral sign always contains a number in the lower limit and a number in the upper limit. Heitor and



Gérson, despite demonstrating in the interview that they had understood how  $F(x)$  was built up, also gave wrong answers: Gérson wrote  $\int_0^x (x+2)dx$  and Heitor wrote  $\int_0^x (t^2 + \sin t)\Delta x$ . They both corrected their answers later, but their first intuition was wrong. Cristina, here as in the rest of the interview did not give any answer at all. Ricardo, who did not do very well in the interview, was the only one who knew how to represent correctly  $F(x)$  straightway, which was rather unexpected. He even made the domain explicit, perhaps remembering that in the activities to plot the area function with Graphic Calculus he always had to specify the domain or use the default domain.

These results have to be examined bearing in mind that the activities with the computer did not directly address the question of variables. While doing the activities with Graphic Calculus the students were always working with one variable:  $x$ . It was through the activities that the students were helped to visualise how the area function  $F(x)$  was built up. The fact of being able to work with it properly did not mean that the students knew the right way to represent it symbolically. Symbolic representation probably involves reconstruction of images and intuitions not only linked to the building up of the concept.

### **The use of the FTC**

The interview had already suggested that in using the FTC the students had memorised it as a rule without reflecting much on the meaning. This will help to explain the results of activities during the interview around the computer in connection with the use of the FTC (and of its corollary as well). In the activities the students were asked to predict the final result of finding the “derivative of the integral” of a function  $f(x)$  and then the “integral of the derivative” of the same function. They were able to predict correctly what the result of the first one would be but none of them predicted the second one correctly. For both processes they used the memorised rule that the final result was  $f(x)$  itself. They did not try to think what would happen at each step of finding the “integral of the derivative”, either geometrically or analytically.

### **An appreciation of proof**

The previous interview had already shown that none of the students had understood the proof of the FTC and in this last interview only Gerson seemed to have been able at least to explain why  $f$  had to be continuous in the FTC. There were some obstacles to this appreciation of the proof. One of them was the concept of a function: as already mentioned,  $F(x)$  was not itself seen as a function with an associated graph. Another obstacle was the definition of the derivative of a function: neither the process of limit involved nor the graphical meaning of the derivative were clear to the students. In fact, definitions were mixed up and trying to accommodate the graphical images to wrong definitions did not help. This was realised particularly in the case of the concepts of continuity and differentiability of a function, essential for understanding the proof of the FTC.

Yet another point concerned what understanding a proof means. Even if one is able to carry out some activities which help in understanding parts of the proof, understanding the proof is more than understanding pieces of it. One such example was Heitor, who seemed to have followed activities 6 and 7 in set 3, with ease; however, he was not able to explain how was used the fact that  $f$  had to be continuous in the proof of FTC.

## CHAPTER 8

### DISCUSSION OF THE RESULTS

This research has investigated students' understanding of the Fundamental Theorem of Calculus, special attention being given to the formal aspects linked to this understanding, namely, definitions and theorems. Undergraduate students from six different classes of scientific-technological areas from the Federal University of Rio de Janeiro were studied. Various methods of collecting data were used: questionnaire, interviews based on responses to this questionnaire, and computer-task based interviews. They were used to provide different insights on the same issues, which proved to be very rewarding for the study.

The questionnaire was administered to 148 students and was divided into two parts. The data obtained from the first part were useful for identifying the main strategies the students use when they face questions related to the FTC and continuity of functions. The data from the second part gave a general idea of the students' position in relation to generalising in mathematics. Using the questionnaire also made it possible to compare the performance of the classes in relation to the results achieved, with three groups being identified with the scores obtained in the first part of the questionnaire.

The main source of data for analysis was the questionnaire, but the interviews also made a decisive contribution. From the first set of interviews (based on the questionnaire), those of 17 of the 26 students interviewed were analysed. These interviews not only helped to enrich the interpretation based only on the questionnaire, but also provided new data for analysis.

In the second set of interviews, the computer provided a different means for observation. The students solved activities related to differentiation, integration and the FTC. The interviews of seven of the nine students were analysed. The main contribution made by these interviews was to provide greater elaboration of conceptual obstacles related to the function  $F(x)$  and to the relation between continuity and differentiability. A graphical window was very important in enabling students to solve a number of activities in which

they met situations which motivated them to express their difficulties in relation to these topics.

The findings obtained in each of these phases of the study are detailed in chapters 5, 6, and 7. They will be described briefly below.

## **8.1 The main findings**

Some of the results obtained in this study are specific to particular topics while others are more general, related to how students deal with the formal aspects of calculus or the way the course itself is structured. The results will be described by topics, using some of the headings of Chapters 5, 6 and 7 as well as a new one: the three groups in this study. However, it should be kept in mind that this is not a rigorous division. The issues are closely related to each other: there is a dynamic relation between them.

### **Images of a derivative and of an integral**

One of the questions of this study was whether the students were able to evoke the visual images associated with the concepts of derivative and integral in order to apply them to solving problems, to interpreting the construction of the graphs of  $f'(x)$  and of  $F(x)$ , and to giving a geometrical interpretation for the FTC.

In relation to the first of these points, the results of this study suggests that although some students did not solve the questions graphically, this did not mean that they did not have these images in their minds. However, they had them as illustrations for the concepts rather than as tools. A review of the literature shows that graphical images tend not to be used to solve problems in the calculus course, being replaced by algorithmic procedures (Mundy, 1985; Heid, 1988; Artigue (1991) and Eisenberg, 1991). The analysis of the questionnaire supports that finding; in fact most of the students did not use the images of an integral as area or of a derivative as the slope of the tangent line, using algorithmic procedures instead of graphical ones. The first interviews, however, revealed that some of the students knew these images and, although a significant number of students (13 in the case of the derivative and 7 in the case of the integral) were able to state the connection between the derivative and the slope of a tangent line and of the

integral and area, this did not mean they were able to use them, particularly in questions they were not used to.

A fact that should be mentioned is that it is not surprising that the students solved some questions procedurally; perhaps this was the first image in their minds or they may even have felt that a procedural answer, being more detailed, was more suitable to give in a questionnaire. What is surprising is their not evoking a graphical image to check the answer. This situation is well exemplified in the question in which they had to solve an integral of a modular function. Most of the students who answered this question wrongly found 0 as the final result, yet if they had thought about the graph of the modular function and what the integral meant graphically they would have been able to see that 0 was not the answer. Mundy (1985 and 1987), in a similar question, observed that 90% of her students gave the same answer to a similar question and she said the reason was that they just dropped the absolute value sign. However, the analysis of the main mistakes made in this question in the main study showed that the students in this sample did not behave exactly in the same way as the students she had worked with, although their final answer might suggest that. The percentage of students who just dropped the absolute value sign, considering  $\frac{x^2}{2}$  as the primitive for  $|x|$ , was 13.5%. The most frequent mistake was made by 17.6% of the students who considered the primitive as

being  $\left| \frac{x^2}{2} \right|$  or  $\frac{|x|^2}{2}$ . This suggests that these students used a rule which works in the case of polynomial functions (the degree of the function increases by 1 after integration) and applied this rule whenever they were integrating a function. In another part of the questionnaire the same rule was applied in some of the justifications, e.g.

$$F(x) = \left[ \frac{f(t)^2}{2} \right]_{-1}^x$$
 . To the students who applied this rule, this is the image they have of integration.

The interviews around the computer showed that most of the students were familiar with the images of the secant lines going to the tangent line and of the area under the graph of a function. However, they were not at all familiar with the images of the graphs of  $f'(x)$  and of  $F(x)$ . It was not just a question of whether they evoked the visual images

associated with the concepts of derivative and integral in order to interpret the construction of the graphs of  $f'(x)$  and  $F(x)$ , but that these images, particularly those of the integral function, seemed to be new for them. However, while using and interpreting these graphs in the context of the interview around the computer, it was realised that these graphs would have been better understood if the images of the tangent lines and area had been in their minds not only as static pictures but as a dynamic construction.

Previous research already reported difficulties with these functions:  $f'(x)$  and  $F(x)$ . One of the obstacles pointed out was that students do not perceive them as representing functions themselves (Tall (1991a) in reference to  $f'(x)$  and Thomas (1995) in reference to  $F(x)$ ). In this study, this aspect was also identified but the results revealed that some characteristics of the integral function make its graphical visualisation and representation more difficult than  $f'(x)$ .

The graph of  $f'(x)$  is constructed by evaluating the value of the slope of the tangent line at each point  $x$  of the domain, a local process. In contrast, the graph of  $F(x) = \int_a^x f(t)dt$ , is made by calculating the value of the area from  $a$  to point  $x$ , a cumulative process. That is in some way unusual for the examples of functions with which they are used to working. Unusual too is how to represent this. That is why only one student was able to represent the integral function correctly in the interviews around the computer (it should be noted that this student did not fare well in the other tasks, so this answer seemed an isolated fact).

The question as to whether the students were able to use these images to give a graphical interpretation of the FTC was not answered. Given the fact that their use of these images in understanding and solving problems dealing only with the concepts themselves was poor, it was not expected that they would be able to connect them in interpreting the FTC. Only one student from the pilot study seemed to have previously thought about that, but encountered difficulties in understanding what would happen at a point, revealing a problem with the limit concept. This problem is related to a dynamic view of limit, which in the literature was identified in other contexts of learning (Tall and

Vinner (1981), Orton (1983a), Heid (1988) and Cornu (1991)). The interviews in the main study around the computer addressed this graphical connection related to the FTC in the last set of activities, but the results showed that if even one or two students were able to follow the activities with this aim in view, they did not seem to be able to reconstruct them in their minds by themselves.

These results suggest that graphical images are used by the teacher mostly to illustrate the concepts rather than as tools to solve problems.

### **The definition of a concept**

In Chapter 1 it was mentioned that the calculus course is where the first contact of the undergraduate student is going to be established with advanced level mathematics. This contact includes clearly formulated definitions and theorems. Although the students have used them throughout their school life, in general it is only at the university that the definitions and theorems will be made explicit rather than just used. The results of this study show that, in the case of definitions there are problems concerning the use of images associated with that particular definition (exemplified here by the definition of a continuous function) and also problems concerning the comprehension of what a definition means.

Considering the use of images, previous studies show that students tend to use concept images which may differ from the formal definition (Tall and Vinner (1981) and Vinner (1991)). The present study supports the conclusions of those studies, adding that the images which students use to replace the definitions may contain only partial aspects of definition, may be graphical ones or make use of arguments related to other definitions.

The reason why students tend to make use of these images is partially related to the way the definitions are presented in the classroom and in their textbooks. One example which helps to illustrate this point is the image of a discontinuous function as one which is not defined at a specific point.

The first indication of this problem appeared in the pilot study when students were asked to sketch the graph of a discontinuous function. Many students gave examples of graphs of functions as  $f(x) = \frac{1}{x}$ , which are not defined at one point (in this case 0). According to



most of the textbooks used in calculus, they were correct, since the definition that these books use states that one of the conditions for  $f$  being continuous at a point  $a$  (not necessarily in the domain of the function) is that  $f$  has to be defined at that point. Although they were correct according to their textbook definition, it was surprising how this image was so strong in their minds. This image tells almost nothing about what a continuous function is, since the point of discontinuity is not even in the domain.

The analysis of this question led to an examination of a number of books on calculus and analysis, the result of which shows that when some authors define what a continuous function is at a point, they include only points in the domain (Lang, 1968; Lima, 1989), while others do not (Hard, 1952; Courant and John, 1965; Spivak, 1967; Leithold, 1994). There is also the problem of using one definition and not being consistent with it (Courant and John, 1965). This problem of inconsistency related to continuity had also been noted by Tall and Vinner (1981) in relation to the definition of continuity as it is stated in the SMP Book 1.

While the image of a discontinuous function as one which is not defined at a specific point is at least consistent with the definition in their textbooks, the image of a continuous function as one which is defined at every point of its domain (which was given by some students) is probably related to using partial aspects of the definition or using arguments related to the concept of a function ( $f$  is a function if every point of the domain has an associated image).

Having in mind only partial aspects of the definition may also explain the finding that it was in connection with the discontinuous functions that the students most used the definition correctly for certain functions: the use of these aspects would be sufficient to classify certain discontinuous functions as such. Another probable reason for this finding was that teachers usually stress more the use of the definition in non-examples, to say that one of the conditions is not true.

In relation to the use of graphical images linked to a continuous function, the analysis of question 3 shows a significant number of students who used graphical arguments for classifying the function as continuous or discontinuous, particularly the argument of being able (or not) to sketch the graph without taking the pencil off the paper. In this case the use of graphical arguments can lead to conflicts with the formal definition, as



Tall and Vinner (1991) observed. The interviews also showed that students did not know to make the bridge between graphical and non-graphical arguments (which could be parts of the formal definition).

It is interesting to observe that while the use of graphical images instead of algorithmic procedures would mean shortening certain solutions, the use of graphical images instead of definition could be interpreted as incomplete knowledge of the definition, and the interviews showed that this was the case for most of the students. Graphical images were used in a situation to classify the functions, but little used in situations to solve problems; they were used to replace the use of the definition but not to replace the use of algorithms.

Besides these points related to the use of definitions, it was also mentioned that there are problems concerning understanding of what a definition means, which were reflected in the use of different and non-equivalent definitions in the questionnaire and in the interview. When asked to compare these arguments in one particular case, students had different reactions, some trying to reconstruct the definition given, while others incorporated particular cases into the general definition given but in such a way as if there were exceptions to it. There was certainly a disconnection between a definition given and its associated image in the students' minds.

### **The statement of the FTC**

Analysis of how the students stated the FTC in the questionnaire showed how partial and fragmented this statement is in their minds. It was a pity that so many students did not answer the question (40.5%), because the answers analysed were very interesting for general discussion about students stating theorems.

The results of the questionnaire revealed that some students did not even state the result that they thought was true (whether related to the FTC or not) in the form of a theorem. They wrote only sentences with no link (no result was given), or took the definition of the defined integral as a Riemann sum and wrote it as if it was the FTC. Others mixed up several definitions and theorems of calculus and joined fragmented parts together. There is clearly some confusion over the difference between the definition of a concept and the statement of a theorem.

The interviews showed that although many students were unable to state the theorem in the questionnaire, this did not mean they did not even hold the idea that differentiation and integration are inverse processes. However, even considering that some of them might not have remembered what the FTC was, the results just mentioned go beyond the scope of not simply knowledge of the name of the theorem or of the theorem itself and show that, as in the case of definitions, there is clearly some confusion of what a theorem means. It is really surprising how students cope with definitions and theorems, the foundations of how to write and read mathematics.

### **The relation between continuity and differentiability**

The confusion over the concepts of continuity and differentiability of a function appeared first in the questionnaire, when students used the argument of  $f$  being differentiable (or not) to classify the functions as continuous or discontinuous. The first interviews indicated that some of the reasons for this confusion is that although the students had in mind that these two concepts are related by a theorem, some of them did not know how to state the theorem correctly. The mathematical language itself was an obstacle, particularly the if-then sentences. Another problem was the absence of crucial examples in their minds which could have helped them to find the correct relation between these concepts.

The activities around the computer, particularly those which the students were asked to zoom in on at some points, indicated one more reason which contributed to this mix-up: misunderstanding the limits in the definitions of these two concepts. A function  $f(x)$  is said to be continuous at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$  and a function  $g(x)$  is said to be differentiable at  $a$  if  $\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$  exists. A common mistake was students using the first limit to explain why a function was (or not) differentiable.

Both reasons show that the graphical images associated with these two concepts - continuity and differentiability - are poor in some students' minds. The second reason also reveals a disconnection between the understanding of a definition of a concept and its graphical meaning.

### **The use of the FTC**

The results of this study showed that the FTC is not very strongly rooted in students' minds. In a situation in which they could choose to apply the FTC or solve the question by a procedural approach (solving first the integral and so finding its integral), most of them used the longer way. This should not be judged severely, given the fact that it was an easy function to integrate. However, it is amazing that most of the students could not apply it in a situation where they had no other option.

These results recall what Artigue (1991) says, as quoted in Chapter 2 (section 2.3). She points out that fundamental ideas involved in the concepts are hidden from the beginning by theorems that reduce theoretical considerations to algebraic techniques. In the case of the FTC, immediately after students are shown the proof, the corollary of the theorem is introduced and they begin to evaluate integrals and learn techniques to do that. The FTC is usually no longer used.

### **The role of variables in the FTC**

The role of variables in the FTC is not clear to some students, as answers in the questionnaire such as  $F'(x) = f(t)$  reveal. This sort of mistake was also observed with the students investigated by Thomas (1995). This problem could be interpreted as students becoming confused with variables only in a situation of applying the FTC, but it was mainly attributed to the way they cope with variables in the function  $F(x)$ .

The reason it had a connection with became clear in situations where students had to find the value of  $F(x)$  for a given  $x$ . One example is that in the interviews most of them did not know how to find  $F(0)$  and  $F(-1)$ : they were very confused as to whether they should put the 0 and the -1 in the place of  $x$  in  $F(x)$  or in the place of  $t$  in the  $f(t)$  in the integrand.

This difficulty is probably related to a general problem described in Chapter 2 (section 2.4), of students not being accustomed to working with different sorts of functions. In the calculus course they will be introduced to the function  $F(x)$  when they learn the definite integral and will use it up to the point where they learn the corollary of the FTC. Thus they remember of it only when they learn the definition of the logarithmic

function  $\ln x = \int_1^x \frac{1}{t} dt$ ,  $x > 0$  and as soon as they learn to differentiate that function, they will not use it anymore. The integral function is used to state a definition but not used to solve further problems.

### **Views of the role of proving**

The results of this study show that considering only the data collected it is probably an exaggeration to say that the students held an experimental view of mathematics and that they did not believe that proofs generalise. However, they definitively were not used to thinking about the process of generalising in mathematics.

The results of the questionnaire reveal that few students, only 5.4%, were used to thinking of proving a fundamental step to generalise a statement. On the other hand, 25.0 % of the students revealed that they knew that proofs are general. The percentage of students who thought there were counter-examples for the FTC is considerable: 25.7%. The interviews showed that some of the students had not understood the aims of the questions, so the questions were asked again in a slightly different way, but the improvement was not as great as expected. Some of the students clearly think of proving as testing, while others think that no counter-example can be given because the theorem was proved a long time ago and since nobody has ever found a counter-example, they will not be the ones to offer one.

The process of proving is also fragmented in their minds: they repeated sentences like “to every  $x$ ” and “to prove the contrary”. In trying to offer a proof (some answers given to question 6 and question 1 of the second part), they showed that they did not know how to state and relate the hypothesis and thesis of the theorem, nor did they know how to use mathematical language. Some of these facts are connected with not understanding logically how a proof works, but are also associated with proof schemes in which the appearance of the argument itself constitutes for the student a predominant factor in the whole argumentation. Harel and Sowder (1998), name this last factor a “ritual proof scheme”. The students who proved in the second part of the questionnaire that  $F'(x) = f''(x)$  provide good examples of such schemes.

Another fact which Harel and Sowder (1998) noticed in some of their students and was also noticed in the interviews in this study was that some students are convinced by a proof because the teacher exhibited it in the classroom or because it is in their textbooks. Some interviewees, when asked how they had convinced themselves, gave one of the two reasons. However, in the questionnaire the results of question 2, part II are contradictory with this interpretation (and in the case of these students this contradiction happened). Although it was stated in the wording of the question that the proof of the FTC could be found in several books on calculus, only 25.0% of the students in the sample affirmed with conviction that they could not offer any counter-example. These two results, in contrast, may lead to the following interpretation: although some students are convinced of a theorem by reason of an external factor (teacher or textbook), this acceptance can be so passive that they do not even think about what to accept to be convinced of the theorem means.

This study suggests that one of the reasons for this weakness in understanding proofs and the processes of proving is the student's attitude towards the whole mathematics course. Most of the interviewees did not care about proofs. In the interviews they said they had studied the theory in their lecture notes, while for doing exercises they had used the textbook. Some of them also made it clear that they did not study proofs while they were preparing for the exams, since they did not need them to pass. It should be noted that the lecture notes are not usually detailed, and the teacher is not always concerned with establishing all the relevant points on the blackboard. Furthermore, most of the exercises in the calculus textbooks do not usually ask questions in which students are required to examine the proof again. So they are not seen as an important component in the calculus course.

### **An appreciation of proof**

One of the first motivations for this research was to see if students had ever thought about how the geometric concepts of area and tangent line were related by the FTC. The proof of the FTC, if well understood, would play the role of explaining this question. However, this required not only understanding the proofs but also having clearly in mind the graphical images associated with a derivative and a integral. It also required that students should be able to interpret the visual meaning of the definition of a continuous

function. As already observed, their graphical images are poor tools (to solve problems and consequently to go further into the theoretical aspects).

One of the key points for students to understand the proof was to see why the function  $f$  had to be continuous. None of the interviewees could explain why  $f$  had to be continuous in the first interview, and in the second, only one student seemed to have reached the right answer. Apart from this problem, the activities around the computer showed that even if they could understand step by step points which were fundamental for understanding the proof, they were not able to use this information later. Even considering that they were not supposed to be learning in so short a time, if they had read and understood the proof before they would probably have made some connections.

### **The three groups in this study**

The scores obtained in the first part of the questionnaire were used to apply some statistical tests in order to compare the performance of the classes. The results of the first part showed that the classes could be divided into three groups. The best group, group 1, was made up of classes of mechanical and industrial engineering students. The middle group, group 2, was made up of mathematics (daytime class), civil engineering and computer science students. The weakest group, group 3, consisted of the mathematics class that is held during the evenings. As regards the second part of the questionnaire, the difference in the scores did not lead to any division into groups.

The results show that group 3 was the one in which most students did not give any answers to the questions. It was also the group which, in general, presented the highest percentage of answers in which procedural arguments were used (it should be observed that in some questions this percentage in group 3 was not the highest because there were many answers which the group left blank). Group 1 presented the highest percentage of students who demonstrated that they knew how to apply the FTC (questions 2 and 5).

However, the division into groups did not add many new points for analysis. The fact that the results were so bad in part II was frustrating. The idea of introducing mathematics classes in the main study was to see if mathematics students had a different attitude towards mathematics. However, the evening mathematics class is very weak, while in the daytime mathematics class most of the students did not choose mathematics

as their first option. In the interviews there were some students from different classes more interested in theoretical aspects of mathematics, but this was more related to a personal motivation than a consequence of the course they were following. Gérson, for example, was the student who was most outstanding in all the phases of the study; he was the only student who used non-procedural and theoretical arguments in all the questions concerned in the first part of the questionnaire; he was very interested in the theoretical aspects of mathematics; and he was following the industrial engineering course.

It should be added that, although the teacher who teaches the mathematics class tries to push the students more towards theoretical aspects of mathematics, the fact of the students from the mathematics classes coming with many deficiencies, interferes negatively with this aim. The curriculum of calculus presupposes that they have come well prepared from the secondary school; there is no time to review topics from the secondary curriculum (except for the evening mathematics classes in which calculus I is given in two semesters). Teachers' realisation that many mathematics students were arriving at the university without having been in contact with deductive reasoning before, led to the introduction of a course in Euclidean geometry in the first semester, from 1987 onwards. However, students acquire the necessary maturity for such reasoning only after a certain time; one semester is not enough. It would be very interesting to apply part II of the test after students having been at least one year in university, to see if there was any difference.

## **8.2 Summary and calculus remarks**

Looking back at the aims of this study (Chapter 1), although there is still more to be investigated in this field, a considerable number of issues has been analysed and some answers proposed.

The results show that there are some conceptual obstacles that hinder students from understanding the FTC which are, in general terms, associated with difficulties related to the concepts of function, continuity, derivative and integral of a function. More specifically, these difficulties are: understanding and representing the cumulative process in the function  $F(x)$  and relating continuous with differentiable functions.



The results also show that although the role of visual imagery in understanding the concepts of continuity, differentiation and integration should be crucial, there is a disconnection between the understanding of a formal definition and its graphical meaning. Graphics are used only to illustrate the concepts, but not as tools to solve problems or to apply in further theoretical developments (such as the FTC).

The tools the students use to solve problems are mainly algorithmic, some of them reflecting the fact that they memorised procedures that work in particular cases and applied them in every case. This tendency to apply algorithms in a rather mechanical way is reflected in their use of the FTC. The FTC has not become an algorithm in their minds as its corollary has, so their first reaction when they see an integral is to solve it.

Besides problems more related to the concepts themselves and their applications, there are others related to the way that the students view and cope with mathematics. These also affect and distance them from understanding the central ideas behind the theorems. As definitions and theorems are fragmented in their minds, there is no way they can appreciate a proof. Apart from not making sense of the proofs, most of them are not sure of their role in mathematics.

The way the course of calculus is structured, what is expected from the students in this course probably has a great influence on their study habits and consequently on the attention they give to the more theoretical aspects of the course. Most of the students only use their lecture notes to study and the textbook to do exercises. They are not required to investigate further theoretical aspects which could be found in the books, and they probably find no need to study details of the theory the teacher probably does not have time to emphasise. Although the calculus course should not follow the same pattern as a course in analysis, it may be wondered if it is not being transformed into a course in “calculus”.

### **8.3 General implications for teaching**

This study focuses basically on two issues: students’ understanding of certain topics in mathematics (continuity, derivative, integral and the FTC) and students’ conceptions of inherent processes of communication and creation in mathematics (definitions, theorems



and proofs). These two points should not be separated: in learning mathematics, students should learn what mathematics is about. This study suggests how close this link has to be for the learning not to be adversely affected. Some of the difficulties behind the understanding of definitions which are prerequisite for the FTC and of the understanding of the FTC itself are related to students not realising what definitions and theorems are about, what they mean geometrically and what the central ideas behind them are.

Although the aim of this study was not to suggest a specific pedagogy to improve the teaching of calculus but, by focusing on the FTC, to examine how calculus has been learnt, some suggestions for teaching it will be given. It is hoped that they will help to overcome the main obstacles, identified in this research, which students find in calculus, and to make the desirable link suggested above which, if missing, is certainly behind many of these obstacles.

The following are some suggestions for the teaching that resulted from this study:

- (1) Instead of only using geometric images to illustrate a definition, it may be advantageous to begin with them to establish a definition;
- (2) In stating a theorem, certain hypotheses that are being used to obtain the result should be made clear, and in this context the word “result” is a key word;
- (3) Definitions and theorems that are being used in the examples and exercises presented during the course should be emphasised;
- (4) Varied and crucial examples, non-examples (in the case of definitions) and counter-examples (in the case of theorems) should be used by which students can be led to read and check the definitions and theorems, working with several aspects and images involved in one or another;
- (5) The learning of the concept of a function should be revised and elaborated and, when possible, emphasis given to the fact that it establishes a relationship between variables and that in some cases this relation has a geometrical interpretation (the functions  $f'(x)$  and  $F(x)$  are privileged in this sense owing to their association with the slope of a tangent line and with area);

(6) Proofs should be worked on with students, rather than only shown and explained. The calculus course is rich in proofs in which the central idea behind the whole argument is essentially geometric. Why not construct the analytical argument on the basis of the geometrical idea?

(7) Topics should be revisited whenever possible. For example, when students learn Green's, Gauss's and Stokes's theorems, attention can be called to the fact that they are all manifestations of the same idea. Although this terminology is not used in calculus (and informally it is possible to say it in simpler words), the idea is that they relate the integral in an  $n$ -manifold with the integral in an  $(n-1)$ -manifold which is the boundary of the original manifold. The FTC is the case in which  $n = 1$ .

#### **8.4 Limitations of this research**

Various strategies were used (questionnaire, interviews and interviews using the computer) to offer different insights into the research, but the students were evaluated only at the end of their courses. Their progress during these courses was not followed, so there was little familiarity with the way each of the classes was taught. Although this study was not aimed specifically at evaluating this, it would probably have helped to interpret some of the results found, particularly in the light of the influence identified of textbooks and study habits.

Another limitation was the time students were able to work with the computer. They should probably have spent more time (one more day) working with the activities planned, or fewer activities should have been planned for them to undertake. They were very tired at the end, particularly because many of the exercises they were required to solve were new to them.

#### **8.5 Implications for future research**

During this study some issues came up which need further research. One of them concerns the calculus course: its curriculum, the way it is taught, its role in the education of the undergraduate student, and teachers' beliefs about it. All these points will be reflected in how students will study calculus and what importance they will attach to its more formal aspects. One of the statements commonly made about this area is that students who take courses in engineering and computer science do not need to learn any

proofs, but will mainly have to learn how to model and solve problems. This is contradictory with the view of the explanatory aspect of the proof discussed in Chapter 2. Others think that the place of more formal aspects, even for mathematics students, is in the analysis course, whereas in calculus they should primarily learn techniques. The foundations for these beliefs would be interesting to discuss, in order for them not to become mere sayings in which teachers and students can believe and passively accept, without reflecting on their consequences.

Another issue is related to the use of the computer. In this research the computer was not used as a teaching device, but it would be interesting to investigate whether its use could help students to visualise some of the more formal aspects of calculus, giving them insights which would ultimately help them to understand the proofs. It would be very interesting to conduct a study with this aim in mind.

## References

- Anton, H. (1995), *Calculus with Analytic Geometry*, 5 ed., Wiley and Sons, New York.
- Artigue (1991), 'Analysis'. In D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, 167-198.
- Balacheff, N. (1988), 'Aspect of proof in pupils' practice of school mathematics', in D. Plim (ed.), *Mathematics, Teachers and Children*, Hodder and Stoughton, London, 216-238.
- Chazan, D. (1993), 'High school geometry students' justifications for their view of empirical evidence and mathematical proof', *Educational Studies in Mathematics* 24, 359-387.
- Cornu, B. (1991), 'Limits', in D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, 153-166.
- Courant, R. and John, F. (1965), *Introduction to Calculus and Analysis*, vol.1, Wiley and Sons, New York.
- Dreyfus, T. (1990), 'Advanced Mathematical Thinking', in P. Nesher and J. Kilpatrick (eds), *Mathematics and Cognition: a Research Synthesis by the International Group for the Psychology of Mathematical Education*, Cambridge University Press, 113-134.
- Eisenberg, T. (1991), 'Functions and associated learning difficulties', in D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, 140-152.
- Ferrini-Mundy, J. and Lauten, D. (1994), 'Learning about Calculus Learning', *The Mathematics Teacher* 87 (2), 115-121.
- Hanna, G. (1989), 'Proofs that prove and proofs that explain', *Proceedings of the Thirteenth PME Conference*, Paris, vol. 2, 45-51.

- Hardy, G. (1952), *A Course of Pure Mathematics*, (10th ed.), Cambridge University Press.
- Harel, G. and Sowder, L. (1996), 'Classifying Processes of Proving', *Proceedings of the Twenth PME Conference*, Valence, vol. 3, 59-65.
- Harel, G. and Sowder, L. (1998), 'Students' Proof Schemes', In E. Dubinsky, A. Schoenfeld and J. Kaput (eds.), *Research on Collegiate Mathematics Education*, American Mathematical Society, USA, 234-283.
- Healy, L. and Hoyles, C. (1998), *Justifying and Proving in School Mathematics: Technical Report on the Nationwide Survey*, Institute of Education, University of London.
- Heid, K. (1988), 'Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool', *Journal for Research in Mathematics Education* 19(1), 3-25.
- Hersh, R. (1993), 'Proving is convincing and explaining', *Educational Studies in Mathematics* 24, 389-399.
- Hoyles, C. (1997), 'The Curricular Shaping of Students' Approaches to Proof', *For the Learning of Mathematics* 17 (1), 7-16.
- Lang, S. (1968), *Analysis I*, Addison-Wesley, London.
- Leithold (1981), *The Calculus with Analytic Geometry*, (4th ed.), Harper & Row, New York.
- Leithold (1994), *O Cálculo com Geometria Analítica*, (3th ed.), Harbra, São Paulo.
- Leron, U. (1983), 'Structuring Mathematical Proofs', *American Mathematical Monthly* 90, 174-185.
- Leron, U. (1985), 'A Direct Approach to Indirect Proofs', *Educational Studies in Mathematics* 16, 321-325.
- Lewis, P. (1990), *Approaching Precalculus Mathematics Discretely*, MIT Press, Cambridge, Massachusetts.

- Lima, E. (1989), *Curso de Análise*, vol.1, (6th ed.) , IMPA - CNPq, Rio de Janeiro.
- Mason, J., Burton, L. and Stacey, K. (1982), *Thinking Mathematically*, Addison-Wesley, London.
- Mundy, J. (1985), 'Analysis of errors of first year calculus students', *Proceedings of ICME 5*, Adelaide, 170-172.
- Neter, J., Wasserman, W. and Kutner, M. (1990), *Applied Linear Statistical Models: regression, analysis of variance, and experimental designs*, (3rd. ed), Irwin, Boston.
- Noss, R. and Hoyles, C. (1996), *Windows on Mathematical Meanings: Learning Cultures and Computers*, Kluwer, Dordrecht.
- Orton, A. (1983a), 'Students' Understanding of Integration', *Educational Studies in Mathematics 14*, 1-18.
- Orton, A. (1983b), 'Students' Understanding of Differentiation', *Educational Studies in Mathematics 14*, 235-250.
- Robert, A. and Schwarzenberger, R. (1991), 'Research in Teaching and Learning Mathematics at an Advanced Level', in D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, 215-230.
- Spivak, M. (1967), *Calculus*, W. A. Benjamin, London.
- Tall, D. (1989), 'The Nature of Mathematical Proof', *Mathematics Teaching 127*, 28-32.
- Tall, D. (1991a), 'The Psychology of Advanced Mathematical Thinking', in D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer, Dordrecht, 3-21.
- Tall, D. (1991b), 'Visualising Differentials in Integration to Picture the Fundamental Theorem of Calculus', *Mathematics Teaching 137*, 29-32.
- Tall, D. (1993), 'Students Difficulties in Calculus: Plenary Address', *Proceedings of ICME 7*, Quebec, 13-28.

Tall, D. (1994), 'A Versatile Theory of Visualisation and Symbolisation in Mathematics', Paper presented at the *Commission Internationale pour l'étude et l'amélioration de l'enseignement des mathématiques*, Toulouse, July, 1994.

Tall, D. (1995), Personal letter.

Tall, D.; Blokland, P. And Kok, D. (1990), *A Graphic Approach to the Calculus*, (software for IBM compatible computers), Sunburst Communications Inc.

Tall, D. and Vinner, S. (1981), 'Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity', *Educational Studies in Mathematics* 12, 151-169.

Thompson, P. (1994), 'Images of rate and operational understanding of the fundamental theorem of calculus', *Educational Studies in Mathematics* 26, 229-274.

Thomas, K. (1995), *The Fundamental Theorem of Calculus: an Investigation into Students' Constructions*, Ph.D. Thesis, Purdue University Graduate School.

Vinner, S. (1991), 'The Role of Definitions in the Teaching and Learning of Mathematics', *Advanced Mathematical Thinking*, Kluwer, Dordrecht, 65-81.

## APPENDIX 1:

### THE PILOT STUDY QUESTIONNAIRE



## Part I

1) Evaluate:

a)  $\int_0^2 (3x^2 - 4x + 1) dx$

b)  $\int_{-1}^1 |x| dx$

2) If  $g(x) = \int_2^x (2t - 1) dt$ , determine:

a)  $g(2) =$

b)  $g'(x) =$

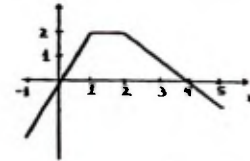
b)  $g'(2) =$

3) a) Define what a continuous function is.

b) Sketch the graph of a discontinuous function.

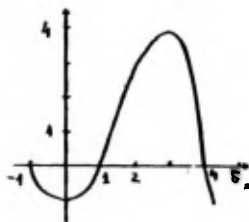
c) Justify why the function you gave as an example in the last item is not continuous.

4) The graph at the right side is the graph of function  $f(x)$  defined in the interval  $[-1, 5]$ .

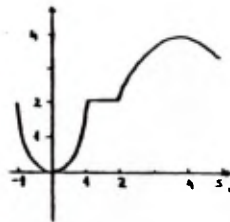


a) Which of the graphs (a), (b), (c), (d) could be the graph of  $F(x) = \int_{-1}^x f(t) dt$ ? ..... Justify your choice

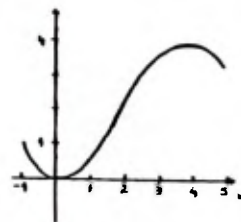
(a)



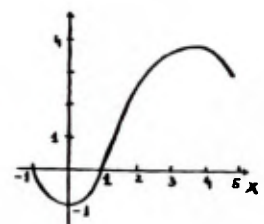
(b)



(c)



(d)

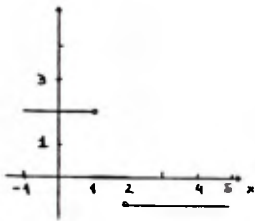


Justification:

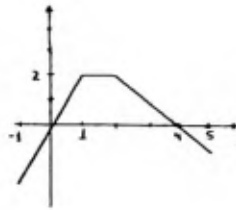
b) Which of the graphs (e), (f), (g), (h) could be the graph of  $F'(x)$  ?...

Justify your choice.

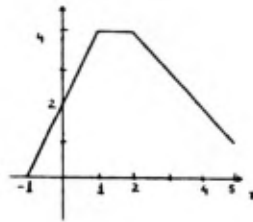
(e)



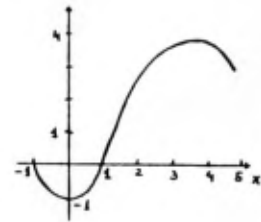
(f)



(g)



(h)



Justification:

5) State the Fundamental Theorem of Calculus.

## Part II

1) It is true that:

a) if  $f(x) = \int_0^x 2t dt$ , then  $f'(x) = 2x$ ;

b) if  $h(x) = \int_{-\frac{\pi}{2}}^x \sin t dt$ , then  $h'(x) = \sin x$ ;

c) if  $m(y) = \int_{-1}^y 3 dt$ , then  $m'(y) = 3$ .

Considering only the data above, is it possible to affirm that for any continuous function,

$$\text{if } g(x) = \int_a^x f(t) dt, \text{ then } g'(x) = f(x) ?$$

2) The Fundamental Theorem of Calculus can be stated in the following way:

"Let  $f$  be a continuous function on  $[a,b]$  and  $x$  be any number in  $[a,b]$ . If  $F$  is the function defined by  $F(x) = \int_a^x f(t) dt$ , so  $F'(x) = f(x)$ ."

Your proof can be found in several calculus textbooks, as for example Leithold's book. Can you give an example of a continuous function  $f$  which does not verify the Fundamental Theorem of Calculus?

---

## APPENDIX 2:

### THE MAIN STUDY QUESTIONNAIRE

Part I

1) Evaluate:

a)  $\int_0^2 (3x^2 + 1) dx$

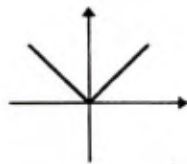
b)  $\int_{-1}^1 |x| dx$

2) If  $g(x) = \int_2^x (2t - 1) dt$ , determine:

a)  $g'(x) =$

b)  $g'(2) =$

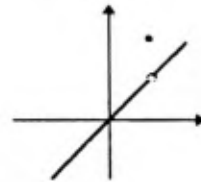
3) Classify the following functions as continuous or discontinuous. Give a brief justification for your answer.



$f_1(x) = |x|$

continuous ☐ discontinuous ☐

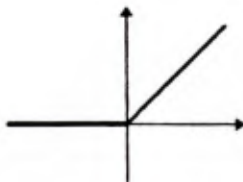
Justification:



$f_2(x) = \begin{cases} x & (x \neq 1) \\ 2 & (x = 1) \end{cases}$

continuous ☐ discontinuous ☐

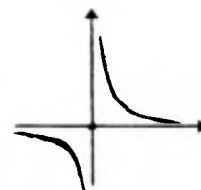
Justification:



$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x > 0) \end{cases}$

continuous ☐ discontinuous ☐

Justification:

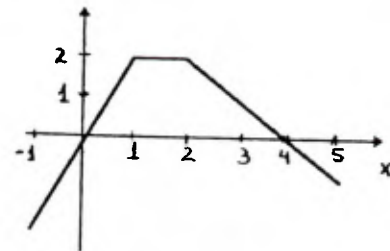


$f_4(x) = \begin{cases} \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$

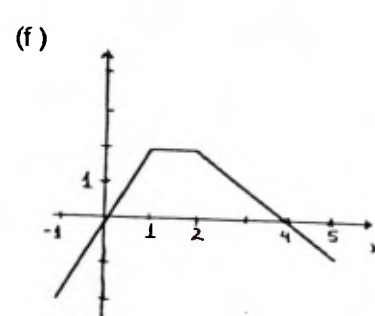
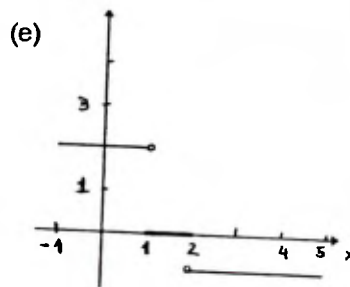
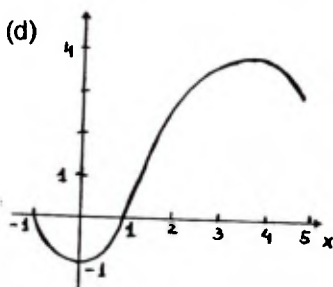
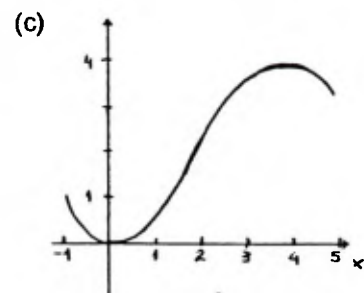
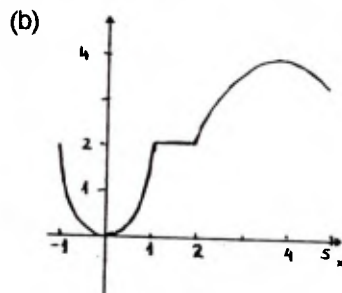
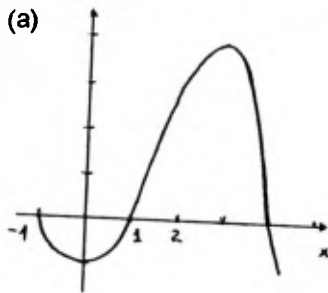
continuous ☐ discontinuous ☐

Justification:

4) The graph at the right side is the graph of function  $f(x)$  defined in the interval  $[-1,5]$ .



Consider the graphs (a), (b), (c), (d), (e), (f) below.



a) Which of the graphs (a), (b), (c), (d), (e), (f) could be the graph of  $F(x) = \int_{-1}^x f(t) dt$  ?

Drawn a circle around the letter corresponding to the right answer. Justify your choice

a   b   c   d   e   f

b) Considering the function  $F(x)$  defined in the item above, which of the graphs (a), (b), (c), (d), (e), (f) could be the graph of  $F'(x)$  ? Drawn a circle around the letter corresponding to the right answer. Justify your choice

a   b   c   d   e   f

5) Let  $h(x) = 5 + \int_0^x e^{-t^2} dt$ . Find the point  $P = (a, b)$  of the graph of  $h$  such that  $h'(a)$  is equal to 1.

6) State the Fundamental Theorem of Calculus.

## Part II

"The first part was devoted to the learning of certain mathematical concepts; in this second part we want to know some of your ideas about mathematics."

1) You are studying mathematics and noticed that:

$$\text{if } F(x) = \int_0^x 2t dt \text{ then } F'(x) = 2x.$$

Suppose that you are the first person in the world to observe this result. On the basis of this example alone, could you publish the following in a book:

" Let  $f$  be a continuous function on the closed interval  $[a,b]$  and  $x$  any number in this interval.

If  $F$  is the function defined as  $F(x) = \int_a^x f(t)dt$  , then  $F'(x) = f(x)$ ." ?

Yes ☐ No ☐

2) The Fundamental Theorem of Calculus can be stated in the following way:

" Let  $f$  be a continuous function in the closed interval  $[a,b]$  and  $x$  any number in this interval. If  $F$  is the function defined as  $F(x) = \int_a^x f(t)dt$  , then  $F'(x) = f(x)$ ."

The proof of this can be found in several books on calculus.

Can you give an example of a continuous function in an interval  $[a,b]$  which does not satisfy the Fundamental Theorem?

Yes ☐ No ☐

## APPENDIX 3:

### THE COMPUTER-BASED ACTIVITIES



## SET 1

In activities 1 to 6 we will work with option 4 of the main menu: Gradient.

1) a) Type function  $f(x) = x^2$  ( $x^2$ ), domain  $D = [-3, 3]$ . Select F1 and then option 1 "drawing chords through a selected point". Give  $x$  the value 1. Press W (Wait) and P (plot). Follow the table which is being created and interpret the values which are being given. You can modify the initial distance from the fixed point ( $x = 1$ ) to the following point, either by touching D and typing the distance you wish or by using  $<$ ,  $>$  or  $-$ . Calculate  $f'(1)$ , trying to obtain an accurate answer.

b) Keep the same function but this time select option 2 to build up the derivative function. Choose two distinct values for  $h$  in order to compare the results obtained. What does  $h$  mean? Evaluate  $f'(1)$ .

2) Repeat the same activities, but this time with the function  $f(x) = x^2 + 2$ . Compare with the results obtained in activity 1.

3)  $f(x) = \sin x$ . Using options 1 and 2, evaluate  $f'(\pi/2)$  and  $f'(0)$ . Use Zoom and interpret the results obtained on the screen.

4)  $f(x) = \text{abs } x$  ( $|x|$ ). Repeat the same activities but this time give  $x$  the values -1.5; 1.5 and 0. Can you predict which graph will appear on the screen as a result of zooming in on each of these cases?

$$5) f(x) = \text{sgn } x = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ 1 & (x > 0) \end{cases}.$$

Repeat the same activities calculating if possible,  $f'(-1)$ ,  $f'(1)$  and  $f'(0)$ . Interpret the results obtained on the screen.

In activity 6 we will work with options 5 and 6 of the menu, respectively: Area (Integration) and Magnify. In activity 7 we will work only with option 5.

6) Find the area of the region in the first quadrant that lies below the given curves and above the  $x$ -axis:

a)  $x + x^2 - x^3$

b)  $3 \sin x - x - 0.5$

7) Find the area of the region that lies below the curve  $f(x) = |x|$  and the  $x$ -axis in the domain  $D = [-1, 1]$ .

## SET 2

1) Sketch with pencil and paper the graph of  $f(x)$  and  $F(x)$  for each  $f$  defined above,  $F(x)$  being the function  $F(x) = \int_0^x f(t)dt$  for (b) and  $F(x) = \int_{-1}^x f(t)dt$  , for (a), (c), (d), (e) and (f).

a)  $f(x) = 2$        $D = [-1, 3]$

b)  $f(x) = 2$        $D = [0, 3]$

c)  $f(x) = x^2$        $D = [-1, 3]$

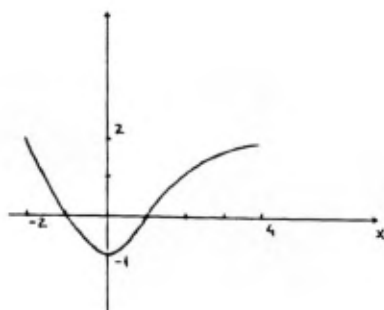
d)  $f(x) = x^2 + 1$        $D = [-1, 3]$

e)  $f(x) = |x|$        $D = [-1, 3]$

f)  $f(x) = \operatorname{sgn}(x)$        $D = [-1, 3]$

Use the computer to draw these same graphs and compare the graphs on the screen with the ones you have drawn.

2) Sketch without using the computer the graph of  $F(x) = \int_{-2}^x f(t)dt$  ,  $f(x)$  being the function the graph of which is drawn above,  $D = [-2, 4]$ .



2) Find all values of  $x$  that satisfy the equations:

a)  $\int_1^x (t^3 - 2t - 1)dt = 0, x > 1.$

b)  $\int_0^x (t^2 + \sin t)dt = 3, x > 0.$

c)  $\int_{-2}^x (|t| - 2)dt = 4, x > -2.$

### SET 3

- 1) Plot area function of  $f(x) = x^2 + \sin x$ ,  $D=[0,3]$ , which we will name  $F(x)$ . How would you symbolically represent  $F(x)$  as an integral?
- 2) Use the program "Area" to draw the graph of  $f(x) = x+2$ ,  $D= [0, 3]$ . Use the Input primitive to verify the analytical expression for the area function obtained. Go to option 4 of the menu, "Gradient", and draw the graph of the area function  $F(x)$  and  $F'(x)$ . Interpret the result.
- 3) Do the same in inverse order: first use "Gradient" to draw  $f'(x)$  and then "Area" to draw the area function of  $f'(x)$ . Compare with the result obtained in activity 3.
- 4) Repeat activities 2 and 3, but this time considering the domain  $[-1,3]$ .
- 5) Repeat activities 2 and 3 for the function  $f(x) = |x|$ ,  $D=[-1,3]$ . Compare with the results obtained.
- 6) The Fundamental Theorem of Calculus relates differentiation and integration, concepts geometrically associated respectively with tangent line and area. Explain how this relation works for a constant function.
- 7) Draw in the computer the graph of  $\sin x$  from  $x = 1$  to  $1.001$  and  $y = -2$  to  $2$ . Discuss the result which appears on the screen. Is the result the same for any function?
- 8) Observe the proof of the Fundamental Theorem of Calculus and try to explain how the relation you have already explored in exercise 5 works for a function which is not necessarily constant.

Does the Fundamental Theorem of Calculus work for any function?

## APPENDIX 4:

### TABLES USED FOR STATISTICAL ANALYSIS

### (1) ANOVA Table

<i>Source of Variation</i>	<i>degrees of freedom</i>	<i>Square Sum</i>	<i>Mean Square</i>	<i>Value of the F-statistic</i>
Classes	5	107.34	21.47	5.85
Residuals	142	521.81	3.67	
Total	147	629.15	4.28	

### (2) Confidence limits (95%) for the analysis of differences in levels between each pair of classes

<i>Classes compared</i>	<i>Inferior limit (I)</i>	<i>Superior limit (S)</i>
em, dm	-2.3083	-0.0917
em, cs	-2.6935	-0.6265
em, ie	-4.5066	-1.7334
em, me	-3.4335	-1.3665
em, ce	-2.6476	-0.4524
dm, cs	-1.4673	0.5473
dm, ie	-3.2872	-0.5528
dm, me	-2.2073	-0.1927
dm, ce	-1.4230	0.7230
cs, ie	-2.7673	-0.1527
cs, me	-1.6644	0.1844
cs, ce	-0.8856	1.1056
ie, me	-0.5873	2.0273
ie, ce	0.2115	2.9285
me, ce	-0.1456	1.8456